Archive for MATH 325, Representation Theory, Spring 2021

Bilkent University, Laurence Barker, 10 June 2021.

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MATH 325, Representation Theory, Spring 2021
Course specification

Laurence Barker, Bilkent University. Version: 2 May 2021.

Course Description: The course is to consist of:

• Ring theory, semisimple algebras.
• Ordinary character theory of finite groups, in other words, representation theory of finite groups over fields of characteristic zero, especially over the field of complex numbers. Techniques for constructing character tables.
• Use of character theory to prove Burnside's $p^\alpha q^\beta$-Theorem and the theorem of Frobenius concerning subgroups $H \leq G$ such that $H \cap {}^x H = 1$ for all $x \in G - H$.

The primary victory condition will be skill at techniques for constructing character tables and a good grasp of the theory behind those techniques.

Course Prerequisites: Knowledge of standard core material in a first course in linear algebra (including the theory of eigenvalues and eigenvectors); knowledge of standard core material in a first course in group theory (quotient groups, conjugacy classes, dihedral groups, symmetric and alternating groups).

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Course Texts: The primary course text is:

> Peter Webb, "A Course in Finite Group Representation Theory", (Cambridge University Press, 2016).

A suggested secondary text on character theory is:

> Jon L. Alperin, Rowen B. Bell, "Groups and Representations", (Springer, Berlin, 1995).

For those with an interest in a deeper treatment of ring theory, a recommended text is

> T.-Y. Lam, "A First Course in Noncommutative Rings", (Springer, Berlin, 1991).

Course Documentation: Various relevant files can be found on my homepage as well as on Moodle.

Syllabus: The format of the following details is *Week number: Monday date: Subtopics.*

1: 25 Jan: Group representations and ordinary characters.

2: 1 Feb: Group algebras. Examples of irreducible characters.

3: 8 Feb: Semisimple modules and semisimple rings.

4: 15 Feb: Some properties of ordinary irriducible characters.

5: 22 Feb: Illustrative examples of ordinary character tables

6: 1 Mar: Centrally primitive idempotents of the group algebra.

7: 8 Mar: Orthogonality properties of the ordinary character table.

8: 15 Mar: Inflation, restriction and induction of characters. Frobenius reciprocity. *No class on Spring Break Wed 17 Mar.*

9: 22 Mar: Constructing character tables using orthogonality.

10: 29 Mar: Symmetric and alternating squares.

11: 5 Apr: The Frobenius–Schur indicator.

12: 12 Apr: Integrality properties of ordinary irreducible characters.

13: 19 Apr: Burnside's $p^\alpha q^\beta$ Theorem. No class on holiday Fri 23 Apr.

14: 26 Apr: Characterization of Frobenius groups.

15: 3 May: Review for Final.

16: 10 May: *No scheduled classes.*

FZ Criterion: Very low homework and midterm marks, dependent on assignment difficulty.

Assessment:

- Homework, 30%,
- Midterm, 30%,
- Final, 40%.

75% attendance is compulsory. Attendance will be assessed through electronic records. Exceptions will be made only for students with documentary evidence of health exemptions or course clashes.

MATH 325, Algebra I, Fall 2020

Homeworks and Solutions

Laurence Barker, Bilkent University. Version: 4 May 2021.

Two guidelines to bear in mind:

Guideline 1: Write in complete sentences, otherwise the meaning will be ambiguous. "Prime p" has no meaning. "So p is prime" and "Let p be a prime" do have meanings, different meanings.

Guideline 2: Define your terms. The meaning of "So p is prime" is unclear if p has not been introduced.

Style tip: In professional literature, a sentence can never begin with a mathematical expression. The sentence must begin with a word, the initial letter of that word, a capital. Without that signal, flow is broken, because the reader has to put effort into recognizing the start of a sentence.

Homeworks

Homework 1

To be submitted as a single PDF file to Moodle by Tuesday, 9 March, 12:00 noon.

Exercise 1.1: How many isomorphism classes of simple $\mathbb{C}S_5$-modules are there? (Hint: this is easy; there is no need to determine the dimensions of the simple modules.)

Exercise 1.2: Let G be a finite group with $|G| \geq 3$. Let V be a simple $\mathbb{C}G$-module. Show that $\dim_{\mathbb{C}}(V)^2 \leq |G| - 2$.

Exercise 1.3: In the case where the coefficient ring is $\mathbb{C}$, prove Maschke's Theorem by first showing that, any $\mathbb{C}G$-module M can be equipped with an inner product such that $\langle gx \,|\, gy \rangle = \langle x \,|\, y \rangle$ for all $g \in G$ and $x, y \in M$.

Exercise 1.4: Prove the following converse to Maschke's Theorem: Let F be a field with prime characteristic p and let G be a finite group with order divisible by p. Show that the group algebra FG is not semisimple. (Hint: Consider the regular module ${}_{FG}FG$ and the submodule A consisting of the elements $\sum_g \lambda_g g$ where $\lambda_g \in F$ for each $g \in G$ and $\sum_g \lambda_g = 0$.)

Incidentally, A is an ideal of FG, called the **augmentation ideal**.

Exercise 1.5: Which of the following statements hold for every finite-dimensional algebra A over a field? Give a proof or a counter-example:
(a) Every simple A-module is isomorphic to a submodule of the regular A-module ${}_AA$.
(b) Every simple A-module is isomorphic to a quotient module of ${}_AA$.

