Archive of past papers, solutions and homeworks for

MATH 325, Representation Theory,

Spring 2013, Laurence Barker

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MATH 325 Group Representations, Spring 2013

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This is an introductory course on groups and representations intended primarily for undergraduates in mathematics, physics, chemistry and engineering.

Outline of topic: Groups are algebraic devices for expressing symmetries of objects. A group is a set of symmetries that is closed under combination and reversibility. Ideally, a water molecule, H_2O, has 4 rigid symmetries, methyl chloride CH_3Cl has 6, methane CH_4 has 24, Buckminsterfullarine C_{60} has 120 rigid symmetries. The corresponding groups are usually denoted V_4 and S_3 and S_4 and $A_5 \times C_2$, respectively.

In many contexts, both in pure and applied mathematics, groups arise as symmetries of vector spaces. Thus, each element of the group is represented by a matrix. There is a multiplication operation on the group which corresponds to matrix multiplication

We shall be concerned with representation theory in the easiest case, namely, where the groups are finite and the group elements are represented by matrices over the complex numbers. These representations are called ordinary representations. One of the fundamental theorems in ordinary representation theory, Maschke's Theorem, says that there is no loss of generality in taking the vector space to be an inner product space.

It turns out that we do not need to do complicated matrix calculations. One consequence of Maschke's Theorem is that, in some sense, the only thing that matters are the traces of the matrices. The function sending each group element to the trace of the associated matrix is called the character of the representation.

All the important information about the representations of given finite group is captured by a table of numbers called the character table of the group. For that reason, ordinary representation theory is sometimes called ordinary character theory.

Aim: The aim of the course will be to learn how to calculate character tables for small groups such as V_4 and S_3 and S_4 and $A_5 \times C_2$. If time permits, we may discuss an application to molecular vibration.

Course Text: J. L. Alperin, R. B. Bell, "Groups and Representations", (Springer, Berlin, 1995).

Syllabus: We shall begin with a discussion of the symmetry groups of the molecules mentioned above. Since the course is intended for a variety of students, background from group theory (conjugacy classes, subgroups) and from linear algebra (eigenvalues and eigenvectors, inner product spaces) will be carefully covered as needed.

Victory criterion: The main criterion for success will be a knowledge of how to construct character tables for groups such as V_4 and S_3 and S_4 and $A_5 \times C_2$.

Prerequisites: It would help if you can already see why a methane molecule has 12 rotational symetries; the hydrogen atoms form a regular tetrahedron with the oxygen atom at the center. (Incidentally, the other 12 of the total 24 symmetries come from combining the rotations with a reflection.)

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Homeworks, Quizzes and Presentations

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Homeworks

Homework 1: Find the number of rigid symmetries of the plane that preserve an ozone molecule (an equilateral triangle, the vertices being oxygen atoms).

Homework 2: Questions 2, 3, 4 in Chapter 1, page 12 of textbook.

Homework 3: Prove the Three Group Isomorphism Theorems and discuss a non-trivial example for each of them. (And prepare for presentations of the material in class.)

Homework 4: Let F be a field, $A = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : a, b, d \in F \right\}$ and $B = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} : a, b, \in F \right\}$. Show that A and B are not semisimple.

Homework 5: Let $1 = e_1 + ... + e_k$ be the sum of the primitive idempotents of $Z(\mathbb{C}G)$, and let χ_1, ..., χ_k be their irreducible characters in respective order. In the case where G is abelian, find a formula expressing e_j, in terms of χ_j, as a linear combination of group elements. Try to generalize your formula to the case where the finite group G is arbitrary.

Homework 6: In the context of the previous homework, deal with the case where G is arbitrary.

Homework 7: Pick a non-abelian group whose character table has not yet been determined in class. Find its character table.

Presentations

Oğuzhan Yörük: "The character table for the non-abelian group with order 21".

Vefa Göksel: "Criteria for semisimplicity and a proof that finite-dimensional simple algebras are semisimple".

Muhammed Said Gündoğan, "Proof of the Burnside $p^\alpha q^\beta$-Theorem".

Mehmet Kişioğlu, "Induction of characters and the character table for S_4."

MATH 325 Representation Theory, Midterm

3 May 2013, LJB, Bilkent University.

Time allowed: 110 minutes.

Please make sure your name is on every sheet of your script.

Throughout, F is an algebraically closed field and A is a finite-dimensional algebra over F.

1: 10% State and prove any version of Maschke's Theorem.

2: 10% Let G be a non-abelian group with order 8. How many simple $\mathbb{C}G$-modules are there, up to isomorphism, and what are their dimensions?

3: 30% Let A be a finite dimensional algebra over a field F.

(a) Show that $\mathrm{End}_A({}_AA) \cong A^\circ$, where ${}_AA$ denotes A as an A-module by left-multiplication and A° denotes the opposite algebra of A.

