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MATH 325, Representation Theory

Bilkent University, Spring 2018, Laurence Barker

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MATH 325 Representation theory
Spring 2018

Course specification

Laurence Barker, Bilkent University.
version: 16 June 2018

Course aims: To acquire knowledge and skill in group representation theory and related areas of algebra.

Course instructor: Laurence Barker, Office SAZ 129.

Classes: Mondays 09:40 - 10:30, SAZ 02, Wednesdays 10:40 - 12:30, SAZ 02.

Office Hours: Mondays, 08:40 - 09:30, SAZ 129.

Course description: The course will consist of:

• Abstract ring theory, firstly on semisimple finite-dimensional algebras, then on semisimple rings in general
• Ordinary character theory of finite groups, in other words, representation theory of finite groups over fields of characteristic zero, especially over the field of complex numbers. Techniques for constructing character tables.
• Use of character theory to prove Burnside's $p^\alpha q^\beta$-Theorem and the theorem of Frobenius concerning subgroups $H \leq G$ such that $H \cap {}^x H = 1$ for all $x \in G - H$.
• Introductory comments on p-local group theory and p-modular representation theory.
• Presentations on topics in representation theory.

The primary victory condition will be skill at techniques for constructing character tables and a good grasp of the theory behind those techniques, including the underlying ring theory.

Course texts:

Primary: Peter Webb, "A Course in Finite Group Representation Theory", (Cambridge University Press, 2016).
Primary: Jon L. Alperin, Rowen B. Bell, "Groups and Representations", (Springer, Berlin, 1995).
Primary: T. Y. Lam, "A First Course in Noncommutative rings", Graduate Texts in Math. 131, (Springer, Berlin, 1991).
Secondary: Michael J. Collins, "Representations and Characters of Finite Groups", Cambridge Studies in Adv. Math. **22**, (Cambridge, Univ. Press, 1990).

Assessment:

- Homeworks and Quizzes, 15%,
- Presentations, 10%
- Midterm, 35%,
- Final, 40%.

Syllabus: Week number: Monday date, subtopics.

1: 29 Jan. Group representations. Group algebras. (First class on Monday.)

2: 5 Feb. Maschke's Theorem. Artin–Wedderburn Theorem for semisimple finite-dimensional algebras, and some direct applications for abelian groups and some groups of small order.

3: 12 Feb. Applications of Zorn's Lemma in algebra. Proof of Artin–Wedderburn Theorem for semisimple rings in general.

4: 19 Feb. The Jacobson radical of an Artinian ring.

5: 26 Feb. Ordinary characters. The character table. The dimension of the centre of a group algebra.

6: 5 Mar. Orthogonality properties of the character table.

7: 12 Mar. Construction of character tables using inflation and orthogonality relations

8: 19 Mar. Restriction of characters. (No class on Wednesday.)

9: 26 Mar. Induction of characters. Frobenius reciprocity. Symmetric and alternating squares.

10: 2 Apr. Construction of character tables using induction and using symmetric and alternating squares.

11: 9 Apr. Further character tables. The existence of a normal complement for Frobenius groups.

12: 16 Apr. Integrality conditions on the character table. Burnside's $p^\alpha q^\beta$-Theorem.

13: 23 Apr. Introduction to p-modular representation theory.

14: 30 Apr. *Presentations.* (No class on Monday.)

15: 7 May. *Presentations.*

16: 14 May. *Presentations.*

17: 21 May. Review. (Last class on Monday.)

Homeworks, Quizzes and Presentations

MATH 325, *Representation Theory*, Spring 2018
MATH 525, *Group Representations*, Spring 2018

Laurence Barker, Mathematics Department, Bilkent University,
version: 1 June 2018.

A good time to ask me about the homework is during Office Hours, before the homework is due to be handed in. Of course, I will expect you to have thought about a homework question before you ask me for help with it.

Office Hours: Mondays, 08:40 – 09:30, Office Room Fen A 129.

Homework Questions

Homework 1, due Monday, 19 February.

HW.1.1: Let A be a unital ring and U an A-module with submodules V, W, X such that $U = V \oplus W$.
(a) Give an example where $X \neq (V \cap X) \oplus (W \cap X)$.
(b) (Modular Law.) Show that if $V \leq X$, then $X = V \oplus (W \cap X)$.