Exercise 1.6: Let G be a finite group, $H \leq Z(G)$, and V a simple $\mathbb{C}G$-module. Show that, viewing V as a $\mathbb{C}H$-module by restriction, V is a direct sum of mutually isomorphic 1-dimensional $\mathbb{C}H$-modules.

Exercise 1.7: Does there exist a positive integer n such that $\dim_{\mathbb{Q}}(M) \leq n$ for all finite abelian groups A and all simple $\mathbb{Q}A$-modules M?

Homework 2

This homework is to be submitted to Moodle by Tuesday, 6 April, by 12:00 noon.

Exercise 2.1: The isomorphically unique non-abelian group with order 21 is a semidirect product $F_{21} = C_3 \ltimes C_7$. Find the character table of F_{21}.

Exercise 2.2: Find the character table of A_6, the alternating group with degree 6. (If you make use of the character tables of any smaller groups, you may just present those character tables without justification.)

Exercise 2.3: Find the character table of S_6, the symmetric group with degree 6. (Again, you may assume, without justification, the character tables of any smaller groups.)

Exercise 2.4: Let n be a positive integer. Let L be a complex vector space with a basis $\{e_1, ...e_n\}$. Let u and v be the endomorphisms of L such that $ue_s = e_{s+1}$ and $ve_s = e^{2\pi is/n}a_s$ where the index $s+1$ is interpreted modulo n.
(a) By a direct argument, show that there does not exist a proper subspace $0 < K < L$ such that $uK = vK = K$.
(b) Show that the group $H = \langle u, v \rangle$ has order n^3.
(c) Introducing suitable clear notation, give a simple formula for the character χ of the $\mathbb{C}H$-module L.
(d) Using (c), give another proof of the conclusion in (a).

Homework 3

This homework is to be submitted to Moodle by Monday, 10 May, by 12:00 noon.

The following comments about quadratic Gauss sums are only incidental to Question 1 below, but they have more of a bearing on Question 2. Let p be an odd prime. Let $\zeta_p = e^{2\pi i/p}$. The **quadratic Gauss sum** for p is defined to be

$$\tau_p = \sum_{a \in \mathbb{Z}/p} \zeta_p^{a^2} .$$

A famous theorem of Gauss asserts that

$$\tau_p = \begin{cases} \sqrt{p} & \text{if } p \equiv_4 1, \\ i\sqrt{p} & \text{if } p \equiv_4 3, \end{cases}$$

where $\equiv_4$ denotes congruence modulo 4. As you may already know if you have taken a first course in classical number theory, that theorem is a key step in a proof of the Law of Quadratic Reciprocity. Recall, the multiplicative group of modulo p units $(\mathbb{Z}/p)^\times = \mathbb{Z}/p - \{0\}$ is cyclic. Let b run over the modulo p quadratic residues, we mean, over the elements of the unique subgroup Q of index 2 in $(\mathbb{Z}/p)^\times$. The algebraic integer

$$\sigma_p = \sum_{b \in Q} \zeta_p^b = \frac{-1 + \tau_p}{2}$$

arises frequently in character tables. Consider, for example,

$$\sigma_5 = \zeta_5 + \zeta_5^4 = \frac{-1+\sqrt{5}}{2} \,, \qquad \sigma_7 = \zeta_7 + \zeta_7^2 + \zeta_7^4 = \frac{-1+i\sqrt{7}}{2} \,.$$

Two of the irreducible characters of $A_5 \cong \mathrm{PSL}_2(5)$, the simple group with order 60, have values in $\mathbb{Z}[\sigma_5] - \mathbb{Z}$. Two of the irreducible characters of $\mathrm{GL}_3(2) = \mathrm{SL}_3(2) = \mathrm{PSL}_3(2) \cong \mathrm{PSL}_2(7)$, the simple group with order 168, have values in $\mathbb{Z}[\sigma_7] - \mathbb{Z}$.

The characters in the following two exercises are ordinary characters, that is, over $\mathbb{C}$.

Exercise 3.1: Let G be a finite group with the following properties:

(1) G is simple.
(2) $|G| = 168$.
(3) Representatives of the conjugacy classes of G can be enumerated such that their orders are 1, 2, 3, 4, 7, 7 and the sizes of their conjugacy classes are 1, 21, 56, 42, 24, 24.
(4) G has a subgroup isomorphic to S_4.
(5) G has a nonabelian subgroup with order 21. (Up to isomorphism, there is only one such group, and we found its character table in Exercise 2.1.)

Find the character table of G, expressing some of the character values in terms of σ_7.

Solutions to Homeworks

Solutions 1

Solution 1.1: The integer partitions of 5 are

$$1+1+1+1+1\,, \quad 2+1+1+1\,, \quad 2+2+1\,, \quad 3+1+1\,, \quad 3+2\,, \quad 4+1\,, \quad 5\,.$$

So S_5 has excactly 7 conjugacy classes. It follows that $\mathbb{C}S_5$ has 7 isomorphism classes of simple modules.

Solution 1.2: We may assume that V is non-trivial. Then

$$|G| = 1 + \dim(V)^2 + \sum\nolimits_U \dim(U)^2$$

where U runs over representatives of the isomorphism classes of simple $\mathbb{C}G$-modules such that U is non-trivial and $U \not\cong V$. Therefore, $\dim(V)^2 < |G|$. But $|G| \geq 3$, so we may assume that $\dim(V) \neq 1$. Since $\dim(V)$ divides $|G|$, we have $\dim(V)^2 \leq |G| - \dim(V) \leq |G| - 2$.

