Archive of past papers, solutions and homeworks for MATH 523, Algebra 1, Fall 2012, Laurence Barker

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MATH 523 Algebra 1, Fall 2012

Handout 1: Course specification

Laurence Barker, Mathematics Department, Bilkent University,
version: 20 September 2012.

Course Aims: The primary aim is to ensure that all students have a thorough grasp of the basics of group theory. A good victory condition for a respectable grade is the ability to find the subgroup lattices of small finite groups, with awareness of the theory that is prerequisite for that: Lagrange's Theorem, Sylow's Theorem, conjugacy classes, centralizers, normalizers, as well as familiarity with some important families of groups, such as symmetric groups. Most of the course credit will be for this material.

A secondary aim, for students already familiar with that material, is fluency in contexts with some spice: contexts of application, use of Zorn's Lemma.

Course Description: The main narrative of the course is to provide a foundation in group theory, together with some fundamental interactions with ring theory. Amazingly, some mathematics PhD programme students were recently found to be unable to find the subgroup lattice of the group D_8. We must make sure such a lapse will never happen again. So the priority must be on thorough grasp of core material rather than breadth of awareness.

However, some students already know this material like the backs of their hands. To keep them awake, the course will sometimes venture out into further realms which, however, will be assigned only a bit of extra credit.

The final two weeks of the course will be spent on 30-minute presentations from the students and discussion of presented material followed by revision for the final exam. Presentations will be compulsory for all participants.

Course Instructor: Laurence Barker, Office SAZ 129.

Course Texts:

Primary: David S. Dummit, Richard M. Foote, "Abstract Algebra", 3rd ed., (Wiley 2003).

Secondary: Thomas W. Judson, "Abstract Algebra: Theory and Applications", (undisclosed online address, 2012).

Some other sources may be supplied for some small components of the syllabus material.

Classes: Tuesdays 10:40 - 11:30 SB-Z10, Thursdays, 13:40 - 15:30 SB-Z10.

Office Hours: Tuesdays, 11:40 - 12:30, SAZ 129.

Syllabus: Week number: Monday date, subtopics.

1: 17 Sept: Symmetry, permutation groups, permutation sets, Lagrange's Theorem, Orbit-Stabilizer Theorem.

2: 24 Sept: Statement of Sylow's Theorem, groups of small order, dihedral groups. The three isomorphism theorems.

3: 1 Oct: Abelian groups, rings, Chinese Remainder Theorem, Structure of Finite Abelian groups.

4: 8 Oct: Automorphisms of cyclic groups, Gauss' Primitive Element Theorem

5: 15 Oct: Set theory, ZFC Axioms, Zorn's Lemma and applications to algebra.

6: 22 Oct: Categories and automorphism groups

Bayram followed by Republic Day, 25 - 29 October.

7: 29 Oct: Symmetric groups. Conjugacy classes, centralizers and normalizers. Simplicity of alternating groups.

8: 5 Nov: Sylow's Theorem. Various proofs of it and applications.

9: 12 Nov: Finite p-groups. Burnside Basis Theorem. Extremal families of finite p-groups.

10: 19 Nov: Automorphisms of groups. Semidirect products. Groups of order pq. Euclidian motions.

11: 26 Nov: Classical linear groups. Finite subgroups of the motions of solid Euclidian space.

12: 3 Dec: Schrier Refinement Theorem, Zassenhaus' Butterfly Lemma, Jordan Holder Theorem. Variants of those theorems for groups with oporators.

13: 10 Dec: Presentations and discussions.

17 Dec: Presentations and discussions.

15: 24 Dec: Review.

Assessment:

• Homeworks and Quizzes 10%.
• Midterm I, 25%, 8 November.
• Midterm II, 25%, 6 December.
• Final, 30%.
• Presentation, 10%
(25% absence from quizzes incurs grade reduction. 50% absence from quizzes results in F or XF. The 30-minute presentation is obligatory.)

Class Announcements: All students, including any absentees from a class, will be deemed responsible for awareness of class announcements.

