Archive for

MATH 325, Representation Theory, Fall 2023

Bilkent University, Laurence Barker, 12 January 2024.

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MATH 325

Representation Theory, Fall 2023

Course specification

Laurence Barker, Bilkent University. Version: 30 December 2023.

Classes: Mondays 15:30 - 17:20, Thursdays 10:30 - 11:20, room SA Z04.

Recitations: Thursdays 11:30 - 12:20, SA Z04.

The Recitations, organized by Mahmut Esat Akın, are an opportunity for discussion of homeworks, course material, background material, or whatever the participants wish.

Office Hours: Wednesdays 17:30 - 18:20, SA 129.

Office Hours is for all the students on the course, regardless of strength.

Instructor: Laurence Barker
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Course Texts: The primary course text is:

> Peter Webb, "A Course in Finite Group Representation Theory", Cambridge University Press 2016. There is a free PDF download of the prepublication version on the homepage of Peter Webb, University of Minnesota.

A suggested secondary text on character theory is:

> Jon L. Alperin, Rowen B. Bell, "Groups and Representations", (Springer, Berlin, 1995).

For those with an interest in a deeper treatment of ring theory, a recommended text is

> T.-Y. Lam, "A First Course in Noncommutative Rings", (Springer, Berlin, 1991).

Homework: You will not pick up course credits for homework, just as athletes do not pick up prizes for training sessions. The homeworks are to help with learning the material and getting into shape for the exams.

Course Documentation: As the course progresses, further documentation will appear on the course Moodle site and my homepage.

Syllabus: *Below is a tentative course schedule.* The format of the following details is *Week number: Monday date: Subtopics (Section numbers).*

Syllabus: The format of the following details is *Week number: Monday date: Subtopics.*

1: 11 Sept: (Thursday only.) Group representations and ordinary characters.

2: 18 Sept: Groups, rings and modules.

3: 25 Sept: Semisimple modules and semisimple rings. The group algebra and Maschke's Theorem.

4: 2 Oct: Conjugacy classes and the centre of a group algebra.

5: 9 Oct: Ordinary irreducible characters.

6: 16 Oct: Ordinary character tables for some small finite groups.

7: 23 Oct: Centrally primitive idempotents of the group algebra.

8: 30 Oct: Orthogonality properties of the ordinary character table.

9: 6 Nov: Inflation, restriction and induction of characters. Frobenius reciprocity.

10: 13 Nov: Constructing character tables.

11: 20 Nov: The character tables of the alternating and symmetric groups.

12: 27 Nov: Symmetric and alternating squares. Further groups and their character tables.

13: 4 Dec: Integrality properties of ordinary irreducible characters.

14: 11 Dec: As applications, Burnside's $p^\alpha q^\beta$-Theorem and characterization of Frobenius groups.

15: 18 Dec: (Monday only.) Review.

Assessment:

- Quizzes, 10%,
- Midterm, 45%, at 20:00 - 22:00, Thursday, 16 November, in SA-Z03.
- Final, 45%, at 09:00 on Friday, 5 January 2024, in SA-Z18.

An FZ grade will be awarded for Midterm marks that are below 20% and that also display outright incomprehension of basic concepts.

75% attendance is compulsory.

Asking questions in class is very helpful. It makes the classes come alive, and it often improves my sense of how to pitch the material. The rule for talking in class is: if you speak, then you must speak to everyone in the room.

Quizzes, with solutions

MATH 325, *Representation Theory*, Fall 2023, Laurence Barker

version: 20 December 2023

Quiz 1: Let $G = C_3 = \{1, a, a^2\}$. Observe that the 1-dimensional $\mathbb{C}$-vector space

$$\mathbb{C} \sum_{g \in G} g = \mathbb{C}(1 + a + a^2)$$

is a $\mathbb{C}G$-submodule of the regular $\mathbb{C}G$module

$${}_{\mathbb{C}G}\mathbb{C}G = \mathbb{C}1 \oplus \mathbb{C}a \oplus \mathbb{C}a^2 \ .$$

Find a basis for a complementary submodule.

Solution: Defining $\omega = e^{2\pi i/3}$, we have

$${}_{\mathbb{C}G}\mathbb{C}G = \mathbb{C}(1 + a + a^2) \oplus \mathbb{C}(1 + \omega^2 a + \omega a^2) \oplus \mathbb{C}(1 + \omega a + \omega^2 a^2)$$

as a direct sum of 1-dimensional $\mathbb{C}G$-modules. So the submodule $\mathbb{C}(1 + a + a^2)$ has complementary submodule $\mathbb{C}(1 + \omega^2 a + \omega a^2) \oplus \mathbb{C}(1 + \omega a + \omega^2 a^2)$. One basis for the complementary submodule is the set $\{1 + \omega^2 a + \omega a^2, 1 + \omega a + \omega^2 a^2\}$.

Another basis for the complementary submodule is $\{1 - 2a + a^2, 1 + a - 2a^2\}$.

Comment 1: The above decomposition of $\mathbb{C}G$ already appeared in the answer to Homework Question 1.1 part (b).

Comment 2: For any finite group G and any field K of characteristic 0, the regular KG-module ${}_{KG}KG$ decomposes as a direct sum of KG-modules

$${}_{KG}KG = K \sum_{g \in G} g \oplus \{ \sum_{g \in G} \lambda_g g : \sum_{g \in G} \lambda_g = 1 \} \ .$$

Quiz 2: Up to isomorphism, how many 12-dimensional semisimple algebras over $\mathbb{C}$ are there?