Solution 1.3: Let $(-|-) : M \times M \to \mathbb{C}$ be any inner product on M. We define

$$\langle x \,|\, y \rangle = \sum_{g \in G} (gx \,|\, gy)\ .$$

Plainly, $\langle -|- \rangle$ is a symmetric sesquilinear form. We have $\langle x \,|\, x \rangle = |G|(x \,|\, x)$, so $\langle -|- \rangle$ is positive definite. Moreover, $\langle -|- \rangle$ is G-invariant in the sense that, given $h \in G$ then, using the substitution $f = gh$, we have

$$\langle hx \,|\, hy \rangle = \sum_{g \in G} \langle ghx \,|\, ghy \rangle = \sum_{f \in G} \langle fx \,|\, fy \rangle = \langle x \,|\, y \rangle\ .$$

Consider a $\mathbb{C}G$-submodule $A \leq M$. Let B be the orthogonal complement of A in M. For all $a \in A$, $b \in B$, $f \in G$, we have $\langle a \,|\, fb \rangle = \langle f^{-1}a \,|\, b \rangle = 0$ because $f^{-1}a \in A$. Therefore, $fb \in B$. So B is a $\mathbb{C}G$-submodule of M. We have shown that M is semisimple. But M is an arbitrary $\mathbb{C}G$-module, so $\mathbb{C}G$ is semisimple.

Solution 1.4: Let $M = {}_{FG}FG$. For a contradiction, suppose FG is a semisimple algebra. Then M is semisimple, and there must exist an FG-submodule $B < M$ such that $M = A \oplus B$. Since $\dim_F(A) = |G| - 1$, we have $\dim_F(B) = 1$. Letting $b \in B - \{0\}$, then $B = Fb$. So, for any $f \in G$, we have $fb = \phi(f)b$ for some $\phi(f) \in F$. Plainly, $\phi : G \to F^\times$ is a group homomorphism. Writing $b = \sum_g \lambda_g g$ with each $\lambda_g \in F$, a comparison of coefficients yields $\phi(f)\lambda_f = \lambda_1$. Therefore, $\lambda_g = \phi(g^{-1})$. The hypotheses on F and G impliy that $b \in A$, which is a contradiction, as required.

Solution 1.5: Part (a), False. Let F be any field and let A be the algebra of upper triangular 2×2 matrices over F. Let

$$e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\ , \qquad f = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\ .$$

Up to isomorphism, A has exactly 2 simple modules, namely S and T, where $eS = S$ and $fT = T$. We have

$$fA = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} F & F \\ 0 & F \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & F \end{bmatrix}$$

which is a 1-dimensional F-submodule of ${}_AA$ but not an A-submodule of ${}_AA$. So T is not isomorphic to any submodule of ${}_AA$.

Part (b), True. Consider a simple A-module S. Let $s \in S - \{0\}$. Since S is simple, $S = As$. So there is an A-map $\theta : {}_AA \to S$ given by $\theta(a) = as$ for $a \in {}_AA$. We have $A/\ker(\theta) \cong S$.

Solution 1.6: Schur's Lemma and the algebraic closure of $\mathbb{C}$ imply that $\mathrm{End}_{\mathbb{C}G}(V) = \mathbb{C}\mathrm{id}_V$. Since $H \leq Z(G)$, the representation $\rho_V : \mathbb{C}G \to \mathrm{End}_{\mathbb{C}}(V)$ restricts to an algebra map $\mathbb{C}H \to \mathrm{End}_{\mathbb{C}G}(V)$. So there exists a group homomorphism $\zeta : H \to \mathbb{C}^\times$ such that $\rho_V(h) = \zeta(h)\mathrm{id}_V$. Therefore, as a $\mathbb{C}H$-module, V is a direct sum of copies of the 1-dimensional $\mathbb{C}H$-module with character ζ.

Solution 1.7: No, there does not exist an upper bound on $\dim_{\mathbb{Q}}(M)$. To see an example, let p be a prime. Put $G = C_p$ and let

$$M = \left\{ \lambda_g g \in \mathbb{Q}G : \sum_g \lambda_g = 0 \right\}$$

as a submodule of the regular $\mathbb{Q}G$-module. We shall show that M is simple. Supposing otherwise, let L be a proper $\mathbb{Q}G$-submodule of M. By Maschke's Theorem, $M = L \oplus K$ for some proper $\mathbb{Q}G$-submodule K. Fix $g \in G - \{1\}$. By extending to $\mathbb{C}$ and diagonalizing the action of g on the $\mathbb{C}$-linear extensions $\mathbb{C}L$ and $\mathbb{C}K$, we see that $\chi_{\mathbb{C}L}(g)$ is the sum of the elements of some proper subset S of the primitive p-th roots of unity in $\mathbb{C}$. But, by Galois theory, the primitive p-th roots of unity are $\mathbb{Q}$-linearly independent. Hence $S \cup \{1\}$ is $\mathbb{Q}$-linearly independent. Therefore, $\chi_{\mathbb{C}L}(g) \notin \mathbb{Q}$. That is impossible, because $\chi_{\mathbb{C}L}(g)$ is the trace of the action of G on the $\mathbb{Q}$-module L. Thus, we have shown that $\mathbb{Q}C_p$ has a simple module with dimension $p - 1$.

