# Archive for
# MATH 227, Introduction to Linear Algebra, Spring 2022

Bilkent University, Laurence Barker, 9 June 2022.

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# Course specification

## MATH 227, *Introduction to Linear Algebra*, Spring 2022

Laurence Barker, Bilkent University. Version: 26 April 2022.

**Course Aims:** To aquire practical knowledge and skill in some important techniques of linear algebra, and to aquire theoretical understanding and ability to critically assess methods and propositions in the area.

**Course Description:** This is an introductory course with an emphasis on methods of calculation, but the theoretical grounding is self-contained and complete. To obtain a satsfactory grade, it will be enough to solve most of the routine problems. To obtain a high grade, it will also be necessary to solve some of the difficult problems and to give clear explanations.

**Course Requirements:** An official prerequisite in the University Catalogue is MATH 106 but, in practice, no knowledge of calculus is needed.

**Instructor:** Laurence Barker, Office SAZ 129, barker at fen dot bilkent dot edu dot tr.

**Assistant:** Serkan Sonel, serkansonel at gmail dot com.

**Textbook:** Howard Anton, Chris Rorres, "Elementary Linear Algebra with Supplemental Applications", 11th edition, Wiley, 2011, 2015. ISBN: 978-1-118-67745-2. (There is an "International Student Edition" of the book, less expensive, printed in greyscale instead of colour.)

*Note:* Those two recommended editions of the book are not the same as the book with the same authors called "Elementary Linear Algebra: Applications Version". (But that book and various other introductory books on linear algebra are suitable too. All homeworks will be specified completely in the Homework file on my homepage.)

**Supplementary material on my homepage:**

- Homeworks, with solutions: homework227spr22.pdf .
- Quizzes, with solutions: quiz227spr22.pdf .
- Course notes on selected topics: PDF files with names of the form: LinAlgPracNotes... .

**Classes:** Tuesdays 13:30 - 15:20, Fridays, 09:30 - 10:20, room V-02.

**Office Hours:** Tuesdays 11:30 - 12:20, Wednesdays 08:40 - 09:30, in my office, room SA-129.

Office hours is for *all* the students on the course, not just the proficient. If you are having difficulty with the course, then it is best to come to see me for advice.

## Weekly Syllabus

The format below is, *Week number; Monday date; Subtopics and textbook section numbers.* The numbering $m.n$ indicates Chapter $m$ Section $n$ in the Anton–Rorres textbook.

**1: Mon 30 Jan - Fri 4 Feb:** Systems of linear equations, 1.1. Sketch of Markov chain scenario and the method of diagonalization.

**2: Mon 7 Feb - Fri 11 Feb:** Gaussian and Gauss–Jordan elimination, 1.2. Matrics, 1.3.

**3: Mon 14 Feb - Fri 18 Feb:** Elementary matrices, matrix inversion of by row operations 1.4, 1.5. Existence and uniqueness of solutions, 1.6. Determinants, 2.1.

**4: Mon 21 Feb - Fri 25 Feb:** Determinants, their algebraic properties, their evaluation by cofactor expansion, 2.1.

**5: Mon 28 Feb - Fri 4 Mar:** Evaluation of determinants by row reduction, 2.2, 2.3. Introduction to Markov chains 4.12.

**6: Mon 7 Mar - Wed 9 Mar:** Euclidian spaces, 3.1.

**7: Mon 14 Mar - Fri 18 Mar:** Norm, dot product, distance in $\mathbb{R}^n$, orthogonality 3.2, 3.3.

**8: Mon 21 Mar - Fri 25 Mar:** Real vector spaces, subspaces, 4.1, 4.2.

**9: Mon 28 Mar - Fri 1 Apr:** Linear independence, spanning, bases, dimension, 4.3, 4.4, 4.5.

**10: Mon 4 Apr - Fri 8 Apr:** Linear transformations, inverses, composition, 8.1, 8.2, 8.3.

**11: Mon 11 Apr - Fri 15 Apr:** Change of basis, 4.6. Row and column spaces, rank-nullity formula, 4.7, 4.8.

**12: Mon 18 Apr - Fri 22 Apr:** Eigenvalues and eigenvectors, 5.1. Diagonalization, 5.2. Complex vector spaces, 5.3.

**13: Mon 25 Apr - Fri 29 Apr:** Applications to Markov chains 10.5. Inner product spaces, 6.1, 6.2.

**14: Thur 5 May - Fri 6 May:** Gram–Schmidt orthogonalization, 6.3.

**15: Mon 9 May - Fri 13 May:** (If time permits, further material on dynamical systems and Markov chains 10.5.) Review for Final.

**Assessment:** The method of assessment is by curve. Grades F and FZ are to be for candidates judged to be incompetent in the routine course material.

- Quizzes 10% (marks for worst two quizzes to be dropped).
- Midterm I, 30%, face-to-face exam, Tuesday, 22 March, 17:30 - 20:00, rooms SA-Z04, SB-Z03, SB-Z04.
- Midterm II, 30%, face-to-face exam, Tuesday, 26 April, 17:30 - 20:00, room V-02.
- Final, 30%, face-to-face exam, Tuesday, 24 May.

FZ criteria: less than 30% in sum of Midterm 1 and Midterm 2 marks. (That is, less than 60 out of the available $100 + 100 = 200$ available marks; each exam is marked out of 100.)

**Masks:** Masks must be worn at all times in class.

**Speaking:** All speaking in class must address everyone in the room. Questions and comments are welcome: they make learning easier, and they make teaching easier too.

**Class Announcements:** All students, including any absentees from a class, will be deemed responsible for awareness of class announcements.

# Homework

MATH 227, *Introduction to Linear Algebra*, Spring 2022, Laurence Barker

version: 19 May 2022

**Style advice:** Your solutions should be presented so as to make the reasoning easily undertandable to others in the class. It will be marked according to how well it would communicate to other students taking the course.

## Homework 1

*Solutions to be discussed in class on or after Tuesday 8 March.*

Some similar questions, and solutions, can be found in the following files on my homepage:
- arch227fall21, Page 5, Homework 1; also page 27, Midterm 1, Questions 1, 2, 3.
- arch227spr19.pdf, pages 7 and 8, Midterm 1, Questions 1 and 2,
- arch227fall18.pdf, pages 10 and 11, Midterm 1, Questions 1 and 2,
- arch227spr17.pdf, pages 16 and 17, Midterm 1, Questions 1 and 2.

**Question 1.1:** Using Gaussian elimination, find all the solutions to

$$2x + y + z = 7\,, \qquad 4x + 3y + z = 13\,, \qquad 6x - y - 2z = 7\,.$$

**Question 1.2:** Using Gauss–Jordan elimination, find all the solutions to

$$2x + y + z = 7\,, \qquad 4x + 3y + z = 13\,, \qquad 8x + 5y + 3z = 27\,.$$

**Question 1.3:** Find the inverse of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 1 \\ 6 & -1 & -2 \end{bmatrix}$,

**(a)** by Gauss–Jordan elimination,

**(b)** by the method of minors and cofactors,

**(c)** and then use that answer to check your answer to Question 1.

**Question 1.4:** Let $A$ be a $4 \times 4$ matrix with $\det(A) = 5$. Let $P$ be an invertible $4 \times 4$ matrix. Evaluate $\det(PAP^{-1})$.