HW.1.2: Let G be a finite group. Show that any finite-dimensional $\mathbb{C}G$-module admits a G-invariant inner product. (That is to say, there exists an inner product $\langle - | - \rangle$ such that $\langle gu \,|\, gv \rangle = \langle u \,|\, v \rangle$ for all $g \in G$ and all elements u and v of the $\mathbb{C}G$-module.) Hence give an alternative proof of Maschke's Theorem in the special case of $\mathbb{C}G$-modules.

Homework 2, due at start of class, Wednesday, 7 March.

For each of the three following groups G, (a) find the conjugacy classes, (b) find the integers $n_1 \leq ... \leq n_k$ such that

$$\mathbb{C}G \cong \mathrm{Mat}_{n_1}(\mathbb{C}) \oplus ... \oplus \mathrm{Mat}_{n_k}(\mathbb{C}) \ .$$

You may use the theorem asserting that k is the number of conjugacy classes of G.
HW.2.1: The quaternion group $G = Q_8$.
HW.2.2: The alternating group $G = A_4$.
HW.2.3: A non-abelian group G with order 21. (It can be shown that, given distinct primes p and q such that there exists a non-abelian group with order pq, then that group is unique up to isomorphism. So people usually speak of *the* non-abelian group with order 21.)

Solutions to Homework

Solution 1.1: Part (a). Let A be a field, let U be a 2-dimensional A-vector space, and let V, W, X be three distinct 1-dimensional subspaces of U. Then $X > \{0\} = (V \cap X) \oplus (W \cap X)$.

Part (b). Plainly, the direct sum $V \oplus (W \cap X)$ can be formed and is contained in X. The equality holds because, given $x \in X$, writing $x = v + w$ with $v \in V$ and $w \in W$, then $w = x - v \in X$, hence $w \in W \cap X$.

Solution 1.2: Let $(-|-) : V \times V \to \mathbb{C}$ be an arbitrary inner product on V. Defining

$$\langle u \,|\, v \rangle = \sum_{g \in G} (gu \,|\, gv)$$

for $u, v \in V$, it is easy to see that $\langle -|- \rangle$ is seqsquilinear, conjugate-symmetric, positive-definite, in other words, $\langle -|- \rangle$ is an inner product. To see that $\langle -|- \rangle$ is G-invariant, observe that, for $f \in G$, making the substitution $h = gf$, we have

$$\langle fu \,|\, fv \rangle = \sum_{g \in G} (gfu \,|\, gfv) = \sum_{h \in G} (hu \,|\, hv) = \langle u \,|\, v \rangle \ .$$

For the rider, we must show that any submodule U of a finite-dimensional $\mathbb{C}G$-module V has complementary submodule. Let W be the orthogonal complement of U with respect to a G-invariant inner product $\langle -|- \rangle$ on V. We are to show that W is a $\mathbb{C}G$-submodule of V, in other words, given $g \in G$ and $w \in W$, then $gw \in W$. For all for all $u \in U$, we have $g^{-1}u \in U$, hence

$$\langle u \,|\, gw \rangle = \langle gg^{-1}u \,|\, gw \rangle = \langle g^{-1}u \,|\, w \rangle \ .$$

Since u is arbitrary, we have $gw \in W$, as required.

Quiz Questions

Q1: Let π be a projection on a finite-dimensional vector space V. (Recall, for a field K and a K-vector space V, a **projection** on V is a K-linear endomorphism π of V such that $\pi^2 = \pi$.) Show that $V = \mathrm{im}(\pi) \oplus \ker(\pi)$.

Q2: Find k and the finite sequence of positive integers $n_1 \leq ... \leq n_k$ such that

$$\mathbb{C}D_8 \cong \mathrm{Mat}_{n_1}(\mathbb{C}) \oplus ... \oplus \mathrm{Mat}_{n_k}(\mathbb{C}) \ .$$

Q3: Up to isomorphism, how many 3-dimensional $\mathbb{C}A_5$-modules are there? What are there characters in terms of the irreducible characters χ_1, ..., χ_5? (In class, the enumeration of the irreducible characters was such that the corresponding degrees are 1, 3, 3, 4, 5.)