MATH 523 Algebra 1, Fall 2012

Homeworks, Quizzes and Presentations

Laurence Barker, Mathematics Department, Bilkent University,
version: 28 December 2012.

Homeworks

Homework 1: Set 20 Sept. Due 27 Sept.

1: Download Judson and read Chapter 6, "Cosets and Lagrange's Theorem".

2: Do questions 3, 4, 9, 10 on pages 100 and 101, weakening hypotheses or strengthening conclusions where possible.

3: Prove Cayley's Theorem (as stated in class).

Homework 2: Set 27 Sept. Due 4 Sept. Read about the Three Isomorphism Theorems for Groups in both of the course texts.

Homework 3: Set 4 Sept. Due 6 Sept. Find the automorphism groups of K_5 and $K_{3,3}$ and the Peterson graph. (All three graphs depicted.)

Homework 4: Set 6 Sept. Due 19 Sept. Judson, Chapter 10, questions 1-15, pages 166-168.

Quizzes

Quiz 1: Show that, given a group G and elements $f, g, h \in G$, then the following three conditions are equivalent: (a) $gf = hf$, (b) $g = h$, (c) $fg = fh$.

Quiz 2: As additive groups, is the index of $\mathbb{Q}$ in $\mathbb{R}$ (a) finite, (b) countably infinite or (c) uncountable?

Quiz 3: 6 September. For $n \geq 3$, what is the automorphism group of the n-cycle graph? (Depicted; the graph with n vertices all of degree 2.)

Quiz 4: Let $H \leq G$. Show that, if $|G : H| = 2$, then $H \trianglelefteq G$. Give an example to show that, for any odd prime p, there exist finite groups $H \leq G$ such that $|G : H| = p$ but $H \not\trianglelefteq G$.

Quiz 5: Let p be a prime. Show that, up to isomorphism, there are precisely two groups with order p^2, both of them abelian.

Quiz 6: Find, up to isomorphism, all the groups G such that $\mathrm{Aut}(G) = 1$.

Presentation titles

Elif Doğan: Lie groups.

Erion Dula: A group-theoretic approach to Gauss' Primitive Element Theorem.

Gezmiş Oğuz: Extremal finite p-groups.

Merve Kaya: Jordan's classification of the finite subgroups of SO(3).

Ismail Alperen Oğut: The Structure Theorem for Finitely Generated Cyclic Groups.

Abdullah Önur: Topological groups.

Recep Özkan: Topological groups.

MATH 523 Algebra I, Midterm 1

8 November 2012, LJB, Bilkent University.

Time allowed: 110 minutes.

Please make sure your name is on every sheet of your script.

1: 10%: Let p be a prime, let $H \leq G \trianglerighteq N$ be groups and suppose that $|G : N| = p$. Show that if $H \not\leq N$ then $|H : H \cap N| = p$.

2: 10% Let F be the set whose elements are the pairs (x, y) with $x \in \mathbb{Z}/7$ (the integers modulo 7) and $y \in \mathbb{Z}/3$. Show that G becomes a group when equipped with the operation given by

$$(x, y)(X, Y) = (x + 2^y X, y + Y) \ .$$

3: 20% Letting F be as in the previous question, find all the subgroups of F. Draw a picture of the subgroup lattice of F. (Hint: you may find it convenient to introduce the notation $a = (1, 0)$ and $b = (0, 1)$. Then $(x, y) = a^x b^y$.)

4: 20% How many conjugacy classes are there in S_6? How many of the conjugacy classes of S_6 are contained in A_6? (Note: you are not required count the conjugacy classes of A_6.)

5: 20% The dihedral group D_{12} is generated by elements a and b such that $a^6 = b^2 = 1$ and $bab^{-1} = a^{-1}$. Find all the conjugacy classes of D_{12}.

6: 20% Without assuming any results about simplicity of the groups A_n, show that, for all $n \geq 2$, there does not exist a transitive A_n-set with size 2. (Hint: consider n ways of regarding A_{n-1} as a subgroup of A_n.)