Solutions 2

Solution 2.1: Write $F_{21} = \langle a, b : a^3 = b^7 = 1, aba^{-1} = b^2 \rangle$. Noting that $\langle b \rangle \lhd F_{21}$, we see that two of the conjuacy classes of F_{21} are $\{b, b^2, b^4\}$ and $\{b^3, b^5, b^6\}$. The other conjugacy classes of F_{21} are now easy to determine. The character table is as follows, where $\omega = e^{2\pi i/3} = (-1 + i\sqrt{3})/2$ and $\sigma = \zeta + \zeta^2 + \zeta^4 = (-1 + i\sqrt{7})/2$ with $\zeta = e^{2\pi i/7}$.

	1	7	7	3	3	$\|[g]\|$
	1	3	3	7	7	$\|\langle g \rangle\|$
$\chi(g)$	1	a	a^2	b	b^6	g
χ_0	1	1	1	1	1	
χ_1	1	ω	$\overline{\omega}$	1	1	
χ_2	1	$\overline{\omega}$	ω	1	1	
χ_3	3	0	0	σ	$\overline{\sigma}$	
χ_4	3	0	0	$\overline{\sigma}$	σ	

The irreducible characters χ_0, χ_1, χ_2 are inflated from the quotient group $C_3 \cong F - 21/\langle b \rangle$. The characters χ_3 and χ_4, easily confirmed to be irreducible using row orthonormality, are induced from the non-trivial irreducible characters of $\langle b \rangle$.

Solution 2.2: Omitted.

Solution 2.3: Omitted.

Solution 2.4: Part (a). For a contradiction, suppose such K exists, and let $x \in K - \{0\}$. Write $x = \sum_s x_s e_s$ with $x_s \in \mathbb{C}$. Replacing x with $v^b x$ for suitable $b \in \mathbb{Z}/n$, we may assume that $x_0 \neq 0$. Then replacing x with x/x_0, we may assume that $x_0 = 1$. Let $\omega = e^{2pii/n}$ and

$$\nu = \frac{1}{n} \sum_{r \in \mathbb{Z}/n} \omega^r v^r \ .$$

Then $\nu e_s = \sum_r \omega^{rs} e_s / n$ which is equal to e_0 or 0 depending on whether $s = 0$ or $s \neq 0$, respectively. Therefore, $\nu x = e_0$. We have shown that $e_0 \in K$. Hence each $e_s = u^s e_0 \in K$ and $K = L$, a contradiction.

Part (b). Define $w = v^{-1} u^{-1} v u$. Then $vu = uvw$. By direct calculation, $we_s = \omega e_s$. So $w \in Z(H)$ and w has order n. It is now clear that every element of H can be written uniquely as $u^a v^b w^c$ with $a, b, c \in \mathbb{Z}/n$. Perforce, $|H| = n^3$.

Part (c). Plainly, for $a, b, c \in \mathbb{Z}/n$, we have $\chi(u^a v^b w^c) = \begin{cases} 0 & \text{if } a \neq 0 \text{ or } b \neq 0 \\ n\omega^c & \text{if } a = b = 0. \end{cases}$

Part (d). We have $\langle \chi \,|\, \chi \rangle = \dfrac{1}{|H|} \displaystyle\sum_{h \in H} |\chi(h)|^2 = \dfrac{1}{n^3} \displaystyle\sum_{c \in \mathbb{Z}/n} n^2 = 1.$

Solutions 3

Solution 3.1: We shall confirm that G has the following character.

	1	21	56	42	24	24	$\lvert[g]\rvert$
$\chi(g)$	1	2	3	4	7	7	$\lvert\langle g\rangle\rvert$
χ_0	1	1	1	1	1	1	
χ_1	3	1	0	1	σ_7	$\overline{\sigma_7}$	
χ_2	3	1	0	1	$\overline{\sigma_7}$	σ_7	
χ_3	6	2	0	0	1	1	
χ_4	7	1	1	1	0	0	
χ_5	8	0	1	0	1	1	

Representatives of the conjugacy classes of S_4 can be enumerated such that the element orders are 1, 2, 2, 3, 4 and the sizes of the conjugacy classes are 1, 6, 3, 8, 6. So the $\mathbb{C}G$-character ψ_1 of the permutation $\mathbb{C}G$-module $\mathbb{C}G/S_4$, we mean to say, the $\mathbb{C}G$-character induced from the trivial $\mathbb{C}S_4$-character, has values $(7, 3, 1, 1, 0, 0)$. Subtracting χ_0, we obtain the $\mathbb{C}G$-character χ_3.

For the non-ablian group F_{21} with order 21, representatives of the conjugacy classes can be enumerated such that the element orders are 1, 3, 3, 7, 7 and the class sizes are 1, 7, 7, 3, 3. So the character of $\mathbb{C}G/F_{21}$ is $(8, 0, 2, 0, 1, 1)$. Subtracting χ_0, we obtain the $\mathbb{C}G$-character χ_4. Let ω be a primitive cube root of unity in $\mathbb{C}$. There is an irreducible $\mathbb{C}F_{21}$-character with values $(1, \omega, \overline{\omega}, 1, 1)$, which induces to a $\mathbb{C}G$-character χ_5 with values $(8, 0, -1, 0, 1, 1)$.

By row orthonomality, χ_3, χ_4, χ_5 are irreducible. We have

$$168 - (\chi_0(1)^2 + \chi_3(1)^2 + \chi_4(1)^2 + \chi_5(1)^2) = 168 - (1 + 36 + 49 + 64) = 18 \ .$$

So the remaining two irreducible $\mathbb{C}G$-characters χ_1 and χ_2 both have degree 3.