Solutions to Quizzes

Sol 1: We must show that $\mathrm{im}(\pi) \cap \ker(\pi) = \{0\}$ and $\mathrm{im}(\pi) + \ker(\pi) = V$. The first of those equalities holds because, given $x \in \mathrm{im}(\pi) \cap \ker(\pi)$, the $x = \pi(y)$ for some $y \in V$, also $\pi(x) = 0$, hence $x = \pi(y) = \pi^2(y) = \pi(x) = 0$. The second of the required equalities holds because, given $z \in V$, then $z = \pi(z) + z - \pi(z)$, also $\pi(z) \in \mathrm{im}(\pi)$ and $z - \pi(z) \in \ker(\pi)$.

Alternative solution: Plainly, $\mathrm{im}(\pi) \cap \ker(\pi) = \{0\}$. So, as a subspace of V, we can form the direct sum $\mathrm{im}(\pi) \oplus \ker(\pi)$. By the rank-nullity formula, the direct sum coincides with V.

Comment: The first solution has the advantage of showing that the finiteness hypothesis on V is unnecessary.

Sol 2: In view of the trivial module, we have $n_1 = 1$. Since $\mathbb{C}D_8$ is non-commutative, $n_k > 1$. We have $n_1^2 + ... + n_k^2 = 8$. Therefore, $k = 5$ and $n_1 = ... = n_4 = 1$ and $n_5 = 2$.

Sol 3: There are precisely 3 such modules up to isomorphisms. Their characters are $3\chi_0$ and χ_1 and χ_2.

Presentations

- Sabri Çetin, Serkan Doğan (together, two hours), "The Burnside $p^\alpha q^\beta$-Theorem".
- Furkan Merdan, "Dirichlet characters".
- Baran Zadeoğlu, "The Jacobson radical as the maxial nilpotent ideal".
- Utku Okur, "The Jacobson radical and semisimplicity".

<u>MATH 325 : *Representation Theory*</u>

<u>MATH 525 : *Group Representations*</u>

Midterm

LJB, 4 April 2018, Bilkent.

Please put your name on every sheet of your manuscript.

Warning: For each question, the length of the solution must be equivalent to, at most, one page in handwriting of the size of this text, with plenty of whitespace. Beyond that length, all excess writing will be ignored.

1: 20% Let K be an algebraically closed field. Up to isomorphism, how many 10-dimensional semisimple algebras over K are there?

2: 20% Let R be a semisimple ring. Show that the center $Z(R)$ is a semisimple ring. Express the number of of isomorphism classes of simple $Z(R)$-modules in terms of $k(R)$, the number of isomorphism classes of simple R-modules.

3: 20% Give an example of a field K and a finite-dimensional algebra A over K such that A not semisimple. For your example, find an ideal J such that J is nilpotent and A/J is semisimple.

4: 20% Let G be a finite group. Let χ_1, ..., χ_k be the irreducible $\mathbb{C}G$-characters. Let g_1, ..., g_k be representatives of the conjugacy classes of G. You may assume the row orthonormality relation $\sum_{\ell=1}^{k} |[g_\ell]| \chi_i(g_\ell^{-1}) \chi_j(g_\ell) = \delta_{i,j} |G|$.

Prove the column orthonormality relation $\sum_{\ell=1}^{k} \chi_\ell(g_i^{-1}) \chi_\ell(g_j) = \delta_{i,j} |C_G(g_i)|$.

5: 20% Let K be a normal subgroup of a finite group G. Let χ be an irreducible $\mathbb{C}G$ character that is not inflated from G/K. Show that, for all $g \in G$, we have

$$\sum_{k \in K} \chi(gk) = 0 = \sum_{k \in K} \chi(kg) \ .$$

(Hint: consider $\sum_k k/|K|$ as an idempotent of $Z(\mathbb{C}G)$.)

Solutions to Midterm.