# Solutions 1

**1.1:** The augmented matrix is $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 7 \\ 4 & 3 & 1 & 13 \\ 6 & -1 & -2 & 7 \end{array}\right]$.

Subtracting multiples of row 1 from rows 2 and 3 gives $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 7 \\ 0 & 1 & -1 & -1 \\ 0 & -4 & -5 & -14 \end{array}\right]$.

Adding 4 times row 2 to row 3 gives $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 7 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -9 & -18 \end{array}\right]$.

Multiplying rows 1 and 3 by factors, $\left[\begin{array}{ccc|c} 1 & 1/2 & 1/2 & 7/2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$.

The equations are $z = 2$ and $y = z - 1 = 1$ and $x = 7/2 - y/2 - z/2 = 7/2 - 1/2 - 1 = 2$. In conclusion, $(x, y, z) = (3, 4, 5)$.

**1.2:** We code the problem as $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 7 \\ 4 & 3 & 1 & 13 \\ 8 & 5 & 3 & 27 \end{array}\right]$.

Subtracting multiples of row 1 from rows 2 and 3 gives $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 7 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & -1 \end{array}\right]$.

Subtracting row 2 from row 3 gives $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 7 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

Introducing a parameter, we put $z = r$. Then $y = -1 + z = r - 1$ and $x = 4 - z = 4 - r$. In conclusion, $(x, y, z) = (r, r - 1, 4 - r)$ where $r$ is a parameter.

**1.3:** Part (a). We set up the problem as $\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & 1 & 0 & 1 & 0 \\ 6 & -1 & -2 & 0 & 0 & 1 \end{array}\right]$.

Subtracting multiples of row 1 from the other two rows, $\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & -4 & -5 & -3 & 0 & 1 \end{array}\right]$.

Adding 4 times row 2 to row 3 gives $\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -9 & -11 & 4 & 1 \end{array}\right]$.

Dividing row 3 by $-9$ gives $\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 11/9 & -4/9 & -1/9 \end{array}\right]$.

Adding multiples of row 3 to the other two rows, $\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & -2/9 & 4/9 & 1/9 \\ 0 & 1 & 0 & -7/9 & 5/9 & -1/9 \\ 0 & 0 & 1 & 11/9 & -4/9 & -1/9 \end{array}\right]$.

Subtracting row 2 from row 1 yields $\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 5/9 & -1/9 & 2/9 \\ 0 & 1 & 0 & -7/9 & 5/9 & -1/9 \\ 0 & 0 & 1 & 11/9 & -4/9 & -1/9 \end{array}\right]$.

Finally, dividing row 1 by 2 gives $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5/18 & -1/18 & 2/18 \\ 0 & 1 & 0 & -7/9 & 5/9 & -1/9 \\ 0 & 0 & 1 & 11/9 & -4/9 & -1/9 \end{array}\right]$.

So $A^{-1} = \dfrac{1}{18}\begin{bmatrix} 5 & -1 & 2 \\ -14 & 10 & -2 \\ 22 & -8 & -2 \end{bmatrix}$.

Part (b). The matrix of minors is $\begin{bmatrix} -6+1 & -8-6 & -4-18 \\ -2+1 & -4+6 & -2-6 \\ 1-3 & 2-4 & 6-4 \end{bmatrix} = \begin{bmatrix} -5 & -14 & -22 \\ -1 & 2 & -8 \\ -2 & -2 & 2 \end{bmatrix}$.

The matrix of cofactors is $\begin{bmatrix} -5 & 14 & -22 \\ 1 & -2 & 8 \\ -2 & 2 & 2 \end{bmatrix}$.

The adjoint matrix is $\operatorname{adj}(A) = \begin{bmatrix} -5 & 14 & -22 \\ 1 & -2 & 8 \\ -2 & 2 & 2 \end{bmatrix}$. We have $\det(A) = 2(-5) + 14 - 22 = -18$ from the (1, 1) entry of the equality $A \operatorname{adj}(A) = \det(A)I$. Therefore, $A = \dfrac{1}{18}\begin{bmatrix} 5 & -1 & 2 \\ -14 & 10 & -2 \\ 22 & -8 & -2 \end{bmatrix}$.

Part (c). We have $A^{-1}\begin{bmatrix} 7 \\ 13 \\ 7 \end{bmatrix} = \dfrac{1}{18}\begin{bmatrix} 35-13+14 \\ -98+130-14 \\ 154-104-14 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

**1.4:** We have $\det(PAP^{-1}) = \det(P)\det(A)\det(P^{-1}) = \det(P)\det(A)\det(P)^{-1} = \det(A) = 5$.

# Homework 2

*Solutions to appear in this file on Tuesday 22 March.*

**Question 2.1:** Let $A = \begin{bmatrix} 1 & 2 & 1 & 11 \\ 3 & 4 & 1 & 25 \\ 5 & 6 & 1 & 39 \end{bmatrix}$. Find a basis for the null space $\mathrm{Null}(A)$.

**Question 2.2:** For $A$ as in the previous question, find a basis for the image $\mathrm{Im}(A)$.

**Question 2.3:** For a positive integer $n$, we write $P_n$ to denote the vector space of polynomial functions $\mathbb{R} \to \mathbb{R}$ that have degree at most $n$. These are the functions $f : \mathbb{R} \to \mathbb{R}$ that have the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$$

where $a_0$, $a_1$, ..., $a_n$ are real numbers. Let $V$ be the subspace of $P_5$ consisting of those functions $f \in P_5$ such that $f(x) = -f(x)$. Find a basis for $V$ and hence evaluate $\dim(V)$.

**Question 2.4:** Let $n$ be a positive integer and let $A$ be an $n \times n$ matrix such that exactly one of the entries of $A$ is non-zero. Find the rank and nullity of $A$.

# Solutions 2

**2.1:** We must solve $\left[\begin{array}{cccc|c} 1 & 2 & 1 & 11 & 0 \\ 3 & 4 & 1 & 25 & 0 \\ 5 & 6 & 1 & 39 & 0 \end{array}\right]$.

Subtracting multiples of row 1 from the other rows, $\left[\begin{array}{cccc|c} 1 & 2 & 1 & 11 & 0 \\ 0 & -2 & -2 & -8 & 0 \\ 0 & -4 & -4 & -16 & 0 \end{array}\right]$.

Dividing row 2 by $-2$ gives $\left[\begin{array}{cccc|c} 1 & 2 & 1 & 11 & 0 \\ 0 & 1 & 1 & 4 & 0 \\ 0 & -4 & -4 & -16 & 0 \end{array}\right]$.