It suffices to show that χ_1 and χ_2 are the complex conjugates of each other. All the values of χ_1 and χ_2 will then follow easily using column orthonormality. Let g be an element of G

with order 7. Let λ_1, μ_1, ν_1 be the eigenvalues of the action of g on a simple $\mathbb{C}G$-module X_1 with character χ_1. We have

$$\chi_1(g) = \lambda_1 + \mu_1 + \nu_1 \ .$$

Defining λ_2, μ_2, ν_2 likewise for χ_2, a similar equality holds for $\chi_2(g)$. By the orthogonality of the column for g and the column for 1, we have

$$\lambda_1 + \mu_1 + \nu_1 + \lambda_2 + \mu_2 + \nu_2 = \chi_1(g) + \chi_2(g) = -1 \ .$$

So at least one of the 6 eigenvalues, without loss of generality λ_1, is a primitive 7-th root of unity. But $\{g, g^2, g^4\}$ is an F_{21}-conjugacy class. Perforce, g and g^2 and g^4 are G-conjugate. So λ_1^2 and λ_1^4 must be eigenvalues of the action of g on X_1. We deduce that

$$\chi_1(g) = \lambda_1 + \lambda_1^2 + \lambda_1^4 \in \{\sigma_7, \overline{\sigma_7} \ .$$

In particular, $\chi_1(g) \not\in \mathbb{R}$. Therefore, χ_1 and χ_2 are complex conjugates, as required.

Comment: It is well-known that, up to isomorphism, there is a unique simple group with order 168. So $G = \mathrm{PSL}_2(7)$. The conjugation action of $\mathrm{GL}_2(7)$ on the normal subgroup $\mathrm{SL}_2(7)$ induces an action as automorphisms on $\mathrm{PSL}_2(7)$. It is easily checked that the elements of order 7 in $\mathrm{GL}_2(7)$ comprise a single conjugacy class. (In the solution to Exercise 3.1, we shall show that, for any odd prime p, the elements of order p in $\mathrm{GL}_2(p)$ comprise a single conjugacy class.) Therefore, G admits an automorphism α that interchanges the two G-conjuacy classes of elements with order 7. On the other hand, since χ_1, χ_3, χ_4, χ_5 are constant on the elements of G with order 7, at least one of χ_1 and χ_2 cannot be constant on those elements. It follows that α interchanges χ_1 and χ_2. From that conclusion, it is again easy to determine χ_1 and χ_2 using column orthonormality.

That argument avoids the use we made of the eigenvalues of the action of g. I award it full marks, though it does invoke more than than the assumptions indicated in the question.

MATH 325 : *Representation Theory*

MATH 525 : *Group Representations*

Midterm

LJB, 16 April 2021, Bilkent.

Please put your name on your manuscript.
All the questions carry equal weight: 25% each for MATH 325; 20% each for MATH 525.
Solutions with excessive details or digressions will be marked down.

1: Using the notation $Q_8 = \{1, i, j, k, z, iz, jz, kz\}$ where z is of order 2, find the character table of $\mathbb{C}Q_8$. (Explain your methods clearly and concisely.)

2: Let $G = C_3 Q_8$ where Q_8 is viewed as a normal subgroup of G and $C_3 = \{1, a, a^2\}$ is viewed as a subgroup of G such that $aia^{-1} = j$ and $aja^{-1} = k$.
(a) Find the conjugacy classes of G. (Hint: for instance $C_G(a) \cong C_6$, so $|[a]| = 4$.)
(b) Find the character table of $\mathbb{C}G$. (Again, explain your methods succinctly.)

3: Express the following group algebras as direct sums of matrix algebras over division rings. (As an illustration, $\mathbb{C}D_8 \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathrm{Mat}_2(\mathbb{C})$. Hint: the finite-dimensional division rings over $\mathbb{R}$ are $\mathbb{R}$ and $\mathbb{C}$ and the ring of quaternions $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}i \oplus \mathbb{C}j \oplus \mathbb{C}k$.)
(a) $\mathbb{C}G$, where G is as in Question 2.
(b) $\mathbb{R}Q_8$,
(c) $\mathbb{R}D_8$,
(d) $\mathbb{R}D_{2^n}$ where $n \geq 4$.

4: Let F be a finite group and $K \trianglelefteq F$.
(a) Given a $\mathbb{C}F/K$-character ψ, let ψ^F denote the function $F \to \mathbb{C}$ such that $\psi^F(f) = \psi(fK)$ for $f \in F$. Briefly explain why ψ^F is a $\mathbb{C}F$-character.
(b) Given a $\mathbb{C}F$-character χ, let $\chi_{F/K}$ denote the function $F/K \to \mathbb{C}$ such that

$$\chi_{F/K}(fK) = \frac{1}{|K|} \sum_{k \in K} \chi(fk) \; .$$

Explain why $\chi_{F/K}$ is a $\mathbb{C}F/K$-character.
(c) Show that, taking inner products in the character rings of $\mathbb{C}F/K$ and $\mathbb{C}F$, we have

$$\langle \chi_{F/K} \,|\, \psi \rangle_{F/K} = \langle \chi \,|\, \psi^F \rangle_F \; .$$

5: For MATH 525 only: (a) Briefly explain why, given a nilpotent ideal N of a finite-dimensional algebra A over a field, the number of isomorphism classes of simple A-modules is equal to the number of isomorphism classes of simple A/N-modules.
(b) Let S be a finite set. Now take A to be a vector space over $\mathbb{C}$ with a basis consisting of the elements $e_{X,Y}$ where $\emptyset \subseteq X \subseteq Y \subseteq S$. We make A become an algebra over $\mathbb{C}$ with multiplication such that $e_{X,Y} e_{X',Y'} = 1$ if $Y = X'$ and $e_{X,Y} e_{X',Y'} = 0$ otherwise. Express the number of isomorphism classes of simple A-modules in terms of $|S|$.