1: There are exactly 4 isomorphism classes of such algebras because the positive integer solutions to $n_1^2 + ...n_k^2 = 10$ are:

$$1^2 + ... + 1^2 = 1^2 + ... + 1^2 + 2^2 = 1^2 + 1^2 + 2^2 + 2^2 = 1^2 + 3^2 = 10 .$$

2: First suppose R is simple as well as semisimple, in other words, $R \cong \mathrm{Mat}_n(\Delta)$ where n is a positive integer and Δ is a division ring. We claim that $Z(R) \cong Z(\Delta)$. Fixing an isomorphism $R \cong \mathrm{Mat}_n(\Delta)$, let $\epsilon_{i,j}$ be the element of R corresponding to the matrix with 1 in the (i,j)-entry and 0 in all the other entries. Let $z \in Z(R)$ and write $z = \sum_{i,j} z_{i,j}\epsilon_{i,j}$ with each $z_{i,j} \in R$. In the case $i \neq j$, a consideration of the equality $\epsilon_{i,i} z \epsilon_{j,j} = z\epsilon_{i,i}\epsilon_{j,j} = 0$ yields $z_{i,j} = 0$. By considering the equality $z\epsilon_{i,j} = \epsilon_{i,j} z$, we see that each $z_{i,i} = z_{j,j}$. Finally, by considering Δ-multiples of the unity element 1_R, we see that each $z_{i,i} \in Z(\Delta)$. The claim is now established.

Generally, write

$$R \cong \bigoplus_{\ell=1}^{k(R)} \mathrm{Mat}_{n_\ell}(\Delta_\ell)$$

as the sum of the Wedderburn components, where each Δ_ℓ is a division ring. By the claim,

$$Z(R) = \bigoplus_{\ell=1}^{k(R)} Z(\Delta_\ell)$$

as a direct sum of fields. In particular, $k(Z(R)) = k(R)$.

3: Let K be any field and let A be a K-module with a basis $\{1, j\}$. We make A become an algebra over K by imposing the multiplication operation whereby 1 is the unity element and $j^2 = 0$. The K-submodule J generated by j is a nilpotent ideal and the quotient A/J is isomorphic to K, which is semisimple.

4: Let X be the $k \times k$ matrix such that the (i,l) entry is $\chi_i(g_\ell)/\sqrt{|C_G(g_\ell)|}$. Bearing in mind that $|G|/|[g_l]| = |C_G(g_\ell)|$ and that $\chi_i(g_\ell^{-1})$ is the complex conjugate of $\chi_i(g_\ell)$, we see that the row orthonormality condition on the character table says precisely that X satisfies the row orthonomality condition in the definition of a unitary matrix, in other words, $X.X^\dagger = 1$. That condition is, of course, equivalent to the column orthonormality condition $X^\dagger.X = 1$, which is precisely the required identity.

5: For any ϕ in the set $\mathrm{Irr}(\mathbb{C}G)$ of irreducible $\mathbb{C}G$-characters, let e_ϕ denote the unity element of the Wedderburn component associated with ϕ. Let $e = \sum_k k/|K|$. Plainly, e is an idempotent of $\mathbb{C}G$. Since $K \trianglelefteq G$, we have $e \in Z(\mathbb{C}G)$. Therefore

$$e = \sum_{\phi \in I} e_\phi$$

for some subset $I \subseteq \mathrm{Irr}(\mathbb{C}G)$.

For any $\phi \in \mathrm{Irr}(\mathbb{C}G)$, we have $ee_\phi = e_\phi$ if and only if e acts as the identity element on a simple $\mathbb{C}G$-module S_ϕ affording ϕ. But each $k \in K$ acts on S_ϕ as a sum of $\phi(1)$ roots of unity.

By the triangle inequality applied to $|K|\phi(1)$ roots of unity, e acts as the identity on S_ϕ if and only if each $k \in K$ acts as the identity on S_ϕ. Therefore, I is the subset of $\mathrm{Irr}(\mathbb{C}G)$ consisting of those ϕ that are inflated from G/K. In particular, $ee_\chi = 0$. But

$$ee_\chi = \frac{\chi(1)}{|G|.|K|} \sum_{f \in G, k \in K} \chi(f^{-1}) k f \ .$$

The required equality now follows by evaluating the coefficient of g^{-1}.

MATH 325 : *Representation Theory*

Final

LJB, 22 May 2018, Bilkent.

Please put your name on every sheet of your manuscript.

Solutions with excessive details or digressions will be marked down.

1: 25% Find the character table of the dihedral group D_{18} of order 18. (Explain your methods clearly and concisely.)

2: 25% Let G be a finite group. Classify the algebra maps $\omega : Z(\mathbb{C}G) \to \mathbb{C}$. Give formulas for $\omega(e_\chi)$ and $\omega([g]^+)$ where e_χ is the primitive idempotent of $Z(\mathbb{C}G)$ associated with an irreducible character χ and $[g]^+$ is the sum of the conjugates of an element $g \in G$.