Adding 4 times row 2 to row 3 gives $\left[\begin{array}{cccc|c} 1 & 2 & 1 & 11 & 0 \\ 0 & 1 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

Subtracting 2 times row 2 from row 3 gives $\left[\begin{array}{cccc|c} 1 & 0 & -1 & 3 & 0 \\ 0 & 1 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

So Null($A$) is the set of vectors $(x_1, x_2, x_3, x_4)$ such that $x_1 = x_3 - 3x_4$ and $x_2 = -x_3 - 4x_4$. Putting $x_3 = r$ and $x_4 = s$, then

$$(x_1, x_2, x_3, x_4) = (r - 3s, -r - 4s, r, s) = r(1, -1, 1, 0), (-3, -4, 0, 1)\} \ .$$

Therefore, $\{(1, -1, 1, 0), (-3, -4, 0, 1)\}$ is a basis for Null($A$).

**2.2:** Subtracting the first column of $A$ from the other columns, we obtain $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & -2 & -2 & -8 \\ 5 & -4 & -4 & -16 \end{bmatrix}$.

Dividing column 2 by $-2$ yields $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & -2 & -8 \\ 5 & 2 & -4 & -16 \end{bmatrix}$.

Subtracting multiples of row 2 from rows 3 and 4 yields $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 5 & 2 & 0 & 0 \end{bmatrix}$.

So a basis for Im($A$) is $\{(1, 3, 5), (0, 1, 2)\}$.

**2.3:** Let $\{e_0, ..., e_5\}$ be the basis for $P^5$ such that $e_i(x) = x_i$. Thus, given $f \in P_5$ and writing $f(x) = a_0 + a_1 x + ..., a_5 x^5$, then $f = a_0 e_0 + a_1 e_1 + ...a_n e_n$. We have $f(x) = f(-x)$ if and only if $a_1 = a_3 = a_5 = 0$. So $\{e_0, e_2, e_4\}$ is a basis for $V$ and $\dim(V) = 3$.

**2.4:** Let $\{e_1, ..., e_n\}$ be the standard basis for $\mathbb{R}^n$. Supposing only the $(i, j)$ entry of $A$ is nonzero, then The image of $A$ is span$\{e_i\}$, which is 1-dimensional. So rank($A$) = 1. By the rank-nullity formula, null($A$) = $n - 1$.

*Comment:* Alternatively, Null($A$) has a basis consisting of those $e_k$ such that $k \neq j$, hence null($A$) = $n - 1$.

# Homework 3

*Solutions to appear in this file on Tuesday 19 April.*

**Question 3.1:** Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 5 \\ 2 & 5 & 7 & 9 \end{bmatrix}$. Find a basis for the image space $\mathrm{Im}(A)$.

**Question 3.2:** Let $\{e_1, e_2, e_3\}$ and $\{f_1, f_2, f_3\}$ be bases for a real vector space $V$. Suppose that

$$f_1 = e_1 + e_2 + e_3\ , \qquad f_2 = e_1 + 2e_2 + 2e_3\ , \qquad f_3 = e_1 + 3e_2 + 4e_3\ .$$

Let $v \in V$, and write $v = x_1e_1 + x_2e_2 + x_3e_3 = y_1f_1 + y_2f_2 + y_3f_3$.

**(a)** Express $e_1$, $e_2$, $e_3$ in terms of $f_1$, $f_2$, $f_3$.

**(b)** Express $(x_1, x_2, x_3)$ in terms of $(y_1, y_2, y_3)$.

**(c)** Express $(y_1, y_2, y_3)$ in terms of $(x_1, x_2, x_3)$.

**Question 3.3:** Express $A = \begin{bmatrix} -1 & 6 & 30 \\ -1 & 4 & 15 \\ 0 & 0 & 1 \end{bmatrix}$ in the form $A = PDP^{-1}$ with $D$ diagonal.

(Hint: an example with more complicated calculations is in the file Linear Algebra Practical Notes Part 4, Eigenvalues and Eigenvectors, on my homepage.)

**Question 3.4:** Express $R = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ in the form $R = PDP^{-1}$ with $D$ diagonal.

(Warning: this is harder than it looks. The eigenvalues are not real numbers.)

**Question 3.5:** Show that the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable.

(Note: this is too hard for an exam question.)

# Solutions 3

**3.1:** Subtracting multiples of column 1 from the other columns yields $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \end{bmatrix}$.

Subtracting column 2 from columns 3 and 4 yields $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}$.

Column operations do not change the image, so a basis for $\mathrm{Im}(A)$ is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

**3.2:** The transformation matrix to $x$-coordinates from $y$-coordinates is $T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}$.

To find $T^{-1}$ we set up the problem as $\left[ \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array} \right]$.

Subtracting row 1 from the other two rows, we obtain $\left[ \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 1 & 3 & -1 & 0 & 1 \end{array} \right]$.

Subtracting row 2 from row 3 gives $\left[ \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array} \right]$.

Subtracting multiples of row 3 from the other two rows, $\left[ \begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & -1 & 3 & -2 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array} \right]$.

Subtracting row 2 from row 3 gives $\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -2 & 1 \\ 0 & 1 & 0 & -1 & 3 & -2 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array} \right]$. So $T^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 3 & -2 \\ 0 & -1 & 1 \end{bmatrix}$.

Part (a). We have $e_1 = f_1 - f_2$ and $e_2 = -2f_1 + 3f_2 - f_3$ and $e_3 = f_1 - 2f_2 + f_3$.

Part (b). We have $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = T \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$.

Part (c). We have $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = T^{-1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 3 & -2 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

**3.3:** The characteristic equation of $A$ is

$$0 = \det(\lambda I - A) = \begin{vmatrix} \lambda+1 & -6 & -30 \\ 1 & \lambda-4 & -15 \\ 0 & 0 & \lambda-1 \end{vmatrix} = (\lambda-1)\big((\lambda+1)(\lambda-4)+6\big)$$

$$= (\lambda-1)(\lambda^2 - 3\lambda + 2) = (\lambda-1)^2(\lambda-2)\ .$$

So the eigenvalues are $\lambda_1$, $\lambda_2$, $\lambda_3$ where $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 2$. Let $e_1$, $e_2$, $e_3$ be eigenvectors such that $Ae_i = \lambda_i A_i$.

Write $e_1 = (x, y, z)$. The equality $Ae_1 = \lambda_1 e_1$ becomes

$$-x + 6y + 30z = x \; , \qquad -x + 4y + 15z = y \; , \qquad z = z \; .$$

That is, $-x + 3y + 15z = 0$. Put $y = 1$ and $z = 0$. Then $x = 3$. Thus, $e_1 = (3, 1, 0)$.

Now write $e_2 = (x, y, z)$. Then $x$, $y$, $z$ satisfy the same equations as before. We must find a solution such that $\{e_1, e_2\}$ is linearly independent. Put $y = 0$ and $z = 1$. Then $x = 15$. Thus, $e_2 = (15, 0, 1)$.