Solutions to Midterm.

Solution 1: Inflating 4 irreducible $\mathbb{C}Q_8$-characters from the quotient group $V_4 \cong Q_8/Z(Q_8)$, then using column orthonormality to determine the remaining irreducible $\mathbb{C}Q_8$-character, we find that the required character table is as follows.

	1	1	2	2	2	$\lvert[g]\rvert$
	1	2	4	4	4	$\lvert\langle g\rangle\rvert$
$\chi(g)$	1	z	i	j	k	g
χ_0	1	1	1	1	1	
χ_1	1	1	1	1	1	
χ_2	1	1	1	1	1	
χ_3	1	1	1	1	1	
χ_4	2	2	0	0	0	

Solution 2: We shall explain why the required character table is as shown.

	1	1	6	4	4	4	4	$\lvert[g]\rvert$
	1	1	4	3	3	6	6	$\lvert\langle g\rangle\rvert$
$\chi(g)$	1	z	i	a	a^2	za	za^2	g
χ_0	1	1	1	1	1	1	1	
χ_1	1	1	1	ω	ω^2	ω	ω^2	
χ_2	1	1	1	ω^2	ω	ω^2	ω	
χ_3	2	2	0	1	1	1	1	
χ_4	2	2	0	ω	ω^2	ω	ω^2	
χ_5	2	2	0	ω^2	ω	ω^2	ω^2	
χ_6	3	3	1	0	0	0	0	

Observe that $Z(G) = \langle z \rangle \cong C_2$. The irreducible characters χ_0, χ_1, χ_2, χ_6 are inflated from the quotient group $G/Z(G) \cong A_4$. Since there are exactly 7 conjugacy classes, the remaining irreducible $\mathbb{C}G$-characters can be indexed as χ_3, χ_4, χ_5. By column orthonormality, at least one of $\chi_3(a)$, $\chi_4(a)$, $\chi_5(a)$ must be nonzero. Therefore, the indexing can be chosen such that $\chi_4 = \chi_1 \otimes \chi_3$ and $\chi_5 = \chi_2 \otimes \chi_5$. It follows that the restrictions of χ_3, χ_4, χ_5 to Q_8 are all equal. Applying column orthonormality again, we easily complete the first three columns and we also obtain $3 + 3|\chi_3(a)|^2 = |G|/|[a]| = 6$, whence $|\chi_3(a)| = 1$. But $\chi_3(a)$ must also be the sum of 2 cube roots of unity. Therefore, $\{\chi_3(a), \chi_4(a), \chi_5(a)\} = \{-1, -\omega, -\omega^2\}$. The indexing can be chosen such that $\chi_3(a) = -1$. It follows that, on the simple $\mathbb{C}G$-module X_3 with character χ_3, the eigenvalues of the action of a are ω and ω^2. The same must hold for the action of a^2 on X_3. Meanwhile, z acts on X_3 as multiplication by -1, so the actions of za and za^2 must both have eigenvalues $-\omega$ and $-\omega^2$. The values of χ_3 are now clear, and the values of χ_4 and χ_5 follow.

Comment: The group G, in this question, is very familiar to finite group theorists. One can easily check that there is an isomorphism $G \cong \mathrm{SL}_2(3)$ given by

$$a \leftrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad i \leftrightarrow \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \quad j \leftrightarrow \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad k \leftrightarrow \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Solution 3: In all the parts below, we use the fact that, given a finite group H then, by

Maschke's Theorem and Wedderburn's Theorem, we have a finite sum $\mathbb{R}H = \bigoplus_i \mathrm{Mat}_{n_i}(\Delta_i)$ where each n_i is a positive integer and Δ_i is a division algebra isomorphic to $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$.

Part (a). From Solution 2, we have $\mathbb{C}G \cong \mathbb{C}^3 \oplus \mathrm{Mat}_2(\mathbb{C})^3 \oplus \mathrm{Mat}_2(\mathbb{C})$ where, for an algebra A and a positive integer n, we write A^n to denote a direct sum of 3 isomorphic copies of A.

Part (b). By Solution 1, all the 1-dimensional irreducible $\mathbb{C}Q_8$-modules can be obtained by extension from $\mathbb{R}$. So $\mathbb{R}^4$ appears as a sum of Wedderburn components of $\mathbb{R}Q_8$. The embedding of Q_8 in the unit group of $\mathbb{H}$ extends to a surjective map $\mathbb{R}Q_8 \to \mathbb{H}$ of algebras over $\mathbb{R}$. Therefore $\mathbb{H}$ appears as a Wedderburn component of $\mathbb{R}Q_8$. By considering dimensions, $\mathbb{R}Q_8 \cong \mathbb{R}^4 \oplus \mathbb{H}$.

Part (c). By viewing D_8 as the group of rigid symmetries of a square in the real plane, we see that $\mathbb{R}D_8$ has an irreducible 2-dimensional representation. So $\mathrm{Mat}_2(\mathbb{R})$ must appear as a Wedderburn component of $\mathbb{R}D_8$. Now arguing as in part (b), we deduce that $\mathbb{R}D_8 \cong \mathbb{R}^4 \oplus \mathrm{Mat}_2(\mathbb{R})$.

