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MATH 523, Algebra 1

Bilkent University, Fall 2016, Laurence Barker

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Course specification

MATH 523, Algebra I, Fall 2016

Laurence Barker, Bilkent University. Version: 16 January 2017

Course Aims: The primary aim of this core course is to ensure that all students have a thorough grasp of the basics of group theory and those basics of ring theory that prepare for a subsequent study of Galois Theory. For a respectable but not distinguished grade, as regards the group theory part, the student should be able find the subgroup lattices of small finite groups such as S_4 and A_5, furthermore, the student should be aware of the theory behind the techniques for finding subgroup lattices, especially Lagrange's Theorem and Sylow's Theorem. For the ring theory, a minimal satisfactory requirement is to know standard theory and routines associated with irreducibility of polynomials in a single variable.

Course Description: The final two-and-a-half weeks of the course are to be spent, firstly, on one-hour presentations and discussion of presented material, secondly, on revision for the final exam.

Course Instructor: Laurence Barker, Office SAZ 129.

Course Texts:

Primary: David S. Dummit, Richard M. Foote, "Abstract Algebra", 3rd ed., (Wiley 2003).

Secondary: Michael Aschbacher, "Finite Group Theory" (Cambridge University Press 1986).

Some other sources may be supplied for small components of the syllabus material.

Classes: Tuesdays 10:40 - 11:30 SB-Z19, Fridays, 09:40 - 10:30 SB-Z19.

Office Hours: Fridays, 08:40 - 09:30, SAZ 129.

Syllabus: The catalogue syllabus material was thoroughly covered, specifically, all the theory of abstract and permutation groups, all the theory of abstract commutative rings. In addition to the marked presentations, activities included discussions and further presentations on background to some research topics that were of concern to the three strong students who took the course.

Assessment: Homework 15%, Presentation 15%, Midterm 30%, Final 40%.

Presentation Titles, two-hour presentations:

Müge Fidan, *The Euler characteristic of the p-subgroup complex.*

Redi Haderi, *Enriched categories.*

Andi Nika, *The Artin–Wedderburn Theorem.*

Homeworks and Presentations

MATH 523, *Algebra 1*, Fall 2016

Laurence Barker, Mathematics Department, Bilkent University,
version: 15 December 2016.

Office Hours: Fridays 08:40 - 09:30, room SA-129.

Homework 1

1.1: Using group theory, show that, given $x, n \in \mathbb{Z}$ with $(x, n) = 1 < n$, then $x^{\phi(n)} \equiv 1$ modulo n, where ϕ denotes the Euler totient function.

1.2: Find, up to isomorphism, all the subgroups of the symmetric group S_4 (of order 24), the alternating group A_5 (of order 60) and the dihedral group D_{2n} (of order $2n$, where n is a positive integer).

1.3: The infinite group SO(3), called the **special orthogonal group of degree** 3, can be regarded as the group of 3×3 real orthogonal matrices with determinant 1. It can also be regarded as the group of rotations around a given point in 3-dimensional Euclidian space. Show that, up to isomorphism, the finite subgroups of SO(3) are precisely those groups that appear as subgroups of S_4 or A_5 or D_{2n}. (Hint: you may assume the classification of the Platonic solids: tetrahedron, octahedron, cube, dodecahedron, icosahedron).

1.4: Find, up to isomorphism, all the groups G such that the automorphism group $\mathrm{Aut}(G)$ is trivial. (Hint: Use a theorem whose standard proof makes use of Zorn's Lemma.)

Homework 2

2.1: Let G be a finite group. Let H be a subgroup of G such that $|G : H|$ is the smallest prime divisor of $|G|$. Show that $H \trianglelefteq G$.

2.2: For a group G, let $\mathbb{Q}B(G)$ denote the Burnside algebra of G over $\mathbb{Q}$. For each $H \leq G$, define a $\mathbb{Q}$-linear map $\epsilon_H : \mathbb{Q}B(G) \to \mathbb{Q}$ such that, given a finite G-set X, then $\epsilon_H[X] = |X^H|$. Show that, for $H, K \leq G$, we have $\epsilon_H = \epsilon_K$ if and only if $H =_G K$.

Homework 3

3.1: Extend, to the context of any principal idea domain, the lectured statement and proof Eisenstein's Criterion.

3.2: Hence show that, given a local principal ideal domain R with characteristic zero field of fractions K and characteristic $p \neq 0$ residue field $R/J(R)$ then, when n is sufficiently large, K cannot have primitive roots of unity of order p^n.

MATH 523: Algebra I. Midterm. LJB, 18 November 2016, Bilkent University.