Solution: Since $\mathbb{C}$ is algebraically closed, any semisimple algebra over $\mathbb{C}$ is isomorphic to a direct sum of matrix algebras over $\mathbb{C}$. Therefore, the answer is the number of ways of expressing 12 as a sum of non-increasing squares. The ways of thus expressing 12 are

$$12 = 9 + 3.1 = 3.4 = 2.4 + 4.1 = 4 + 8.1 = 12.1 \ .$$

Therefore, the answer is 5.

Quiz 2: Advanced version: How many 12-dimensional semisimple algebras over $\mathbb{R}$ are there? You may use a theorem of Frobenius which asserts that every finite-dimensional division algebra over $\mathbb{R}$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$.

Solution: Let m denote the answer.

For any natural number n, we define $f(n)$ to be the number of ways of expressing n as a sum of non-increasing squares. A table of values of $f(n)$, for $n \leq 12$, is as follows.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$f(n)$	1	1	1	1	2	2	2	2	3	4	4	4	5

Given any division ring Δ, then $f(n)$ is the number of isomorphism classes of n-dimensional algebras over Δ that can be decomposed as direct sums of matrix algebras. Any 12-dimensional algebra A over $\mathbb{R}$ decomposes as $A = A_{\mathbb{H}} \oplus A_{\mathbb{C}} \oplus A_{\mathbb{R}}$ where each A_Δ is a direct sum of matrix algebras over Δ. As parameters of A, we introduce $a = \dim_{\mathbb{H}}(A_{\mathbb{H}})$ and $b = \dim_{\mathbb{C}}(A_{\mathbb{C}})$ and $c = \dim_{\mathbb{R}}(A_{\mathbb{R}})$. We have $4a+2b+c = 12$. For each (a, b, c), the number of possible isomorphism classes for A is $f(a)f(b)f(c)$. Therefore,

$$m = \sum_{a,b,c \in \mathbb{N}\,:\, a+b+c=12} f(a)f(b)f(c) \ .$$

The possibilities for (a, b, c) and the values of $f(a)$, $f(b)$, $f(c)$ and $f(a)f(b)f(c)$ are as shown.

a	3	2	2	2	1	1	1	1	1	0	0	0	0	0	0	0
b	0	2	1	0	4	3	2	1	0	6	5	4	3	2	1	0
c	0	0	2	4	0	2	4	6	8	0	2	4	6	8	10	12
$f(a)$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$f(b)$	1	1	1	1	2	1	1	1	1	2	2	2	1	1	1	1
$f(c)$	1	1	1	2	2	1	2	2	3	2	2	4	2	3	4	5
$f(a)f(b)f(c)$	1	1	1	2	2	1	2	2	3	2	2	4	2	3	4	5

Summing the entries of the bottom row, we conclude that $m = 37$.

Comment: When I set the advanced version of the quiz, I underestimated the answer. When I later solved the quiz, it did take me more than ten minutes.

Quiz 3: Let $G = A_5$, the alternating group of order 60. You may assume that the group algebra $\mathbb{C}G$ has exactly 5 simple modules, up to isomorphism, with dimensions 1, 3, 3, 4, 5. Up to isomorphism, how many simple 6-dimensional $\mathbb{C}G$-modules are there?

Solution: Write S_0, ..., S_4 for representatives of the isomorphism classes of simple $\mathbb{C}G$-modules, enumerated such that their dimensions are 1, 3, 3, 4, 5, respectively. Any $\mathbb{C}G$-module M is determined by the multiplicities m_0, ..., m_4, where $M \cong m_0S_0 \oplus ... \oplus m_4S_4$. Now supposing that $\dim(M) = 6$, then

$$6 = m_0 + 3m_1 + 3m_2 + 4m_3 + 5m_4 \ .$$

The number of possibilities for theisomorphism class of M is the number of natural number solutions $m_0...m_4 = (m_0, ..., m_4)$ to that equation. The solutions are

10001 , 20010 , 00200 , 02000 , 01100 , 30100 , 31000 , 60000 .

Thus, the answer is 8.

Quiz 4: The ordinary character table of the group $S_3 = \langle a, b : a^3 = b^2 = (ab)^2 \rangle$ is as shown. Evaluate the natural numbers λ, μ, ν where $(\chi_2)^2 = \lambda\chi_0 + \mu\chi_1 + \nu\chi_2$.

	1	3	2	$\lvert[g]\rvert$
	1	2	3	$\lvert\langle g\rangle\rvert$
$\chi(g)$	1	b	a	g
χ_0	1	1	1	
χ_1	1	-1	1	
χ_2	2	0	-1	

Solution: Let $\psi = (\chi_2)^2$. Now $(\psi(1), \psi(b), \psi(a)) = (4, 0, 1)$. By inspection, $\psi = \chi_0 + \chi_1 + \chi_2$. So $\lambda = \mu = \nu = 1$.

Comment: We could also directly calculate $\lambda = \langle \chi_0 \,|\, \psi \rangle$ and similarly for μ and ν.

Quiz 5: Let $H \leq G$ be finite groups and χ an irreducible $\mathbb{C}G$-character. Show that there exists an irreducible $\mathbb{C}H$-character ψ such that $\langle \chi \,|\, \mathrm{ind}_H^G(\psi) \rangle > 0$.