Finally, write $e_3 = (x, y, z)$. The equality $Ae_3 = \lambda_1 e_3$ becomes

$$-x + 6y + 30z = 2x \; , \qquad -x + 4y + 15z = 2y \; , \qquad z = 2z \; .$$

hence, $z = 0$ and $x = 2y$. Putting $y = 1$, then $x = 2$. Thus $e_2 = (2, 1, 0)$.

We have $A = PDP^{-1}$ where $P = \begin{bmatrix} 3 & 15 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

*Comment:* To check the asserted diagonalization, observe that $P^{-1} \begin{bmatrix} 1 & -2 & -15 \\ 0 & 0 & 1 \\ -1 & 3 & 15 \end{bmatrix}$.

**3.4:** The characteristic equation of $R$ is

$$0 = \det(\lambda I - R) = \begin{vmatrix} \lambda - 2 & 1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^2 + 1 = \lambda^2 - 4\lambda + 5 \; .$$

The solutions to that quadratic equation are $\lambda = (4 \pm \sqrt{16 - 20})/2 = 2 \pm i$. So the eigenvalues of $R$ are $\lambda_1 = 2 + i$ and $\lambda_2 = 2 - i$.

Let $e_1$ and $e_2$ be eigenvectors such that $Re_j = \lambda_j e_j$ for $j \in \{1, 2\}$. Write $e_1 = (x, y)$. Then $2x - y = (2 + i)x$ and $x + 2y = (2 + i)y$. In other words, $x = iy$. Putting $y = 1$, then $x = i$. Thus, $e_1 = (i, 1)$. Replacing $i$ with $-i$ in the above calculations, we see that we can put $e_2 = (-i, 1)$. Thus, we can put

$$P = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \; , \qquad D = \begin{bmatrix} 2 + i & 0 \\ 0 & 2 - i \end{bmatrix} \; .$$

*Comment 1:* To check the diagonalization, first note that $P^{-1} = \dfrac{1}{2} \begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix}$.

*Comment 2:* For any real numbers $a$ and $b$, the matrix

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

is a multiple of a matrix having the form

$$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \; .$$

The matrix $R_\theta$ represents a rotation of the plane through an angle of $\theta$. If $b \neq 0$, then $\theta$ is not an integer multiple of half a full revolution. In that case, a calculation much as above shows that there are two distinct eigenvalues and every eigenvector is a non-zero multiple of either $(i, 1)$ or else $(-i, 1)$; in particular, the eigenvectors do not depend on $\theta$, in other words, the eigenvectors do not depend on $a$ and $b$.

**3.5:** The characteristic equation of $A$ is

$$\begin{vmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 \, .$$

So the only eigenvalue of $A$ is 1. Let $e$ be an eigenvector and write $e = (x, y)$. By considering the first coordinate of the equality $Ae = e$, we have $x + y = x$, in other words, $y = 0$. Therefore, every eigenvector of $A$ has the form $(x, 0)$ with $x \neq 0$. Evidently, $A$ does not have a basis consisting of eigenvectors. So $A$ is not diagonalizable.

# Quizzes, with solutions

MATH 227, *Introduction to Linear Algebra*, Spring 2022, Laurence Barker

version: 19 May 2022

For some past quizzes and solutions, see the following archives on my homepage:

- pages 20 to 26 of arch227fall21.pdf .
- pages 5 and 6 of arch227spr19.pdf .
- pages 7, 8, 9 of arch227fall18.pdf .

**Quiz 1:** Using Gaussian elimination or Gauss–Jordan elimination, solve:

$$x + 2y + 3z = 11 \ , \qquad y + 4z = 10 \ , \qquad 2y - z = 4 \ .$$

*Solution:* Coding the problem as $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 11 \\ 0 & 1 & 4 & 10 \\ 0 & 2 & -1 & 4 \end{array}\right]$.

and subtracting 2 times row 2 from row 3, we obtain $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 11 \\ 0 & 1 & 4 & 10 \\ 0 & 0 & -9 & -16 \end{array}\right]$.

Dividing the last row by $-9$ yields $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 11 \\ 0 & 1 & 4 & 10 \\ 0 & 0 & 1 & 16/9 \end{array}\right]$.

So $z = 16/9$. Hence, $y = 10 - 4z = (90 - 64)/9 = 26/9$ and $x = 11 - 2y - 3z = (99 - 52 - 48)/9 = -1/9$. In conclusion, $(x, y, z) = (-1/9, 26/9, 16/9)$.

*Comment:* You do not get full marks if you neglect to simplify in a satisfactory way, for instance, if you just express the value of $x$ as $11 - 100/9$. Leaving work for the reader, in that way, is failing to complete the treatment of the problem.

**Quiz 2:** Find $\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}^{-1}$.

*Solution:* The inverse is $\dfrac{1}{2} \begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix}$ .

*Comment:* We used the formula $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, which holds when $ad - bc \neq 0$.

**Quiz 3:** Let $A = \begin{bmatrix} 1 & 5 & 4 \\ 2 & 4 & 5 \\ 4 & 5 & 3 \end{bmatrix}$. Evaluate $\det(A)$.

*Solution:* Subtracting multiples of row 1 from the other rows, $\det(A) = \begin{vmatrix} 1 & 5 & 4 \\ 0 & -6 & -3 \\ 0 & -15 & -13 \end{vmatrix}$.

Dividing row 2 by $-6$ yields $\det(A) = -6 \begin{vmatrix} 1 & 5 & 4 \\ 0 & 1 & 1/2 \\ 0 & -15 & -13 \end{vmatrix}$.

Adding 15 times row 2 to row 3 gives $\det(A) = -6 \begin{vmatrix} 1 & 5 & 4 \\ 0 & 1 & 1/2 \\ 0 & 0 & -11/2 \end{vmatrix}$.

So $\det(A) = (-6)(-11/2) = 33$.

**Quiz 4:** Do we have $\text{span}\{(1,2,4), 2,4,1), (4,1,2)\} = \mathbb{R}^3$?

*Solution 1:* We have

$$\begin{vmatrix} 1 & 2 & 4 \\ 2 & 4 & 1 \\ 4 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix} - 2 \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} + 4 \begin{vmatrix} 2 & 4 \\ 4 & 1 \end{vmatrix} = 7 - 2.0 - 4.14 = 7 - 56 - -49 \neq 0\ .$$

So the matrix $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 1 \\ 4 & 1 & 2 \end{bmatrix}$ is invertible and the answer is: yes.

*Solution 2:* The answer is yes if and only if the equation $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 1 \\ 4 & 1 & 2 \end{bmatrix} \underline{x} = \underline{0}$ has nonzero solutions.

We code the problem as $\left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 2 & 4 & 1 & 0 \\ 4 & 1 & 2 & 0 \end{array}\right]$.

Subtracting multiples of the top row from the other two rows, $\left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 0 & -9 & 0 \\ 0 & -7 & -14 & 0 \end{array}\right]$.

Plainly, there are no non-zero solutions, so the answer is: yes.