1: Given a group G, a G-set Ω and an element $g \in G$, we define the **support** of g in Ω to be

$$\mathrm{supp}(g) = \{\omega \in \Omega : g\omega \neq \omega\} \ .$$

When $\mathrm{supp}(g) = \Omega$, we say that g **acts fixed-point-freely** on Ω. Show that, if G acts transitively on Ω and $1 < |\Omega| < \infty$, then some element of G acts fixed-point-freely on Ω.

2: By the following steps, show that the alternating group A_n is simple for all integers $n \geq 5$. (No marks will be awarded for proving the theorem by a different method and then retrospectively deducing the steps.)

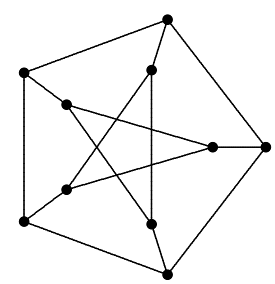
(a) In at most ten words, indicate a quick easy way of proving the simplicity of A_5.

(b) Inductively, now suppose that $n \geq 6$ and A_{n-1} is simple. For a contradiction, suppose also that A_n has a proper normal subgroup L. Show that L acts transitively on $\{1, ..., n\}$ and that every non-trivial element of L acts fixed-point-freely.

(c) Deduce that $|L| = n$. Hence obtain a contradiction.

3: Let Ω be a set, and let $\mathrm{Sym}(\Omega)$ denote the group of permutations of Ω. Find all the simple normal subgroups of $\mathrm{Sym}(\Omega)$. (Hint: Of course, the answer depends on the cardinality $|\Omega|$. To deal with cases where Ω is infinite, consider the elements of $\mathrm{Sym}(\Omega)$ that have finite support.)

4: Homomorphisms, in the category of graphs, are edge-preserving functions between vertex sets. The depicted graph is called the **Peterson graph**.



(a) Find a way of labelling the 10 vertices of the Peterson graph with the 10 subsets of size 2 in $\{1, 2, 3, 4, 5\}$ such that any two adjacent vertices are labelled with disjoint subsets. Hence prove that the automorphism group of the graph is S_5.

(b) Describe a surjective homomorphism from the graph of the dodecahedron to the Peterson graph.

(c) What is the isomorphism class of the group of rotational symmetries of the dodecahedron? Justify your answer carefully.

(d) What is the isomorphism class of the group of rigid (distance-preserving) symmetries of the dodecahedron? Justify your answer carefully.

5: Prove the following theorem, called Goursat's Theorem: Let F and G be groups. Consider the quintuples (I, U, θ, V, J) such that $F \geq I \trianglerighteq U$ and $V \trianglelefteq J \leq G$ and θ is an isomorphism $I/U \leftarrow J/V$. The condition $A = \{(i, j) \in I \times J : iU = \theta(jV)\}$ characterizes a bijective correspondence $(I, U, \theta, V, J) \leftrightarrow A$ between the quintuples and the subgroups $A \leq F \times G$.

MATH 523: Algebra I. Final. LJB, 10 January 2016, Bilkent University.

1: Let $K \trianglelefteq G$ be a finite groups.
(a) Let p be a prime, S a Sylow p-subgroup of K. Show that $G = N_G(S)K$.
(b) State and prove a generalization of part (a) for a G-set X upon which K acts transitively.

2: The regular 4-dimensional polytope called the **600-cell** or the **tetraplex** has 120 vertices. Given $\theta \in [0, \pi]$, letting $n(\theta)$ be the number of vertices subtending, at the centre, an angle of θ from a given vertex, then the non-zero values of $n(\theta)$ are as shown in the following table.

θ	0	$\pi/5$	$\pi/3$	$2\pi/5$	$\pi/2$	$3\pi/5$	$2\pi/3$	$4\pi/5$	π
$n(\theta)$	1	12	20	12	30	12	20	12	1

For each of the 4 angles $\pi/5$, $2\pi/5$, $3\pi/5$, $4\pi/5$, the 12 vertices at that angle comprise a regular icosahedron. For each of the 2 angles $\pi/3$ and $2\pi/3$, the 20 vertices form a regular dodecahedron. The 30 vertices at angle $\pi/2$ form an icosidodecahedron (the polyhedron constructed by taking the vertices to be the midpoints of the edges of an icosahedron or, alternatively, a dodecahedron).

The rotational symmetry group G acts transitively on the vertices. Evaluate $|G|$. Evaluate the number of Sylow 5-subgroups of G.