Solution: The regular $\mathbb{C}G$-character χ_{reg}^G is given by

$$\chi_{\mathrm{reg}}^G = \sum_{\chi \in \mathrm{Irr}(\mathbb{C}G)} \chi(1)\chi \ .$$

From the formula $\chi_{\mathrm{reg}}^G(g) = |G|\delta_{g,1}$, with $g \in G$, we see that $\chi_{\mathrm{reg}}^G = \mathrm{ind}_H^G(\chi_{\mathrm{reg}}^H)$. So

$$\sum_{\psi \in \mathrm{Irr}(\mathbb{C}H)} \psi(1) \langle \chi \,|\, \mathrm{ind}_H^G(\psi) \rangle = \langle \chi \,|\, \mathrm{ind}_H^G(\chi_{\mathrm{reg}}^H) \rangle = \langle \chi \,|\, \chi_{\mathrm{reg}}^G \rangle = \chi(1) \ .$$

It follows that $\langle \chi \,|\, \mathrm{ind}_H^G(\psi) \rangle > 0$ for some ψ.

Quiz 6: Consider the group $D_8 = \langle a, b \rangle$ where a is a rotation through a quarter of a revolution and b is a reflection. The character table of the subgroup $C_4 = \langle a \rangle$ is as follows.

$\chi(g)$	1	a	a^2	a^3	g
ϕ_0	1	1	1	1	
ϕ_1	1	i	-1	$-i$	
ϕ_2	1	-1	1	-1	
ϕ_3	1	$-i$	-1	i	

Fill in the entries of the following table of characters induced to D_8 from C_4.

	1	1	2	2	2	$\lvert[g]\rvert$
	1	2	4	2	2	$\lvert\langle g \rangle\rvert$
	1	a^2	a	b	ab	g
$\mathrm{ind}(\phi_0)$	?	?	?	?	?	
$\mathrm{ind}(\phi_1)$	?	?	?	?	?	
$\mathrm{ind}(\phi_2)$	?	?	?	?	?	
$\mathrm{ind}(\phi_3)$	?	?	?	?	?	

Solution: Using the formula for induced characters, we obtain the following table.

	1	a^2	a	b	ab	g
$\mathrm{ind}(\phi_0)$	2	2	2	0	0	
$\mathrm{ind}(\phi_1)$	2	-2	0	0	0	
$\mathrm{ind}(\phi_2)$	2	2	-2	0	0	
$\mathrm{ind}(\phi_3)$	2	-2	0	0	0	

Quiz 7: Let $V = \mathbb{R}^3$ as an $\mathbb{R}S_4$-module with S_4 transitively permuting the vertices of a regular tetrahedron in V. Enter, into the following table, the values of the $\mathbb{C}S_4$-character $\chi_{\mathbb{C}V}$ of the $\mathbb{C}S_4$-module $\mathbb{C}V = \mathbb{C} \otimes_{\mathbb{R}} V$.

	1^4	2.1^2	2^2	3.1	4	g
$\chi_{\mathbb{C}V}$	?	?	?	?	?	

Solution: We shall show that the entries are as follows.

	1^4	2.1^2	2^2	3.1	4	g
$\chi_{\mathbb{C}V}$	3	1	-1	0	-1	

The dimension of $\mathbb{C}V$ is $\chi_{\mathbb{C}V}(1) = 3$. The eigenvalues of each reflection 2.1^2 are 1, 1, -1, which sum to $\chi_{\mathbb{C}V}(2.1^2) = 1$. The eigenvalues of each rotation 2^2 are 1, -1, -1, which sum to $\chi_{\mathbb{C}V}(2^2) = -1$. The eigenvalues of each rotation 3.1 are 1, ω, ω^2, where $\omega = e^{2\pi i/3}$, hence $\chi_{\mathbb{C}V}(3.1) = 0$. The eigenvalues of the reflections with shape 4 are -1, i, $-i$, which sum to $\chi_{\mathbb{C}V}(4) = -1$.

Alternative solution: Let χ_0 denote the trivial $\mathbb{C}S_4$-character. The $\mathbb{C}S_4$-character $\chi_{\mathbb{C}V} + \chi_0$, being the $\mathbb{C}S_4$-character of the $\mathbb{C}S_4$-module associated with the natural S_4-set, has values 4, 2, 0, 1, 0 at 1^4, 2.1^2, 2^2, 3.1, 4, respectively.

Quiz 8: The group $\mathrm{SL}_2(3)$ is the group of $2{\times}2$ matrices over the field with order 3. We have a semidirect product $\mathrm{SL}_2(3) = C_3 \ltimes Q_8$. Let $\omega = e^{2\pi i/3}$. Write a for a generator of the subgroup C_3. Write $Q_8 = \{1, i, j, k, z, iz, jz, kz\}$ in the usual way. We saw in class that part of the character table for $\mathrm{SL}_2(3)$ is as follows. (The first 4 rows are inflated from the quotient group $A_4 \cong \mathrm{SL}_2(3)/\langle z \rangle$. The first entries of χ_4, χ_5, χ_6 rows come from column orthonormality. The second entries of those three rows come from column orthonormality together with the fact that the only possible eigenvalues of the action of z are ± 1.) Determine the entries labelled s, s', s'', t, t', t''.