**Quiz 5:** Does $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$ span $\mathbb{R}^3$?

*Solution:* We have $\begin{vmatrix} 1 & 1 & 2 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 1(1+1) - 1(-1-1) + 2(1-1) = 4 \neq 0.$

So the specified set is linearly independent and the answer is: yes.

**Quiz 6:** Find a basis for $\text{Null}(A)$ where $A = \begin{bmatrix} 3 & 6 & -3 \\ 1 & 2 & -1 \\ 3 & 6 & -3 \end{bmatrix}$.

*Solution:* The null space orf $A$ is the set of vectors $\underline{x} = (x, y, z)$ such that $A\underline{x} = \underline{0}$. Subtracting multiples of the first row from the other two rows, then dividing the first row by 3, the augmented matrix becomes

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

Thus, $x + 2y - z = 0$. Introducing parameters $y = r$ and $z = s$, then $x = -2r + s$. Hence

$$(x, y, z) = (-2r + s, r, s) = r(-2, 1, 0) + s(1, 0, 1) .$$

So a basis for Null$(A)$ is $\{(-2, 1, 0), (1, 0, 1)\}$.

**Quiz 7:** Let $f_1 = (2, 5)$ and $f_2 = (1, -3)$. Suppose $(x_1, x_2) = y_1 f_1 + y_2 f_2$. Express $x_1$ and $x_2$ in terms of $y_1$ and $y_2$. (In other words, writing $e_1 = (1, 0)$ and $e_2 = (0, 1)$, suppose $x_1 e_1 + x_2 e_2 = y_1 f_1 + y_2 f_2$. Express $(x_1, x_2)$ in terms of $(y_1, y_2)$.)

*Solution:* The transformation matrix to $x$-coordinates from $y$-coordinates is $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$.
So $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

**Quiz 8:** Find the eigenvalues of $\begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$.

*Solution:* The characteristic equation is $0 = \begin{vmatrix} 2-\lambda & 3 \\ 0 & 4-\lambda \end{vmatrix} = (2-\lambda)(4-\lambda)$.
So the eigenvalues are 2 and 4.

**Quiz 9:** Find an invertible matrix $P$ and a diagonal matrix $D$ such that $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = PDP^{-1}$.

*Solution:* The characteristic equation is

$$0 = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1) .$$

So the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 1$. Letting $e_1 = (x, y)$ be an eigenvector with eigenvalue $\lambda_1$, then

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix} .$$

That equality can be expressed as $2x + y = 3x$ and $x + 2y = 3y$. We can put $x = y = 1$. Thus, $e_1 = (1, 1)$. Letting $e_2 = (x, y)$ be an eigenvector with eigenvalue $\lambda_2$, then

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} .$$

In other words, $2x + y = x$ and $x + 2y = y$. We can put $x = 1$ and $y = -1$. Thus, $e_2 = (1, -1)$. Taking the $i$-th column of $P$ to be the coordinate vector $e_i$, we have

$$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} , \qquad D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} .$$

*Comment:* As a check, the specified values of $P$ and $D$ yield

$$PDP^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & 1 \\ 3 & -1 \end{bmatrix} \frac{1}{-2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 3 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} .$$

**Quiz 10:** Apply the Gram–Schmidt process to the basis $\{e_1, e_2\}$ of $\mathbb{R}^2$, where $e_1 = (2, 1)$ and $e_2 = (1, -1)$, to obtain an orthonormal basis $\{f_1, f_2\}$.

*Solution:* We have $f_1 = e_1 / \parallel e_1 \parallel = (2, 1)/\sqrt{5}$. Let $f_2' = e_2 - \langle f_1 \,|\, e_2 \rangle f_1$. Then

$$f_2' = (1, -1) - \frac{1}{\sqrt{5}}\big((2, 1).(1, -1)\big)\frac{1}{\sqrt{5}}(2, 1) = (1, -1) - \frac{2-1}{5}(2, 1) = (3/5, -6/5) .$$

Then $f_2 = f_2' / \parallel f_2' \parallel = (1, -2)/ \parallel (1, -2) \parallel = (1, -2)/\sqrt{5}$.

## MATH 227: Introduction to Linear Algebra. Spring 2022. Midterm 1

LJB, 22 March 2022, Bilkent University.

Exam duration: 17:40 - 19:30.

Use a maximum of 2 pages for each question. Further pages will not be marked.

Please put your name and section on EVERY sheet of your manuscript.

The use of telephones, calculators or other electronic devices is prohibited. The use of red pens or very faint pencils is prohibited too.

Please do not hand in the question sheet. You may take the question sheet home.

Remember to state your results clearly and to show your working clearly.

**1: 25 marks.** By any method, find all the solutions to

$$3x + 2y + z = 14 , \qquad x + 6y + 5z = 20 , \qquad 9x + 22y + 17z = 88 .$$

**2: 25 marks** Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 16 \end{bmatrix}$. Find the inverse of $A$ by:

**(a)** Gauss–Jordan elimination,

**(b)** the method of minors and cofactors.

**3: 25 marks** Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{bmatrix}$.

**(a)** Find a basis for the null space $\mathrm{Null}(A)$.

**(b)** Evaluate $\mathrm{null}(A)$ and $\mathrm{rank}(A)$.

**4: 25 marks** For which values of $t$ does the equation $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & t \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 15 \\ 24 \end{bmatrix}$ have:

**(a)** no solutions?

**(b)** exactly one solution?

**(c)** infinitely many solutions?

# Midterm 1 Solutions

**1:** The augmented matrix is $\left[\begin{array}{ccc|c} 3 & 2 & 1 & 14 \\ 1 & 6 & 5 & 20 \\ 9 & 22 & 17 & 88 \end{array}\right]$.

Interchanging rows 1 and 2 gives $\left[\begin{array}{ccc|c} 1 & 6 & 5 & 20 \\ 3 & 2 & 1 & 14 \\ 9 & 22 & 17 & 88 \end{array}\right]$.

Subtracting multiples of row 1 from the others, $\left[\begin{array}{ccc|c} 1 & 6 & 5 & 20 \\ 0 & -16 & -14 & -46 \\ 0 & -32 & -28 & -92 \end{array}\right]$.

Dividing row 2 by $-16$ gives $\left[\begin{array}{ccc|c} 1 & 6 & 5 & 20 \\ 0 & 1 & 7/8 & 23/8 \\ 0 & -32 & -28 & -92 \end{array}\right]$.

Adding 32 times tow 2 to row 3 gives $\left[\begin{array}{ccc|c} 1 & 6 & 5 & 20 \\ 0 & 1 & 7/8 & 23/8 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

Subtracting 6 times row 2 from row 1 gives $\left[\begin{array}{ccc|c} 1 & 0 & -1/4 & 11/4 \\ 0 & 1 & 7/8 & 23/8 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

The equations are $x - z/4 = 11/4$ and $y + 7z/8 = 23/8$. Introducing a parameter $s$, we can put $z = s$. The solutions are

$$(x, y, z) = (s/4 + 11/4, -7s/8 + 23/8, s)$$

where $s$ is any real number.