(b) Show that $\mathrm{Mat}_n(F)^\circ \cong \mathrm{Mat}_n(F)$ for any positive integer n.

(c) Show that $(FG)^\circ \cong FG$ for any finite group G. (Hint: consider inverses of group elements.)

4: 40% Let G be a finite group. Suppose that G has order $|G| = 125$ and G has exactly 29 conjugacy classes and G has a unique smallest non-trivial normal subgroup N. Suppose furthermore that G/N is an abelian group with order $|G/N| = 25$.

(a) Explain why the number of 1-dimensional $\mathbb{C}G$-modules, up to isomorphism, is at least 25.

(b) Let S be a 1-dimensional $\mathbb{C}G$-module. Let K be the set of elements $g \in G$ such that $gs = s$ for all $s \in S$. Show that K is a non-trivial normal subgroup of G.

(c) Deduce that the number of 1-dimensional $\mathbb{C}G$-modules, up to isomorphism, is exactly 25.

(d) How many $\mathbb{C}G$-modules are there, up to isomorphism, and what are their dimensions?

5: 10% Let A be a finite-dimensional algebra over $\mathbb{C}$ and let M be an A-module. Recall that the ring $\mathrm{End}_A(M)$ is the set of linear maps $\theta : M \to M$ such that $\theta(am) = a\theta(m)$ for all $a \in A$ and $m \in M$. Let $B = \mathrm{End}_A(M)$ and $C = \mathrm{End}_B(M)$ and $D = \mathrm{End}_C(M)$. Show that $B = D$.

MATH 325 Representation Theory, Final

20 May 2013, LJB, Bilkent University.

Time allowed: 2 hours.

Please make sure your name is on every sheet of your script.

1: 10% Write down, without proof, the character table for the group C_3.

2: 34% Let G be a finite group with the following properties: The order is $|G| = 12$. There are 4 conjugacy classes in G, namely

$$[1] = \{1\} \ , \quad [x] = \{x, \, y, \, z\} \ , \quad [a] = \{a, \, xa, \, ya, \, za\} \ , \quad [a^2] = \{a^2, \, xa^2, \, ya^2, \, za^2\} \ .$$

The elements x and a have orders 2 and 3, respectively. The set $\{1, x, y, z\}$ is a normal subgroup. Construct the character table of G, carefully explaining how each irreducible character is obtained.

3: 28% Let χ and ψ be distinct irreducible characters for a finite group G.

(a) Let e_χ be the idempotent of $Z(\mathbb{C}G)$ such that e_χ acts as the identity function on simple $\mathbb{C}G$-modules with character χ while e_χ annihilates all other simple $\mathbb{C}G$-modules. State, without proof, a formula for e_χ as a linear combination of group elements.

(b) Using the formulas for e_χ and e_ψ, show that $\sum_{g \in G} \chi(g^{-1}) \, \psi(g) = 0$.

(c) Show that $\chi(g^{-1})$ is the complex conjugate of $\chi(g)$.

(d) Using the formula for e_χ together with part (c), evaluate $\sum_{g \in G} |\chi(g)|^2$.

4: 28% Let A be a semisimple algebra over a field F. Given an A-module V with representation $\rho_V : A \to \mathrm{End}_F(V)$, we define the **character** of V to be the function $\chi_V : A \to F$ such that $\chi(a) = \mathrm{tr}(\rho_V(a))$ for $a \in A$. In other words, $\chi_V(a)$ is the trace of the action of a on V.

(a) Show that, given A-modules V_1 and V_2 such that $V_1 \cong V_2$, then $\chi_{V_1} = \chi_{V_2}$.

(b) Show that, given A-modules U and V, then $\chi_{U \oplus V} = \chi_U + \chi_V$.

(c) Show that, given a simple A-module S, then there exists an element $a_S \in A$ such that $\chi_S(a_S) = 1$ and $\chi_T(a_S) = 0$ for all simple A-modules T not isomorphic to S.

(d) Deduce that, given A-modules V_1 and V_2 such that $\chi_{V_1} = \chi_{V_2}$, then $V_1 \cong V_2$.

Syllabus

The originally offered syllabus was modified to accommodate the requirements of the class.

Week 1: Symmetries of molecules. Statement of matrix version of Maschke's Theorem.

Week 2: Review of prerequisite linear algebra. Proof of matrix version of Maschke's Theorem

Week 3: Groups, homomorphisms, representations.

Week 4: Representation theoretic examples of groups, subgroups, conjugacy classes.

Week 5: Rings and ideals.

Week 6: Modules and associated isomorphism theorems.

Week 7: Module homomorphisms and endomorphism rings.

Week 8: Semisimple modules.

Week 9: Artin–Wederburn Structure Theorem.

Week 10: Centrally primitive idempotents of the group algebra.

Week 11: Orthogonality properties for character tables.

Week 12: Constructing character tables.

Week 13: Presentations.

Week 14: Presentations and review for Final.