MATH 523 Algebra I, Midterm 2

13 December 2012, LJB, Bilkent University.

Time allowed: 110 minutes.

Please make sure your name is on every sheet of your script. Please do not hand the question sheets in (the examiner already knows what the questions are).

1: 20%: Let G be the group of rigid symmetries of a cube (the group of distance-preserving transformations of solid Euclidian space that permute the vertices of a given cube). What are the simple composition factors of G, and what are the multiplicities of each composition factor?

2: 20% Let G be a finite group and let P be a non-trivial p-subgroup of G. Let X be any set of subgroups of G such that the following two conditions hold:
(A) for all $Q \in X$, we have $P \not\leq N_G(Q)$,
(B) for all $Q \in X$ and $x \in P$, we have ${}^xQ \in X$.

(1) Show that p divides $|X|$.

(2) Let $\mathrm{Syl}_p(G)$ be the set of Sylow p-subgroups of G. Show that, putting $X = \mathrm{Syl}_p(G) - \{P\}$, then conditions (A) and (B) hold.

3: 20% State Sylow's Theorem. Use it to show that there does not exist a simple group with order 140.

4: 20% Let A, B, C be subgroups of a group. Suppose $A \leq B$. Show that $A(B \cap C) = B \cap AC$.

5: 20% Recall that, given a group G, the **Frattini subgroup** $\Phi(G)$ is defined to be the intersection of the maximal subgroups of G.

(a) Let $M < G$. Show that $\Phi(G)M < G$.

(b) Let $N \trianglelefteq G$. Show that, given a maximal subgroup M of G, then $\Phi(N) \leq M$. (Hint: Argue by contradiction. Use Question 4.)

(c) Deduce that $\Phi(N) \trianglelefteq \Phi(G)$.

Solutions and comments to Midterm 2

1: The stabilizer of a vertex is isomorphic to S_3 so, by the Orbit-Stabilizer Equation, $|G| = 8|S_3| = 48$. Recall that, by considering the diagonals of the cube, the group of rotational symmetries is isomorphic to S_4. Therefore G has a composition series

$$1 \lhd C_2 \lhd V_4 \lhd A_4 \lhd S_4 \lhd G\,.$$

All the quotients in this series are isomorphic to C_2 except that $A_4/V_4 \cong C_3$. So the simple composition factors are C_2 and C_3 with multiplicities 4 and 1, respectively.

Comment: In fact, $G \cong S_4 \times C_2$. Indeed, letting g be the involution sending each vertex to its opposite vertex, then $S_4 \cap \langle g \rangle = 1$ and $S_4 \,.\, \langle g \rangle = G$. Hence, via the Direct Product Recognition Theorem, $G \cong S_4 \times \langle g \rangle$.

2: Part 1: Condition (B) says that P acts on X by conjugation. Condition (A) says that P does not fix any element of X, in other words, all of the P-orbits have size greater than 1. But P is a finite p-group, so the size of each P-orbit is a power of p. Therefore every P-orbit in X has size divisible by p.

Part 2: Plainly, condition (B) holds. Denying condition (A), let Q be a Sylow p-subgroup of G such that $P \leq N_G(Q)$. Then PQ is a subgroup of G. Since $PQ/Q \cong P/(P \cap Q)$ and $P \cap Q < P$, we deduce that $|PQ|_p > |G|_p$, which conflicts with Lagrange's Theorem.

Comment: It is a mistake to argue that the congruence $|\mathrm{Syl}_p(G)| \equiv 1$ modulo p immediately implies condition (A) for $X = \mathrm{Syl}_p(G)$. Conceivably, $\mathrm{Syl}_p(G)$ could have $kp + 1$ singleton P-orbits for some positive integer k. To establish that this cannot happen, some particular features of the P-set $\mathrm{Syl}_p(G)$ have to be used.