Part (d). We shall show that $\mathbb{R}D_{2^n} \cong \mathbb{R}^4 \oplus \mathrm{Mat}_2(\mathbb{R})^m$ where $m = 2^{n-2} - 1$. Let a be an element of D_{2^n} with order 2^{n-1}. For each integer $0 < i < 2^{n-2}$, there is an irreducible 2-dimensional $\mathbb{R}D_{2^n}$-module X_i upon which a acts as a rotation through an angle of $\theta_i = 2\pi i/2^{n-1} = \pi i 2^{2-n}$. The trace of the action of a on X_i is $2\cos(\theta_i)$. So the X_i are mutually non-isomorphic. Therefore, $\mathrm{Mat}_2(\mathbb{R})$ appears as a Wedderburn compotent of $\mathbb{R}D_{2^n}$ with multiplicity at least m. On the other hand, D_{2^n} has a quotient group isomorphic to V_4, so $\mathbb{R}$ must appear as a Wedderburn component of $\mathbb{R}D_{2^n}$ with multiplicity at least 4. The asserted conclusion now follows by considering dimensions.

Solution 4: Part (a). Any $\mathbb{C}F/K$-module with character ψ becomes a $\mathbb{C}F$-module with charaacter ψ^F when we allow each f to act as fK.

Part (b). Let X be a $\mathbb{C}F$-module affording χ. Let e be the primitive idempotent of $\mathbb{C}K$ corresponding to the trivial $\mathbb{C}K$-character. In other words, $e = \sum_k k/|K|$. Then $X = eX \oplus (1-e)x$ as a sum of $\mathbb{C}F$-modules. The $\mathbb{C}F$-character of eX is $f \mapsto \chi(fe)$. But K acts trivially on eX, so we can view eX as a $\mathbb{C}F/K$-module with fK acting as f. The $\mathbb{C}F/K$-character of eX is $fK \mapsto \chi(fe)$.

Part (c). We have

$$\frac{1}{|F/K|} \sum_{fK \subseteq F} \overline{\chi}_{F/K}(fK)\psi(fK) = \frac{1}{|F|} \sum_{fK \subseteq K, k \in K} \overline{\chi}(fk)\psi(fK) = \frac{1}{|F|} \sum_{f \in F} \overline{\chi}(f)\psi^F(f) \ .$$

Comment: The underlying module theoretic construction in part (a) and part (b) are, respectively, the inflation functor and the deflation functor

$${}_F\mathrm{Inf}_{F/K} \ : \ \mathbb{C}F\text{–mod} \leftarrow \mathbb{C}F/K\text{–mod} \ , \qquad {}_{F/K}\mathrm{Def}_F \ : \ \mathbb{C}F/K\text{–mod} \leftarrow \mathbb{C}F\text{–mod} \ .$$

They can be unified with induction and restruction by defining generalized induction and generalized restriction with respect to any group homomophism. We may discuss this in class.

MATH 325 : *Representation Theory*

Final

LJB, 23 May 2021, Bilkent.

Time allowed: 2 hours.

Please put your name on every sheet of your manuscript.

Solutions with excessive details or digressions will be marked down.

The four questions all have the same number of marks.

1: The **dicyclic group with order** 12 is defined to be

$$D = \langle a, b : a^4 = b^3 = 1, aba^{-1} = b^2 \rangle \ .$$

You may assume that 1, a, a^2, a^3, b, $a^2 b$ are representatives of the conjugacy classes of D. Find the character table of $\mathbb{C}D$. Justify your answer.

2: What are the possible orders $|G|$ of a finite group G when:
(a) Exactly 1 of the irreducible $\mathbb{C}G$-characters has degree 2 and all the other irreducible $\mathbb{C}G$-characters have degree 1?
(c) Exactly 2 of the irreducible $\mathbb{C}G$-characters have degree 2 and all the other irreducible $\mathbb{C}G$-characters have degree 1?

3: Let E and F be finite groups. Let ξ be the character of a $\mathbb{C}E$-module X. Let η the character of a $\mathbb{C}F$-module Y. Let χ be the character of the $\mathbb{C}(E \times F)$-module $X \otimes_{\mathbb{C}} Y$ where the action is such that $(e, f)(x \otimes y) = (ex) \otimes (fy)$ for $e \in E$, $f \in F$, $x \in X$, $y \in Y$.
(a) Show that $\chi(e, f) = \xi(e)\eta(f)$.
(b) Show that χ is irreducible if and only if ξ and η are irreducible.
(c) Recall, $S_3 = \langle r, s : r^3 = s^2 = 1, srs^{-1} = r^2 \rangle$. Write down the character table of $\mathbb{C}(S_3 \times S_3)$.

4: Let G be a finite group. We define a relation $\equiv$ on G such that, given $g, h \in G$, then $g \equiv h$ provided $g = ahxa^{-1}$ for some $a \in G$ and $x \in Z(G)$.
(a) Show that $\equiv$ is an equivalence relation.
(b) Let g_1, ..., g_n be representatives of the equivalence classes with respect to $\equiv$. Let χ be a $\mathbb{C}G$-character. State and prove, in terms of $\chi(g_1)$, ..., $\chi(g_n)$ and whatever else may be appropriate, a necesary and sufficient criterion for χ to be irreducible.

Solutions to Final.

Sol 1: The character table of D is as shown.