*Comment:* That was by Gauss–Jordan elimination. The Gaussian method would be to omit the last row operation, and to write down the equations $x+6y+5z = 20$ and $y+7z/8 = 23/8$. Again putting $z = s$, we would immediately obtain $y = -7s/8+28/8$. Then we would calculate $x$ from

$$x = -6y - 5z + 20 = -6\left(\frac{-7s}{8} + \frac{28}{8}\right) - 5s + 20 = \frac{6.7 - 5.8}{8}s + \frac{-6.28 + 20.8}{8} = \frac{s}{4} + \frac{11}{4}\,.$$

**2:** Part (a). We set up the problem as $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 1 & 0 \\ 1 & 4 & 16 & 0 & 0 & 1 \end{array}\right]$.

Subtracting row 1 from rows 2 and 3 gives $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 3 & 15 & -1 & 0 & 1 \end{array}\right]$.

Subtracting 3 times row 2 from row 3 gives $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 6 & 2 & -3 & 1 \end{array}\right]$.

Dividing row 3 by 6 gives $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1/3 & -1/2 & 1/6 \end{array}\right]$.

Subtracting multiples of row 3 from rows 1 and 2 gives $\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 2/3 & 1/2 & -1/6 \\ 0 & 1 & 0 & -2 & 5/2 & -1/2 \\ 0 & 0 & 1 & 1/3 & -1/2 & 1/6 \end{array}\right]$.

Subtracting row 2 from row 1 gives $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8/3 & -2 & -1/3 \\ 0 & 1 & 0 & -2 & 5/2 & -1/2 \\ 0 & 0 & 1 & 1/3 & -1/2 & 1/6 \end{array}\right]$.

Therefore, $A^{-1} = \dfrac{1}{6}\begin{bmatrix} 16 & -12 & 2 \\ -12 & 15 & -3 \\ 2 & -3 & 1 \end{bmatrix}$.

Part (b). The matrix of minors is $\begin{bmatrix} 32-16 & 16-4 & 4-2 \\ 16-4 & 16-1 & 4-1 \\ 4-2 & 4-1 & 2-1 \end{bmatrix} = \begin{bmatrix} 16 & 12 & 2 \\ 12 & 15 & 3 \\ 2 & 3 & 1 \end{bmatrix}$.

Hence, the matrix of cofactors is $\begin{bmatrix} 16 & -12 & 2 \\ -12 & 15 & -3 \\ 2 & -3 & 1 \end{bmatrix}$.

The adjoint matrix $\mathrm{adj}(A)$ is the same as the matrix of cofactors. Using the first row of $A$ and the first column of $\mathrm{adj}(A)$, we have $\det(A) = 1.16 - 1.12 + 1.2 = 6$. Finally, using the formula $A^{-1} = \mathrm{adj}(A)/\det(A)$, we obtain the same value for $A^{-1}$ as in part (a).

**3:** Part (a). The null space $\mathrm{Null}(A)$ is the set of vectors $\underline{x} = (x_1, x_2, x_3, x_4)$ such that $A\underline{x} = 0$. To find $\mathrm{Null}(A)$, we perform row operations on $A$ to arrive at a matrix in row echelon form.

Subtracting row 1 from the other two rows, we obtain $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 2 & 8 & 26 \end{bmatrix}$.

Subtracting 2 times row 2 from row 3 gives $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 2 & 12 \end{bmatrix}$.

Dividing row 3 by 2 gives $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 1 & 6 \end{bmatrix}$.

Subtracting multiples of row 3 from rows 1 and 2 gives $\begin{bmatrix} 1 & 1 & 0 & -5 \\ 0 & 1 & 0 & -11 \\ 0 & 0 & 1 & 6 \end{bmatrix}$.

Subtracting row 2 from row 1 gives $\begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -11 \\ 0 & 0 & 1 & 6 \end{bmatrix}$.

Therefore, $\mathrm{Null}(A)$ consists of the vectors $(x_1, x_2, x_3, x_4)$ that satisfy

$$x_1 + 6x_4 = x_2 - 11x_4 = x_3 + 6x_4 = 0 .$$

Putting $x_4 = s$, then $(x_1, x_2, x_3, x_4) = (-6s, 11s, -6s, s) = s(-6, 11, -6, 1)\}$. Thus

$$\mathrm{Null}(A) = \mathrm{span}\{(-6, 11, -6, 1)\} .$$

We conclude that $\{(-6, 11, -6, 1)\}$ is a basis for $\mathrm{Null}(A)$.

Part (b). By part (a), $\mathrm{null}(A) = 1$. By the rank-nullity formula, $\mathrm{rank}(A) = 4 - (A) = 3$.

*Comment:* As an alternative way of expressing the calculations in part (a), Null$(A)$ is the set of solutions to the system of linear equations with augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 4 & 8 & 0 \\ 1 & 3 & 9 & 27 & 0 \end{array}\right].$$

One can solve this system of linear equations using the same row operations as before. The entries in the right-hand column will all remain 0 throughout.

**4:** The equation has a unique solution if and only if the specified $3 \times 3$ matrix is invertible. We have

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & t \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 8 & t \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & t \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 5t - 48 - 2(4t - 42) + 3(32 - 35) = -3t + 27 = 3(9 - t)\ .$$

So the invertibility holds if and only if $t \neq 9$.

When $t = 9$, the augmented matrix of the equation is $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 4 & 5 & 6 & 15 \\ 7 & 8 & 9 & 24 \end{array}\right]$.

Subtracting multiples of row 1 from the other rows, $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -3 & -6 & -9 \\ 0 & -6 & -12 & -18 \end{array}\right]$.

Two further row operations yield $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$, which has infinitely many solutions.

Therefore, case (a) never holds, case (b) holds when $t \neq 9$, case (c) holds when $t = 9$.

# MATH 227: Introduction to Linear Algebra. Spring 2022. Midterm 2

LJB, 26 April 2022, Bilkent University.

Exam duration: 17:40 - 19:40.

Use a maximum of 2 pages for each question. Further pages will not be marked.

Please put your name and section on EVERY sheet of your manuscript.

The use of telephones, calculators or other electronic devices is prohibited. The use of red pens or very faint pencils is prohibited too.

Please do not hand in the question sheet. You may take the question sheet home.

Remember to state your results clearly and to show your working clearly.

**1: 15 marks.** Let $A = \begin{bmatrix} 2 & 1 & 3 & 2 \\ 3 & 4 & 4 & 5 \\ 7 & 6 & 10 & 9 \end{bmatrix}$.

Find a basis for the image space $\mathrm{Im}(A)$. What is the rank of $A$?

**2: 35 marks.** Let $B = \begin{bmatrix} 3 & 0 & 0 \\ 6 & -6 & 6 \\ 8 & -12 & 11 \end{bmatrix}$.

Find invertible $P$ and diagonal $D$ such that $B = PDP^{-1}$.