3: In this question, you may assume standard versions of standard results about finite abelian groups. You are to formulate a suitable elegant definition and then prove that your definition has the required features. You are to define, for any finitely generated abelian group A, the notion of a *quasibasis* of A. The definition must be such that:

A: Any quasibasis of A is a finite subset of A.
B: Given a quasibasis S of A, then $A \cong \prod_{s \in S} \langle s \rangle$ where $\langle s \rangle$ denotes the cyclic subgroup generated by s.
C: Given quasibases S and T of A, then there exists a bijection $S \leftrightarrow T$ such that, when $S \ni s \leftrightarrow t \in T$, we have $\langle s \rangle \cong \langle t \rangle$.
D: Every finitely generated abelian group has a quasibasis.

Solutions to Final exam

1: Part (a). The uniqueness part of Sylow's Theorem implies that, given $g \in G$, then there exists $k \in K$ such that $S^g = S^k$. Hence $gk^{-1} \in N_G(S)$.

Part (b). Given $x \in X$, then $G = N_G(x)K$. The proof is the same as for part (a), but with S replaced by x and with the equality $S^g = S^k$ replaced by $g^{-1}x = k^{-1}x$.

Comment: These arguments, and other variants, are sometimes called **Frattini arguments**. According to legend (I have not checked the primary source), part (a) is the original Frattini argument, used by Giovanni Frattini in 1885.

2: Recall, the regular icosahedron and regular dodecahedron both have rotational symmetry group A_5. The icosidodecahedron, by its construction, also has rotational symmetry group A_5.

In particular, the G-stabilizer of a vertex is A_5. By the Orbit–Stabilizer Equation,

$$|G| = 120\,|A_5| = 2^5.3^2.5^2 = 7200\ .$$

Let n be the number of Sylow 5-subgroups of G. We shall prove that $n = 36$.

By Sylow's Theorem, $n \equiv 1$ modulo 5 and n divides $2^5.3^2 = 288$. So

$$n \in \{1, 16, 6, 96, 36\}\ .$$

Let us say that an element $g \in G$ is **good** provided $|\langle g \rangle| = 5$ and g fixes a vertex. Consider a good element g and a g-fixed vertex x. stabilized by g. Let $m(\theta)$ be the set of g-fixed vertices subtending, at the centre, an angle of θ from x. A rotation of the dodecahedron with order 5 fixes 2 opposite vertices. But, for the other two polygons mentioned in the question, the vertices have degree less than 5, hence a rotation with order 5 cannot fix any vertices. We deduce that the values of m are as shown in the next table.

θ	0	$\pi/5$	$\pi/3$	$2\pi/5$	$\pi/2$	$3\pi/5$	$2\pi/3$	$4\pi/5$	π
$m(\theta)$	1	2	0	2	0	2	0	2	1

In particular, each good element of G fixes precisely 10 vertices. So, letting s be the number of good elements, there are precisely $10s$ pairs (h, y) such that h is a good element fixing vertex y. On the other hand, since A_5 has precisely 24 elements with order 5, the number of such pairs is 120.24. Therefore $s = 120.24/10 = 2^5.3^2 = 288$. Since there are only 2 conjugacy classes of elements with order 5 in A_5, there must be at most 2 conjugacy classes of good elements in G. So $[g]_G$ has order 288 or 144, in other words, $C_G(g)$ has order $|G|/288 = 25$ or $|G|/144 = 50$. Either way, $C_G(g)$ evidently contains a unique Sylow 5-subgroup of G. But the Sylow 5-subgroups of G have order 25, so they are abelian. We deduce that each good element belongs to a unique Sylow 5-subgroup.

Let us say that a subgroup $H \leq G$ is **good** provided $|H| = 5$ and some element of H is good, equivalently, all 4 non-trivial elements of H are good. The set of good subgroups is stable under G-conjugation, so each Sylow p-subgroup contains the same number, k say, of good subgroups. The number of good subgroups is $s/4 = 72 = kn$. Plainly, $k \neq 5$. A group of order 25 can have at most 6 subgroup with order 5, so $1 \leq k \leq 6$. Therefore

$$n \in \{72/1, 72/2, 72/3, 72/4, 72/6\} = \{72, 36, 24, 18, 12\}\ .$$

Comparing with the previously obtained constraint on n, we conclude that $n = 36$.

3: We define a **quasibasis** of A to be a subset $S \subseteq A$ such that: S spans A; if $\sum_s \lambda_s\, s = 0$ with$\lambda_s \in \mathbb{Z}$, then each $\lambda_s\, s = 0$; the annihilator of each s in $\mathbb{Z}$ is a primary ideal, we mean, (n) where $n = 0$ or n is a prime power.

This can be generalized to the scenario where $\mathbb{Z}$ is replaced by any principal ideal domain Θ. A primary ideal of Θ is an ideal I such that, in the quotient ring, Θ/I, every zero-divisor is nilpotent.

The justification is immediate by appeal to a suitable version of the Structure Theorem for Finitely Generated Modules of a Principal Ideal Domain.