	1	1	6	4	4	4	4	$\lvert[g]\rvert$
	1	2	4	3	3	6	6	$\lvert\langle g\rangle\rvert$
	1	z	i	a	a^2	az	a^2z	g
χ_0	1	1	1	1	1	1	1	
χ_1	1	1	1	ω	ω^2	ω	ω^2	
χ_2	1	1	1	ω^2	ω	ω^2	ω	
χ_3	3	3	-1	0	0	0	0	
χ_4	2	-2	s	t	?	?	?	
χ_5	2	-2	s'	t'	?	?	?	
χ_6	2	-2	s''	t''	?	?	?	

Solution: By column orthogonality, $|s|^2 + |s'|^2 + |s''|^2 = 0$. Therefore, $s = s' = s'' = 0$.

By column orthonormality, t and t' and t'' cannot all be 0. By considering tensor products with χ_1 and χ_2, we may assume that $t' = \omega t$ and $t'' = \omega^2 t$. Column orthonormality now gives $|t| = 1$. But t must also be the sum of two cube roots of unity. We deduce that, numbering χ_4, χ_5, χ_6 suitably, then $t = -1$ and $t' = -\omega$ and $t'' = -\omega^2$.

Comment: The rest of the character table can now be determined easily, and it is as follows.

	1	z	i	a	a^2	az	a^2z	g
χ_4	2	-2	0	-1	-1	1	1	
χ_5	2	-2	0	$-\omega$	$-\omega^2$	ω	ω^2	
χ_6	2	-2	0	$-\omega^2$	$-\omega$	ω^2	ω	

To see this, first note that, for the simple module S with character χ_4, the eigenvalues of the action of a must be ω and ω^2, both with multiplicity 1. The eigenvalues of the action of a^2 must be the same. Since z acts on S as negation, the eigenvalues of the action of az must be $-\omega$ and $-\omega^2$, with both multiplicities 1. A similar comment holds for a^2z. All the values for χ_4 are now clear. Using tensor products by χ_1 and χ_2 again, we obtain the remaining entries.

MATH 325, Representation Theory, Fall 2023

Homeworks

Laurence Barker, Bilkent University. Version: 12 January 2024.

Two guidelines to bear in mind:

Guideline 1: Write in complete sentences, otherwise the meaning will be ambiguous. "Prime p" has no meaning. "So p is prime" and "Let p be a prime" do have meanings, different meanings.

Guideline 2: Define your terms. The meaning of "So p is prime" is unclear if p has not been introduced.

Homework 1

Recall, an element e of a ring is called an **idempotent** provided $e^2 = e$.

Exercise 1.1: Find all the idempotents of:

(a) the group algebra $\mathbb{C}C_2$,

(b) the group algebra $\mathbb{C}C_3$,

(c) the group algebra $\mathbb{C}C_n$, where n is any positive integer.

Recall that, for a ring R, an R-module M is said to be **simple** provided M has exactly 2 submodules, namely $\{0\}$ and M.

Exercise 1.2: Using Exercise 1.1, show that every $\mathbb{C}C_n$-module is 1-dimensional.

Recall, Maschke's Theorem asserts that, given a finite group G and a field F such that $\mathrm{char}(F)$ does not divide $|G|$, then the group algebra FG is semisimple. The next exercise gives an alternative proof of that theorem in the special case where $F = \mathbb{C}$.

Exercise 1.3: Let G be a finite group and let U be a finite-dimensional $\mathbb{C}G$-module. Let

$$U \times U \ni (x, y) \mapsto \langle x \,|\, y \rangle \in \mathbb{C}$$

be any inner product on U. Define

$$\langle x \,|\, y \rangle' = \frac{1}{|G|} \sum_{g \in G} \langle gx \,|\, gy \rangle \ .$$

Show that $\langle - | - \rangle'$ is an inner product on U. By considering orthogonal complements with respect to $\langle - | - \rangle'$, show that $\mathbb{C}G$ is semisimple.

Exercise 1.4: Consider the unit group $\mathbb{H}^\times$ of the ring of quaternions $\mathbb{H}$. Which of the following groups are isomorphic to a subgroup of $\mathbb{H}^\times$? (make sure you justify your answers clearly.)

(a) the group C_4? (The cyclic group with order 4.)

(b) the group V_4? (The non-cyclic group with order 4.)

(c) the group C_8? (The cyclic group with order 8.)

(d) the group Q_8? (The quaternion group with order 8.)

(e) the group D_8? (The dihedral group with order 8.)

The next question is not on examinable material. The notion of a rng is not on the examinable syllabus.

Exercise 1.5: A rng R is said to be **locally unital** provided, for all $x, y \in R$, there exists an idempotent e of R such that $x, y \in eRe$. For a locally unital rng R, find a definition of an R**-module** that reduces to the usual notion of an R-module in the case where R is a ring.

Comment: Actually, Exercise 1.5 is quite easy, but it is background for the following more difficult problem, which will become understandable when we have defined semisimplicity.

• A locally unital rng R is said to be **strongly locally semisimple** provided there exists a set of idempotents $\mathcal{E}$ of R satisfying the following three conditions:

Orthogonality: For any two distinct elements e and f of $\mathcal{E}$, we have $ef = fe = 0$,

Completeness: We have $R = \bigoplus_{e,f \in \mathcal{E}} eRf$,

Partial semisimplicity: For any finite subset $\mathcal{D} \subseteq \mathcal{E}$, writing d for the sum of the elements of $\mathcal{D}$, the ring dRd is semisimple.