$\chi(g)$	1	3	1	3	2	2	$\|[g]\|$
	1	4	2	4	3	6	$\|\langle g\rangle\|$
	1	a	a^2	a^3	b	a^2b	g
χ_0	1	1	1	1	1	1	
χ_1	1	1	1	1	1	1	
χ_2	1	i	1	i	1	1	
χ_3	1	i	1	i	1	1	
χ_4	2	0	2	0	1	1	
χ_5	2	0	2	0	1	1	

The 4 irreducible characters χ_0, χ_1, χ_2, χ_3, χ_4 are inflated from the group $D/\langle b\rangle \cong C_4$. Since D has 6 conjugacy classes, $\mathbb{C}D$ has exactly 2 other irreducible characters, χ_5 and χ_6. Orthonomality of the column for a^2b shows that $\chi_4(a^2)$ and $\chi_5(a^2b)$ cannot both be 0. It follows that $\chi_6 = \chi_2 \otimes \chi_5$. The values $\chi_4(g)$ and $\chi_5(g) = \chi_2(g)\chi_4(g)$ now follow using column orthonormality.

Sol 2: Part (a). We shall show that the possible values of $|G|$ are 6 and 8. Let n be the number of 1-dimensional $\mathbb{C}G$-characters. Then $|G| = n + 2^2 = n + 4$. In view of the trivial $\mathbb{C}G$-character, $n \geq 1$. The degree of any irreducible complex character divides the order of the group, so n is even. On the other hand, n is the index of the derived subgroup of G, so n divides $|G|$, hence n divides 4. We have shown that $n \in \{2, 4\}$, hence $|G| \in \{6, 8\}$. Finally, 6 and 8 are realized as $|G|$ in the cases $G \cong D_6$ and $D \cong D_8$, respectively.

Part (b). We shall show that the possible values of $|G|$ are 10 and 12 and 16. Again, let n be the number of 1-dimensional characters. We now have $|G| = n + 2^2 + 2^2 = n + 8$. Arguing as before, we deduce that $n \in \{2, 4, 8\}$, hence $|G| \in \{10, 12, 16\}$. Again, all three of those cases are realized by dihedral groups.

Sol 3: Part (a). Let $\mathcal{X}$ and $\mathcal{Y}$, respectively, be bases of X and Y consisting of eigenvectors for the actions of e and f, then $\mathcal{X} \otimes \mathcal{Y}$ is a basis of $X \otimes Y$ consisting of eigenvectors for the action of $e \otimes f$. With respect to those three bases, the matrices representing the actions of e and f and (e, f) are diagonal and the diagonal entries are the eigenvalues. Letting λ be the eigenvalue of e associated with eigenvector $x \in \mathcal{X}$, letting μ be the eigenvalue of f associated with eigenvector $y \in \mathcal{Y}$, then $\lambda\mu$ is the eigenvalue of (e, f) associated with eigenvector $x \otimes y$.

Part (b). By part (a), $\langle \chi \,|\, \chi \rangle = \langle \xi \,|\, \xi \langle\rangle \eta \,|\, \eta \rangle$.

Part (c). Recall, the character table for S_3 is as shown.

$\chi(g)$	1	s	r	g
θ	1	1	1	
ϕ	1	1	1	
ψ	2	0	1	

Hence, applying parts (a) and (b), we obtain the following the character table for $\mathbb{C}(S_3 \times S_3)$.

$\chi(g)$	$(1,1)$	$(1,s)$	$(1,r)$	$(s,1)$	(s,s)	(s,r)	$(r,1)$	(r,s)	(r,r)	g
$\theta\otimes\theta$	1	1	1	1	1	1	1	1	1	
$\theta\otimes\phi$	1	1	1	1	1	1	1	1	1	
$\theta\otimes\psi$	2	0	1	2	0	1	2	0	1	
$\phi\otimes\theta$	1	1	1	1	1	1	1	1	1	
$\phi\otimes\phi$	1	1	1	1	1	1	1	1	1	
$\phi\otimes\psi$	2	0	1	2	0	1	2	0	1	
$\psi\otimes\theta$	2	2	2	0	0	0	1	1	1	
$\psi\otimes\phi$	2	2	2	0	0	0	1	1	1	
$\psi\otimes\psi$	4	0	2	0	0	0	2	0	1	

Sol 4: Part (a). Writing $[g]_G$ for the G-conjugacy class of g, then $g \equiv h$ if and only if $[g]_G Z(G) = [h]_G Z(G)$. So $\equiv$ is an equivalence relation, in fact, the equivalence class of g under $\equiv$ is $[g]_G Z(G)$.

Part (b). We shall show that χ is irreducible if and only if the function $G \ni g \mapsto |\chi(g)|$ is constant on each equivalence class $[g]_\equiv$ and

$$\sum_{i=1}^{n} |[g]_\equiv|.|\chi(g_i)|^2 = |G| \ .$$

Suppose χ is irreducible. Then $\sum_g |\chi(g)|^2 = |G|$ summed over $g \in G$. When $g = ahxa^{-1}$ in the notation of the question, we have $g = aha^{-1}x$ and $\chi(h) = \chi(aha^{-1})$. But x acts on a simple $\mathbb{C}G$-module as multiplication by a root of unity, so $|\chi(aha^{-1})| = |\chi(aha^{-1}x)|$. Hence, the function $g \mapsto |\chi(g)|$ is constant on the equivalence classes under $\equiv$ and the specified sum is equal to $|G|$.

Conversely, suppose $g \mapsto |\chi(g)|$ is constant on each of those equivalence classes. Then the specified sum is equal to $|G|\langle\chi\,|\,\chi\rangle$, which is equal to $|G|$ if and only if χ is irreducible.