3: Sylow's Theorem: Given a finite group G, then G acts transitively by conjugation on the Sylow p-subgroups of G, and the number of Sylow p subgroups divides the p'-part of $|G|$ and is congruent to 1 modulo p.

It follows that, if $|G| = 140$, then there is a unique Sylow 5-subgroup P. Then $P \trianglelefteq G$ and G is not simple. Alternatively, the conclusion can be obtained by observeing that there is a unique Sylow 7-subgroup.

4: Plainly, $B \geq A(B \cap C) \leq AC$. So $A(B \cap C) \leq B \cap AC$. For the reverse inequality, consider an element $b \in B \cap AC$. Write $b = ac$ with $a \in A$ and $c \in$. We have $c = a^{-1}b \in B$, hence $c \in B \cap C$. Therefore $b \in A(B \cap C)$, as required.

Comment: The formal manipulation $A(B \cap C) = AB \cap AC = B \cap AC$ is not valid. Although those equalities do hold under the hypothesis $A \leq B$, one has to explain why. Dropping that hypotheisis, the inequality $A(B \cap C) < AB \cap AC$ can be strict, for instance, when A, B, C are subgroups of S_3 such that $|A| = 3$ and $|B| = |C| = 2$ and $B \neq C$.

5: Part (a). Let L be a maximal subgroup of G such that $M \leq L$. Since $\Phi(G) \leq L$, we have $\Phi(G)M \leq L$.

Part (b). Suppose, for a contradiction, that $\Phi(N) \not\leq M$. Since $\Phi(N)$ is a characteristic subgroup of N and since $N \trianglelefteq G$, we have $\Phi(N) \trianglelefteq G$. So we can form the subgroup $\Phi(N)M$ of G. But M is maximal, so $\Phi(N)M = G$. By Question 4, $\Phi(N)(N \cap M) = N \cap G = N$.

Applying part (a) with N in place of G, we deduce that $N \cap M = N$. Hence, $\Phi(N) \leq N \leq M$, which is a contradiction, as required.

Part (c). We have $\Phi(N) \leq \Phi(G)$ by part (b). As we saw in the proof of part (b), $\Phi(N) \trianglelefteq G$. Therefore $\Phi(N) \trianglelefteq \Phi(G)$.

Comment: Two candidates apparently felt that $\Phi(G)$ must contain any strict normal subgroup of G. Again, the group S_3 is a counter-example. Well, mistakes happen, but they ought not to happen when there is a counter-example with order less than 12.

MATH 523 Algebra I, Final

10 January 2013, LJB, Bilkent University.

Time allowed: 2 hours.

Please make sure your name is on every sheet of your script. Please do not hand the question sheets in.

1: 10% Let G be a group, and let M and N be normal subgroups of G. Show that $G/(M \times N)$ is isomorphic to a subgroup of $G/M \times G/N$.

2: 20% Let G and H be finite groups.

(a) Show that, given a prime p, then every Sylow p-subgroup of $G \times H$ has the form $S \times T$ where S and T are Sylow p-subgroups of G and H, respectively.

(b) Give a counter-example to show that part (a) fails if the term "Sylow p-subgroup" is replaced by the term "p-subgroup".

3: 30%: Prove that, up to isomorphism, there are exactly 2 groups with order 8. For each of those groups, find all the subgroups and draw a diagram showing the inclusion relations between the subgroups.

4: 40% Let G be a simple group with order 60. In this question, you are to prove that $G \cong A_5$ by an argument outlined in the following steps. You may assume that the group A_5 is simple.

(a) Show that, for all $H < G$, we have $|G : H| \geq 5$.

(b) Suppose that $P \cap Q = 1$ for all distinct Sylow 2-subgroups P and Q of G. Show that $|G : N_G(P)| = 5$.

(c) Suppose that G has Sylow 2-subgroups P and Q such that the $|P \cap Q| = 2$. Let K be the subgroup of G generated by P and Q. Show that $|G : K| = 5$. (Hint: note that $P \cap Q \leq Z(K)$.)

(d) Deduce that $G \cong A_5$.