• A locally unital rng R is said to be **weakly locally semsimple** provided, for every idempotent e of R, the ring eRe is semisimple.

Given a locally unital rng R, show that, if R is strongly semisimple, then R is weakly semisimple.

I do not know whether the converse holds. I would be interested in the answer because I have used the strong version as a definition of "locally semisimple" in one paper, and the weak version as the definition of "locally semisimple" in other papers. Has my terminology been consistant?

Homework 2: Do the Exercises in Sections 2 and 3 of the notes.

Homework 3: Do the Exercises in Sections 4 and 5 of the notes.

Homework 4

Exercise 4.1: Show that, up to isomorphism, there exists a unique non-abelian group F_{21} with order 21. Find the character table of F_{21}.

Exercise 4.2: Find the ordinary character table of the alternating group A_6. (As well as the midterm techniques, you may also make use of products of characters, symmetric and alternating squares, induction. Note that A_6 has a subgroup isomorphic to A_5 and a subgroup isomorphic to S_4.)

Exercise 4.3: Find the ordinary character table of the symmetric group S_6.

For $g \in G$ and $H \leq G$ and a $\mathbb{C}H$-module M, we define the **conjugate $\mathbb{C}^gH$-module** ${}_{^gH}\mathrm{Con}^g_H(M)$, sometimes written more briefly as ${}^gM$, such that ${}^gM = M$ as $\mathbb{C}$-vector spaces and, given $h \in H$, then the action of ${}^gh$ on ${}^gM$ coincides with the action of h on M. We write the associated map on characters as ${}_{^gH}\mathrm{con}^g_H : \mathbb{C}R_{\mathbb{C}}({}^gH) \leftarrow \mathbb{C}R_{\mathbb{C}}(H)$.

Exercise 4.4: Let $K \trianglelefteq G$ be finite groups and ϕ a $\mathbb{C}K$-character. Show that ${}_K\mathrm{res}_G\mathrm{ind}_K(\phi)$ is a sum of G-conjugates of ϕ.

Exercise 4.5: Let G be a finite group and $F, H \leq G$. Let M be a $\mathbb{C}H$-module. Show that

$$ {}_F\mathrm{Res}_G\mathrm{Ind}_H(M) \cong \bigoplus_{FgH \subseteq G} {}_F\mathrm{Ind}_{F\cap {}^gH}\mathrm{Con}^g_{F^g\cap H}\mathrm{Res}_H(M) $$

where the notation indicates that FgH runs over the F-H-double consets in G.

Exercise 4.5: Let $H \leq G$ be finite groups, V a $\mathbb{C}H$-module, U a $\mathbb{C}G$-module. Recall, the Frobenius Reciprocity Theorem asserts that

$$ \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathbb{C}G}({}_G\mathrm{Ind}_H(V), U)) = \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathbb{C}H}(V, {}_H\mathrm{Res}_G(U))) $$

or equivalently, in character theoretic terms

$$ \langle {}_G\mathrm{ind}_H(\chi_V) \,|\, \chi_U \rangle_G = \langle \chi_V \,|\, {}_H\mathrm{res}_G(\chi_U) \rangle_H \ . $$

In class, we directly proved the former equality using Schur's Lemma, and we directly proved the latter equality using the formula for the inner product on the character algebra. As a third proof, explicitly find a $\mathbb{C}$-linear isomorphism

$$ \mathrm{Hom}_{\mathbb{C}G}({}_G\mathrm{Ind}_H(V), U) \cong \mathrm{Hom}_{\mathbb{C}H}(V, {}_H\mathrm{Res}_G(U)) \ . $$

MATH 325: Representation Theory

Midterm

16 November 2023, LJB

The duration of the exam is 120 minutes. It is a closed book exam.

1: (20 marks.) Let A be a 5-dimensional non-commutative semisimple algebra over $\mathbb{R}$. How many isomorphism classes of 8-dimensional A-modules are there?

2: (20 marks.) Find the ordinary character table of the group Q_8, the quaternion group with order 8.

3: (20 marks.) Find the ordinary character table of the group D_{16}, the dihedral group with order 16.

4: (20 marks.) Two idempotents e and f of a ring R are said to be **conjugate** provided $e = ufu^{-1}$ for some unit u of R. Let G be a finite group, and let χ_1, ..., χ_k be the irreducible $\mathbb{C}G$-characters. In terms of k and the degrees $\chi_i(1)$:

(a) How many primitive idempotents does $Z(\mathbb{C}G)$ have?

(b) How many idempotents does $Z(\mathbb{C}G)$ have?

(c) How many conjugacy classes of primitive idempotents does $\mathbb{C}G$ have?

(d) How many conjugacy classes of idempotents does $\mathbb{C}G$ have?

5: (20 marks.) Let d be a positive integer and let A_1, ..., A_m be mutually commuting $d \times d$ matrices over $\mathbb{C}$. That is, $A_i A_j = A_j A_i$ for all i and j. Suppose there exist positive integers n_1, ..., n_r such that $A_i^{n_i}$ is the identity matrix for all i. Using the representation theory of finite groups, without using any general theorems about commuting matrices, show that there exists an invertible $d \times d$ matrix P such that, for all i, the matrix PA_iP^{-1} is diagonal.