**3: 35 marks.** Let $e_1$, $e_2$, $e_3$, $f_1$, $f_2$, $f_3$ be vectors in a 3-dimensional real vector space such that

$$f_1 = 9e_1 + 3e_2 + e_3 \ , \qquad f_2 = 4e_1 + 2e_2 + e_3 \ , \qquad f_3 = e_1 - e_2 + e_3 \ .$$

Let $x_1$, $x_2$, $x_3$, $y_1$, $y_2$, $y_3$ be real numbers such that $x_1e_1 + x_2e_2 + x_3e_3 = y_1f_1 + y_2f_2 + y_3f_3$.

**(a)** Express $e_1$, $e_2$, $e_3$ in terms of $f_1$, $f_2$, $f_3$.

**(b)** Express $x_1$, $x_2$, $x_3$ in terms of $y_1$, $y_2$, $y_3$.

**(c)** Express $y_1$, $y_2$, $y_3$ in terms of $x_1$, $x_2$, $x_3$.

**(d)** Now suppose $(y_1, y_2, y_3) = (3, 2, 1)$. Evaluate $(x_1, x_2, x_3)$.

**4: 15 marks.** Find all the eigenvalues and all the eigenvectors for the matrix

$$\begin{bmatrix} i & i \\ i & i \end{bmatrix}$$

where $i$ denotes a square root of $-1$. (By convention, we do not consider the zero vector $(0, 0)$ to be an eigenvector.)

# Midterm 2 Solutions

**1:** Rearranging the columns, we obtain $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 4 & 3 & 4 & 5 \\ 6 & 7 & 10 & 9 \end{bmatrix}$.

Subtracting multiples of column 1 from the other columns, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & -5 & -8 & -3 \\ 6 & -5 & -8 & -3 \end{bmatrix}$.

Multiplying the columns by scalars, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 1 & 1 \\ 6 & 1 & 1 & 1 \end{bmatrix}$.

Subtracting column 2 from columns 3 and 4 yields $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 6 & 1 & 0 & 0 \end{bmatrix}$.

The latest matrix is in column echelon form, so a basis for $\mathrm{Im}(A)$ is $\left\{ \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

**2:** The characteristic equation is

$$0 = \det(\lambda I - B) = \begin{vmatrix} \lambda - 3 & 0 & 0 \\ -6 & \lambda + 6 & 0 \\ -8 & 12 & \lambda - 11 \end{vmatrix} = (\lambda - 3) \begin{vmatrix} \lambda + 6 & -6 \\ 12 & \lambda - 11 \end{vmatrix}$$

$$= (\lambda - 3)\big((\lambda + 6)(\lambda - 11) + 72\big) = (\lambda - 3)(\lambda^2 - 5\lambda + 6) = (\lambda - 2)(\lambda - 3)^2 \ .$$

So the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = \lambda_3 = 3$. We are to find a basis $\{e_1, e_2, e_3\}$ such that $Be_i = \lambda_i e_i$.

Write $e_1 = (x, y, z)$. We have

$$3x = 2x \ , \qquad 6x - 6y + 6z = 2y \ , \qquad 8x - 12y + 11z = 2z \ .$$

Hence $x = 0$ and $3z = 4y$. Putting $y = 3$, then $z = 4$. Thus, $e_1 = (0, 3, 4)$.

Now write $e_2 = (x, y, z)$. We have

$$3x = 3x \ , \qquad 6x - 6y + 6z = 3y \ , \qquad 8x - 12y + 11z = 3z \ .$$

Hence $2x + 2z = 3y$. Putting $x = 0$ and $z = 3$, then $y = 2$. Thus, $e_2 = (0, 2, 3)$.

Finally, write $e_3 = (x, y, z)$. The equations for $x$, $y$, $z$ are the same as for $e_2$. Putting $x = 1$ and $z = 2$, then $y = 2$. Thus, $e_3 = (1, 2, 2)$. We have

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 2 & 2 \\ 4 & 3 & 2 \end{bmatrix} \ , \qquad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \ .$$

**3:** Part (a). The transformation matrix to $x$-coordinates from $y$-coordinates is

$$T = \begin{bmatrix} 9 & 4 & 1 \\ 3 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix} \ .$$

The matrix of minors is $\begin{bmatrix} 2+1 & 3+1 & 3-2 \\ 4-1 & 9-1 & 9-4 \\ -4-2 & -9-3 & 18-12 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 1 \\ 3 & 8 & 5 \\ -6 & -12 & 6 \end{bmatrix}$.

The matrix of cofactors is $\begin{bmatrix} 3 & -4 & 1 \\ -3 & 8 & -5 \\ -6 & 12 & -6 \end{bmatrix}$. Hence $\mathrm{adj}(T) = \begin{bmatrix} 3 & -3 & -6 \\ -4 & 8 & 12 \\ 1 & -5 & -6 \end{bmatrix}$.

By considering the $(1,1)$ entry of the equality $T.\mathrm{adj}(T) = \det(T)I$, we have

$\det(T) = 27 - 16 + 1 = 12$. Hence, $T^{-1} = \dfrac{1}{12}\begin{bmatrix} 3 & -3 & -6 \\ -4 & 8 & 12 \\ 1 & -5 & 6 \end{bmatrix}$. Therefore,

$$e_1 = (3f_1 - 4f_2 + f_3)/12\ , \quad e_2 = (-3f_1 + 8f_2 - 5f_3)/12\ , \quad e_3 = (-f_1 + 2f_2 - f_3)/2\ .$$

Part (b). We have $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = T\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 & 4 & 1 \\ 3 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$.

Part (c). We have $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = T^{-1}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \dfrac{1}{12}\begin{bmatrix} 3 & -3 & -6 \\ -4 & 8 & 12 \\ 1 & -5 & 6 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$.

Part (d), in this case, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = T\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 27+8+1 \\ 9+4-1 \\ 3+2+1 \end{bmatrix} = \begin{bmatrix} 36 \\ 12 \\ 6 \end{bmatrix}$.

**4:** The characteristic equation is

$$0 = \begin{vmatrix} i-\lambda & i \\ i & i-\lambda \end{vmatrix} = (i-\lambda)^2 - i^2 = \lambda^2 - 2i\lambda = \lambda(\lambda - 2i)\ .$$

So the eigenvalues are 0 and $2i$. The eigenvectors for eigenvalue 0 are the vectors $(s, -s)$ where $s$ is a nonzero complex number. The eigenvectors for eigenvalue $2i$ are the vectors $(t, t)$ where $t$ is a nonzero complex number.

# MATH 227: Introduction to Linear Algebra. Spring 2022. Final

LJB, 24 May 2022, Bilkent University.

Exam duration: 15:20 - 17:20.

Use a maximum of 2 pages for each question. Further pages will not be marked.

Please put your name and section on EVERY sheet of your manuscript.

The use of telephones, calculators or other electronic devices is prohibited. The use of red pens or very faint pencils is prohibited too.

Please do not hand in the question sheet. You may take the question sheet home.

Remember to state your results clearly and to show your working clearly.