3: 25% Let $K \trianglelefteq G$ be finite groups. Given $g \in G$ and a simple $\mathbb{C}K$-module V, we write ${}^gV$ to denote the $\mathbb{C}K$-module such that $V = {}^gV$ as $\mathbb{C}$-vector spaces and, for $k \in K$, the action of ${}^gk$ on ${}^gV$ coincides with the action of k on V. Let U be a simple $\mathbb{C}G$-module. Let V_1 and V_2 be two simple $\mathbb{C}K$-modules appearing in the restriction of U to $\mathbb{C}K$. Prove that $V_2 \cong {}^gV_1$ for some $g \in G$.

4: 25% A $\mathbb{C}G$-module M is said to be **faithful** provided the representation $\rho : G \to \mathrm{GL}(M)$ is injective. Show that, if there exists a simple faithful $\mathbb{C}G$-module, then $Z(G)$ is cyclic.

Solutions to Final.

Sol 1: Write $D_18 = \langle a, b : a^9 = b^2 = (ab)^2 = 1 \rangle$. The character table is as follows, where $c_k = 2\cos(2\pi k/9)$. Note that $c_3 = -1$.

1	2	2	2	2	9	$\lvert[g]\rvert$
1	9	9	3	9	2	$\lvert\langle g\rangle\rvert$
1	a	a^2	a^3	a^4	b	g
1	1	1	1	1	1	
1	1	1	1	1	1	
2	1	1	2	1	0	
2	c_1	c_2	1	c_4	0	
2	c_2	c_4	1	c_1	0	
2	c_4	c_1	1	c_2	0	

The two 1-dimensional $\mathbb{C}D_{18}$-characters were found by noting that the abelianization of D_{18} is $C_2 \cong D_{18}/C_9$. The other four irreducible $\mathbb{C}D_{18}$-characters were obtained by inducing the non-trivial 1-dimensional $\mathbb{C}C_9$-characters. To confirm the irreducibility of those latter four, we need only observe that none of them are linear combinations of the two 1-dimensional $\mathbb{C}D_{18}$-characters.

Sol 2: We have $Z(\mathbb{C}G) = \bigoplus_\chi \mathbb{C}e_\chi$, where χ runs over the irreducible $\mathbb{C}G$-characters and e_χ is the associated primitive idempotent of $Z(\mathbb{C}G)$. So the algebra maps $\omega_\chi : Z(\mathbb{C}G) \to \mathbb{C}$ are in a bijective correspondence with the χ and they are given by

$$\omega_\chi(e_{\chi'}) = \begin{cases} 1 & \text{if } \chi = \chi', \\ 0 & \text{otherwise.} \end{cases}$$

Given $\zeta \in Z(\mathbb{C}G)$, then $\zeta = \sum_\chi \omega_\chi(\zeta) e_\chi$. So, letting V_χ be a simple $\mathbb{C}G$-module affording χ, then ζ acts on V_χ as multiplication by $\omega_\chi(\zeta)$. By considering the trace of the action of $[g]^+$ on V_χ, we obtain

$$\omega_\chi([g]^+) = |[g]|\chi(g)/\chi(1) \ .$$

Sol 3: Since U is simple, $U = \sum_{g \in G} gV_1$. We have an isomorphism of simple $\mathbb{C}K$-modules $gV_1 \cong {}^gV_1$. By the theory of semisimple modules, we have a $\mathbb{C}K$-isomorphism $U \cong \bigoplus_{g \in S} {}^gV_1$ for some $S \subseteq G$. By the uniqueness of decompositions of semisimple modules, $V_2 \cong {}^gV_1$ for some $g \in S$.

Sol 4: Let M be a simple faithful $\mathbb{C}G$-module. Then M restricts to a faithful $\mathbb{C}Z(G)$-module. But, by Schur's Lemma, $Z(G)$ acts on M as scalar multiplication via a group homorphism $\theta : Z(G) \to \mathbb{C}$. All the finite subgroups of $\mathbb{C}$ are cyclic. The faithfulness of M as a $\mathbb{C}Z(G)$-module implies that θ is injective. Therefore $Z(G)$ is cyclic.