Solutions to Midterm

1: A theorem of Frobenius asserts that the only finite-dimensional division algebras over $\mathbb{R}$ are $\mathbb{R}$ and $\mathbb{C}$ and $\mathbb{H}$. So, by the semisimplicity of A, we have

$$A \cong \mathrm{Mat}_a(\mathbb{H}) \oplus \mathrm{Mat}_b(\mathbb{C}) \oplus \mathrm{Mat}_c(\mathbb{R})$$

for some $a, b, c \in \mathbb{N}$. Since $\dim_{\mathbb{R}}(A) = 5$ and A is non-commutative, the only possibilities for A are $A \cong \mathbb{H} \oplus \mathbb{R}$ or $A \cong \mathrm{Mat}_2(\mathbb{R}) \oplus \mathbb{R}$. Let M be an 8-dimensional A-module.

In the first case, A has exactly 2 simple modules up to isomorphism, say, S and T, of dimensions 4 and 1, respectively. Up to isomorphism, there are exactly 3 possibilities for M, namely $S \oplus S$ and $S \oplus 4T$ and $8T$.

In the second case, A again has exactly 2 simple modules, S and T, but now of dimensions 2 and 1, respectively. The possibilities for M are $sS \oplus tT$ where s and t are natural numbers satisfying $2s + t = 8$. Hence $0 \leq s \leq 4$, and there are exactly 5 possibilities.

Therefore, the answer is 3 or 5.

2: We write $Q_8 = \{1, i, j, k, -1, -i, -j, -k\}$ with the usual notation. The conjugacy classes of Q_8 are

$$[1] = \{1\}\ , \qquad [-1] = \{-1\}\ , \qquad [i] = \{i, -i\}\ , \qquad [j] = \{j, -j\}\ , \qquad [k] = \{k, -k\}\ .$$

The ordinary character table of Q_8 is as shown.

	1	1	2	2	2	$\lvert[g]\rvert$
	1	2	4	4	4	$\lvert\langle g\rangle\rvert$
	1	-1	i	j	k	g
χ_0^G	1	1	1	1	1	
χ_1^G	1	1	1	-1	-1	
χ_2^G	1	1	-1	1	-1	
χ_3^G	1	1	-1	-1	1	
χ_4^G	2	-2	0	0	0	

The first four irreducible charcaters are inflated from the quotient group $Q_8/Z(Q_8) \cong V_4$. Since Q_8 has exactly 5 conjugacy classes, there is only one more row. That last row is obtained using the column orthonormality between the first column and all the other columns.

3: First we shall construct the character table for D_8, then we shall use it to obtain the character table for D_{16}. Generally, $D_{4m} = \langle a, b : a^{2m} = b^2 = (ab)^2 = 1\rangle$ for any positive integer m. We claim that the character table for D_8 is as follows.

	1	1	2	2	2	$\lvert[g]\rvert$
	1	2	4	4	4	$\lvert\langle g\rangle\rvert$
	1	a^2	a	b	ab	g
ψ_0	1	1	1	1	1	
ψ_1	1	1	1	-1	-1	
ψ_2	1	1	-1	1	-1	
ψ_3	1	1	-1	-1	1	
ψ_4	2	-2	0	0	0	

An easy manipulation of the relations in the presentation of D_{4m}, in general, yields $ba = a^{-1}b$. So the conjugacy classes of D_{4m} are

$$[1] = \{1\}\ , \qquad [b] = \{a^{2r}b : r \in \mathbb{Z}\}\ , \qquad [ab] = \{a^{2r+1} : r \in \mathbb{Z}\}\ , \qquad [a^s] = \{a^s, a^{-s}\}$$

for $s \in \mathbb{Z}$. The above table for D_8 is now straightforward to obtain by the same steps as in Question 2. We now prove that the character table for D_{16} is as follows.

	1	4	4	2	2	2	1	$\|[g]\|$
	1	2	2	8	4	8	2	$\|\langle g\rangle\|$
	1	b	ab	a	a^2	a^3	a^4	g
χ_0	1	1	1	1	1	1	1	
χ_1	1	-1	-1	1	1	1	1	
χ_2	1	1	-1	-1	1	-1	1	
χ_3	1	-1	1	-1	1	-1	1	
χ_4	2	0	0	0	-2	0	2	
χ_5	2	0	0	$\sqrt{2}$	0	$-\sqrt{2}$	0	
χ_6	2	0	0	$-\sqrt{2}$	0	$\sqrt{2}$	0	

The irreducible characters χ_0 to χ_4 are inflated from the group $D_{16}/Z(D_{16}) \cong D_8$. The irreducible chartacters χ_5 and χ_6 are the characters of the complexifications of the real representations obtained by letting D_{16} act on a regular octogon with a acting as a rotation by $\pi/4$ or $3\pi/4$, respectively. Indeed, the trace of the action of a for those two representations is $2\cos(\pi/4) = \sqrt{2}$ and $2\cos(3\pi/4) = -\sqrt{2}$, respectively.

4: Part (a). There is a bijective correspondence $\chi \leftrightarrow e_\chi$ between the irreducible characters χ of G and the primitive idempotents e_χ of $Z(\mathbb{C}G)$. So the number of primitive idempotents of $Z(\mathbb{C}G)$ is k.