**1: 20 marks.** Let $e_1 = (1, 1, 1)$ and $e_2 = (1, 2, 0)$ and $e_3 = (3, 2, 4)$. Let $\{f_1, f_2, f_3\}$ be the orthonormal basis for $\mathbb{R}^3$ obtained from $\{e_1, e_2, e_3\}$ by the Gram–Schmidt process. Evaluate $f_1$ and $f_2$ and $f_3$.

**2:** Let $A = \begin{bmatrix} -7 & -4 & 6 \\ -9 & -2 & 6 \\ -19 & -9 & 15 \end{bmatrix}$.

You do not need to consider the characteristic equation in this question, and there will be no marks for work involving the characteristic equation.

**(a) 10 marks.** Show that $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 3 \\ 7 \end{bmatrix}$ are eigenvectors of $A$, and find their corresponding eigenvalues.

**(b) 10 marks.** Let $P = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 2 & 5 & 7 \end{bmatrix}$. Find a diagonal matrix $D$ such that $A = PDP^{-1}$.

**(c) 20 marks.** By any method, evaluate $P^{-1}$.

**(d) 10 marks.** Using parts (b) and (c), give a formula for the entry in row 2 and column 2 of $A^n$.

**3: 20 marks.** Let $M = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 4 & 3 & 2 & 1 \\ 1 & 3 & 5 & 7 \end{bmatrix}$.

**(a)** Find a basis for the image of $M$.

**(b)** Find a basis for the null space of $M$.

**(c)** What are the rank and nullity of $M$?

**4: 10 marks.** Regard the set of complex numbers $\mathbb{C}$ as a vector space over the real numbers $\mathbb{R}$, where the vector space addition is the usual addition of complex numbers and the scalar multiplication is the usual multiplication of a real number and a complex number.

**(a)** Write down a basis for $\mathbb{C}$ as a vector space over $\mathbb{R}$.

**(b)** What is the dimension of $\mathbb{C}$ as a vector space over $\mathbb{R}$?

# Solutions to Final

**1:** Put $f_1 = e_1/\parallel e_1 \parallel = (1,1,1)/\sqrt{3}$. Write $f_2' = e_2 - \langle f_1 \,|\, e_2 \rangle f_1$. Then

$$f_2' = (1,2,0) - \frac{1}{3}\big((1,1,1).(1,2,0)\big)(1,1,1) = (1,2,0) - (1,1,1) = (0,1,1)\ .$$

Put $f_2 = f_2'/\parallel f_2' \parallel = (0,1,-1)/\sqrt{2}$. Put $f_3' = e_3 - \langle f_1 \,|\, e_3 \rangle f_1 - \langle f_2 \,|\, e_3 \rangle f_2$. Then

$$\begin{aligned} f_3' &= (3,2,4) - \frac{1}{3}\big((1,1,1).(3,2,4)\big)(1,1,1) - \frac{1}{2}\big((0,1,-1).(3,2,4)\big)(0,1,-1) \\ &= (3,2,4) - (3,3,3) + (0,1,-1) = (0,0,0)\ . \end{aligned}$$

The set $\{e_1, e_2, e_3\}$ is not a basis, indeed, a nontrivial linear relation is $4e_1 - e_2 - e_3 = 0$. So the calculated set $\{f_1', f_2', f_3'\}$ cannot be an orthogonal basis, and we cannot obtain an orthogonal basis by the method.

In conclusion, $f_1 = (1,1,1)/\sqrt{3}$ and $f_2 = (0,1,-1)/\sqrt{2}$, but the process does not return a value for $f_3$ since $\{e_1, e_2, e_3\}$ is not a basis.

*Comment:* The question has a mistake. I gave 18 marks for correctly following the Gram–Schmidt process as far as it can be taken, full marks for correctly explaining why the process does not yield an orthonormal basis in this case.

The mistake was not deliberate. Nevertheless, I maintain that it does serve well as a test of critical thinking.

**2:** Part (a). We have

$$A \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -7-4+12 \\ -9-2+12 \\ -19-9+30 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\ ,$$

$$A \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} -14-12+30 \\ -18-6+30 \\ -38-27+74 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 10 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}\ ,$$

$$A \begin{bmatrix} 3 \\ 3 \\ 7 \end{bmatrix} = \begin{bmatrix} -21-12+42 \\ -27-6+42 \\ -37-27+105 \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ 21 \end{bmatrix} = 3 \begin{bmatrix} 3 \\ 3 \\ 7 \end{bmatrix}\ .$$

So the three specified vectors are eigenvectors, and the corresponding eigenvalues, in order, are 1, 2, 3.

Part (b). By part (a), $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

We set up the problem as $\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 3 & 3 & 0 & 1 & 0 \\ 2 & 5 & 7 & 0 & 0 & 1 \end{array}\right]$.

Subtracting multiples of row 1 from the other rows, $\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 & 1 \end{array}\right]$.

Subtracting row 2 from row 3 gives $\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$.

Subtracting 3 times row 3 from row 1 gives $\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 4 & 3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$.

Subtracting 2 times row 2 from row 1 gives $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & 1 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$.

So $P^{-1} = \begin{bmatrix} 6 & 1 & -3 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$.

Part (d). We have $A^n = PD^nP^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 2 & 5 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{bmatrix} \begin{bmatrix} 6 & 1 & -3 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 2.2^n & 3.3^n \\ 1 & 3.2^n & 3.3^n \\ 2 & 5.2^n & 7.3^n \end{bmatrix} \begin{bmatrix} 6 & 1 & -3 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

whose $(2,2)$ entry is $1 + 3.2^n - 3.3^n = 1 + 3(2^n - 3^n)$.

**3:** Part (a). Subtracting multiples of the first column from the others, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & -2 & -3 \\ 4 & -5 & -10 & -15 \\ 1 & 1 & 2 & 3 \end{bmatrix}$.

Further column operations yield $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 4 & 5 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$.

which is in column echelon form. It follows that $\mathrm{Im}(M)$ has basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 5 \\ -1 \end{bmatrix} \right\}$.

Part (b). We obtain $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -5 & -10 & -15 \\ 0 & 1 & 2 & 3 \end{bmatrix}$ by subtracting multiples of row 1.

Further row operations yield $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, then $\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

The null space of $A$ consists of the vectors $(x_1, x_2, x_3, x_4)$ such that

$$x_1 - x_3 - 2x_4 = x_2 + 2x_3 + 3x_4 = 0\ .$$

Introducing parameters $s$ and $t$, putting $x_3 = s$ and $x_4 = t$, then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} s + 2t \\ -2s - 3t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} .$$

So the null space of $A$ has basis $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Part (d). Evidently, $\text{rank}(A) = \text{null}(A) = 2$.

**4:** Part (a). One basis for $\mathbb{C}$ over $\mathbb{R}$ is $\{1, i\}$, where $i^2 = -1$.
Part (b). The dimension of $\mathbb{C}$ over $\mathbb{R}$ is $|\{1, i\}| = 2$.