Part (b). The idempotents of $Z(\mathbb{C}G)$ are the elements of $Z(\mathbb{C}G)$ that have the form $\sum_\chi z_\chi e_\chi$ where each $z_\chi \in \{0, 1\}$. So the number of such idempotents is 2^k.

Part (c). Given any poistive integer n, then $\mathrm{Mat}_n(\mathbb{C})$ has exactly $n+1$ conjugacy classes of idempotents, indeed, two idempotents i and j of $\mathrm{Mat}_n(\mathbb{C})$ are conjugate if and only if i and j have the same rank. Moreover, i is primitive if and only if i has rank 1. Hence, $\mathrm{Mat}_n(\mathbb{C})$ has a unique conjugacy class of primitive idempotents. We have $\mathbb{C}G \cong \bigoplus_\chi \mathbb{C}Ge_\chi$ and $\mathbb{C}Ge_\chi \cong \mathrm{Mat}_{\chi(1)}(\mathbb{C})$. Every idempotent i of $\mathbb{C}G$ decomposes as a sum of mutually orthogonal idempotents $i = \sum_\chi ie_\chi$.

When i is primitive, we have $ie_\chi = i$ for some χ and all the other terms of the summation are zero. So the number of conjugacy classes of primitive idempotents of $\mathbb{C}G$ is k.

Part (d). By the first paragraph of the response to part (c), the number of conjugacy classes of idempotents of $\mathbb{C}G$ is the product $\prod_\chi(\chi(1) + 1)$.

5: Let $\mathcal{A}$ be the multiplicative abelian group generated by the A_i. Let $\rho : \mathbb{C}\mathcal{A} \to \mathrm{Mat}_n(\mathbb{C})$ be the $\mathbb{C}$-linear extension of the identity map on $\mathcal{A}$. Identifying $\mathrm{Mat}_n(\mathbb{C}) = \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)$ in the usual way, we regard $\mathbb{C}^n$ as a $\mathbb{C}\mathcal{A}$-module with representation ρ. By Maschke's Theorem, $\mathbb{C}^n = V_1 \oplus ... \oplus V_n$ as a direct sum of 1-dimensional $\mathbb{C}\mathcal{A}$-modules. Let v_i be a nonzero element of V_i. With respect to the basis $\mathcal{V} = \{v_1, ..., v_n\}$, the matrix representing the action of any element of $\mathbb{C}\mathcal{A}$ is diagonal. Therefore, letting P be the transformation matrix from coordinates with respect to the standard basis of $\mathbb{C}^n$ to coordinates with respect to $\mathcal{V}$, then PAP^{-1} is a diagonal matrix for all $A \in \mathcal{A}$.

Midterm Makeup

12 December 2023, LJB

The duration of the exam is 120 minutes. It is a closed book exam.

1: (20 marks.) Find the set of all positive integers n such that n is the number of isomorphism classes of simple modules for some 9-dimensional semisimple algebra over $\mathbb{R}$.

2: (20 marks.) Find the ordinary character table of the group $C_2 \times S_3$, where C_2 denotes the cyclic group with order 2 and S_3 denotes the symmetric group with order 6.

3: (20 marks.) Find the ordinary character table of the group D_{12}, the dihedral group with order 12.

4: (20 marks.) Let G be a finite group and χ the $\mathbb{C}G$-character of a $\mathbb{C}G$-submodule M of the regular $\mathbb{C}G$-module ${}_{\mathbb{C}G}\mathbb{C}G$. Let $A = \mathrm{End}_{\mathbb{C}G}(M)$. (Thus, A is the algebra of $\mathbb{C}$-linear maps $M \to M$ that commute with the action of G.) Describe, in terms of χ and the irreducible $\mathbb{C}G$-characters:

(a) The number of isomorphism classes of simple $Z(\mathbb{C}G)$-submodules of M.

(b) The dimensions of the simple $Z(\mathbb{C}G)$-submodules of M.

(c) The number of isomorphism classes of simple A-submodules of M.

(d) The dimensions of the simple A-submodules of M.

5: (20 marks.) Find all the simple submodules of the regular $\mathbb{R}C_5$-module ${}_{\mathbb{R}C_5}\mathbb{R}C_5$.

MATH 325: Representation Theory

Final

5 January 2024, LJB

The duration of the exam is 120 minutes. It is a closed book exam.

Please write your name on every sheet of paper that you submit.

1: (20 marks.) Let G be a finite group and H a subgroup of G such that $H \neq G$. Show that the permutation $\mathbb{C}G$-module $\mathbb{C}G/H$ is not simple.

2: (30 marks.) **(a)** Construct the ordinary character table of the dihedral group D_{18} with order 18.

(b) Show that, for every irreducible $\mathbb{C}D_{18}$-character χ, there exists a subgroup $C \leq D_{18}$ and an irreducible $\mathbb{C}C$-character ψ such that $\chi = \mathrm{ind}_C^{D_{18}}(\psi)$.

3: (30 marks.) Let H be the non-abelian group generated by elements u, v, w such that:

- the elements u and v and w all have order 3,
- the element w is in the centre of H,
- we have $vu = uvw$.

(a) Show that, $Z(H) = \{1, w, w^2\}$ and, for every $h \in H - Z(H)$, the conjugacy class of h is $[h] = \{h, hw, hw^2\}$.

(b) Construct the ordinary character table of H.

4: (20 marks.) Let $G = S_6$, the symmetric group with degree 6. Up to isomorphism, how many algebras are there that have the form $Z(\mathrm{End}_{\mathbb{C}G}(M))$ where M is a non-zero finite-dimensional $\mathbb{C}G$-module?

Solutions to Final

1: The 1-dimensional subspace V of $\mathbb{C}G/H$ spanned by the sum of the elements of G/H is a $\mathbb{C}G$-submodule of $\mathbb{C}G/H$. Yet the dimension of $\mathbb{C}G/H$ is $|G : H|$, which is greater than 1 because $H \neq G$. So V is a proper submodule of $\mathbb{C}G/H$.

Sketch of alternative: Using the formula for the inner product on the character algebra, it can be shown that the trivial $\mathbb{C}G$-character has multiplicity 1 in the $\mathbb{C}G$-character of $\mathbb{C}G/H$.

2: Write $D_{18} = \langle a, b : a^9 = b^2 = (ab)^2 = 1 \rangle$.

Part (a). The character table is as follows, where $\xi_m = e^{2\pi i m/9} + e^{-2\pi i m/9} = 2\cos(2\pi m/9)$.

	1	9	2	2	2	2	$\|[g]\|$
	1	2	9	9	9	9	$\|\langle g\rangle\|$
	1	b	a	a^2	a^3	a^4	g
χ_0	1	1	1	1	1	1	
χ_1	1	-1	1	1	1	1	
χ_2	2	0	ξ_1	ξ_2	ξ_3	ξ_4	
χ_3	2	0	ξ_2	ξ_4	ξ_3	ξ_1	
χ_4	2	0	ξ_4	ξ_1	ξ_3	ξ_2	
χ_5	2	0	-1	-1	2	-1	

The irreducible characters χ_0 and χ_1 are inflated from the quotient group $D_{18}/\langle a \rangle \cong C_2$. Since $\langle a \rangle$ is the derived subgroup of D_{18}, there are no other irreducible $\mathbb{C}D_{18}$-characters of degree 1. The other 4 irreducible characters of D_{18} are induced from the non-trivial irreducible characters of the subgroup $\langle a \rangle$. They must be irreducible, because their inner products with the two degree 1 irreducibles are 0.

Part (b). For χ_0 and χ_1, we can put CD_{18}. For the other 4 irredicible $\mathbb{C}D_{18}$-characters, the above construction shows that we can put $C = \langle a \rangle$.

3: Part (a). Plainly $\{1, w, w^2\} \leq Z(H)$. Since H is a non-abelian 3 group with order 3^3, we must have $|Z(H)| = 3$. Since $H/Z(H) \cong C_3 \times C_3$, all the non-singleton conjugacy classes must be of order 3 and must be contained in a coset of $Z(H)$. But those cosets have order 3, so the non-singleton conjugacy classes must coincide with the non-trivial cosets.

Part (b). The character table is as shown, where $\omega = e^{2\pi i/3}$.

	1	1	1	3	3	3	3	3	3	3	3	$\|[g]\|$
	1	3	3	3	3	3	3	3	3	3	3	$\|\langle g\rangle\|$
	1	w	w^2	v	v^2	u	uv	uv^2	u^2	u^2v	u^2v^2	$\|\langle g\rangle\|$
χ_0	1	1	1	1	1	1	1	1	1	1	1	
χ_1	1	1	1	ω	ω^2	1	ω	ω^2	1	ω	ω^2	
χ_2	1	1	1	ω^2	ω	1	ω^2	ω	1	ω^2	ω	
χ_3	1	1	1	1	1	ω	ω	ω	ω^2	ω^2	ω^2	
χ_4	1	1	1	ω	ω^2	ω	ω^2	1	ω^2	1	ω	
χ_5	1	1	1	ω^2	ω	ω	1	ω^2	ω^2	ω	1	
χ_6	1	1	1	1	1	ω^2	ω^2	ω^2	ω	ω	ω	
χ_7	1	1	1	ω	ω^2	ω^2	1	ω	ω	ω^2	1	
χ_8	1	1	1	ω^2	ω	ω^2	ω	1	ω	1	ω^2	
χ_9	3	3ω	$3\omega^2$	0	0	0	0	0	0	0	0	
χ_{10}	3	$3\omega^2$	3ω	0	0	0	0	0	0	0	0	

The 9 irreducibles with degree 1 are inflated from $H/Z(H)$. The characters χ_9 and χ_{10} are induced from the two irreducible characters of $\langle v, w \rangle$ upon which v acts trivially and w acts non-trivially.

4: There are exactly 11 simple $\mathbb{C}G$-modules up to isomorphism, because that is the number of conjugacy classes of G, indeed, the 11 partitions of 6 are:

$$1+1+1+1+1+1,\ 2+1+1+1+1,\ 2+2+1+1,\ 2+2+2,$$

$$3+1+1+1,\ 3+2+1,\ 3+3,\ 4+1+1,\ 4+2,\ 5+1,\ 6\ .$$

The algebra $Z(\mathrm{End}_{\mathbb{C}G}(M))$ is the direct sum of n copies of $\mathbb{C}$, where n is the number of isomorphism classes of simple $\mathbb{C}G$-modules occuring in M. The possible values of n are the positive integers less than or equal to 11. Thus, there are 11 possible values for n. So the number of possible isomorphism classes for $Z(\mathrm{End}_{\mathbb{C}G}(M))$ is 11.