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MATH 227, Introduction to Linear Algebra

Bilkent University, Spring 2019, Laurence Barker

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Course specification

MATH 227, *Introduction to Linear Algebra*, Spring 2019

Laurence Barker, Bilkent University. Version: 5 March 2019.

Course Aims: To aquire practical knowledge and skill in some important techniques of linear algebra, and to aquire theoretical understanding and ability to critically assess methods and propositions in the area.

Course Description: This is an introductory course with an emphasis on methods of calculation, but with a theoretical grounding that is self-contained and complete. Some victory conditions, for the student, include: an understanding of the notion of a vector space as something more than just a system of coordinates; an ability to apply the method of diagonalization, a clear grasp of the theory behind that method.

Course Requirements: An official prerequisite in the University Catalogue is MATH 106 but, in practice, no knowledge of calculus is needed.

Instructor: Laurence Barker, Office SAZ 129, barker at fen dot bilkent dot edu dot tr.

Textbook: Howard Anton, Chris Rorres, "Elementary Linear Algebra", 11th edition, Wiley, 2011, 2015. ISBN: 978-1-118-67745-2.

Warning: The Main course text is not the book with the same authors, same title, same edition number, same publisher but with a different ISBN. The main course text has the text "International Student Version" on the front cover, whereas the other book does not. The two books have different exercises.

Midterm 1 revision notes: see the file on my homepage: midterm1revision227Spr19.pdf
Notes on diagonalization: see the file on my homepage: diagonalization.pdf .

Classes: Tuesdays 10:40 - 12:30, Fridays 09:40 - 10:30, room MA-301.

Office Hours: Fridays 08:40 - 09:30, in my office, room SA-129.

Office hours is for *all* the students on the course, not just the proficient. If you are having difficulty with the course, then it is best to come to see me for advice.

Weekly Syllabus

The format below is, *Week number; Monday date; Subtopics and textbook section numbers.* The numbering $m.n$ indicates Chapter m Section n in the Anton–Rorres textbook.

1: 4 Feb: Systems of linear equations, 1.1. Sketch of Markov chain scenario and the method of diagonalization.

2: 11 Feb: Gaussian and Gauss–Jordan elimination, 1.2. Matrics, 1.3.

3: 18 Feb: Elementary matrices, inversion of matrices by row operations 1.4, 1.5. Existence and uniqueness of solutions, 1.6.

4: 25 Feb: Determinants, their algebraic properties, their evaluation by row reduction and by cofactor expansion, 2.1, 2.2, 2.3.

5: 4 Mar: Euclidian spaces, norm, dot product, distance, angle, 3.1, 3.2.

6: 11 Mar: Pearson correlation coefficient (special notes). Introduction to Markov chains, 4.12.

7: 18 Mar: (No class on Friday.) Real vector spaces, subspaces, 4.1, 4.2.

8: 25 Mar: Linear independence, spanning, bases, dimension, 4.3, 4.4, 4.5.

9: 1 Apr: Linear transformations, inverses, composition, 8.1, 8.2, 8.3.

10: 8 Apr: Change of basis, 4.6. Row and column spaces, rank-nullity formula, 4.7, 4.8.

11: 15 Apr: Eigenvalues and eigenvectors, 5.1

12: 22 Apr: (No class on Tuesday.) Complex vector spaces, 5.3.

13: 29 Apr: Diagonalization, 5.2. Applications to Markov chains, 10.5.

14: 6 May: Inner product spaces, 6.1, 6.2.

15: 13 May: Gram–Schmidt orthogonalization, 6.3. Review for Final.

Assessment: The method of assessment is by curve. Grades F, FX, FZ are to be for candidates judged to be thoroughly incompetent in the routine course material.

- Quizzes and Participation 10%.
- Midterm I, 30%, Wednesday, 13 March, 18:00.
- Midterm II, 30%, Tuesday, 30 April, 18:00.
- Final, 30%, Friday, 31 May, 09:00.

FZ criteria: (1) less than 30% in sum of Midterm 1 and Midterm 2 marks, or (2) less than 50% in that sum and less than 75% attendance.

Class Announcements: All students, including any absentees from a class, will be deemed responsible for awareness of class announcements.

Midterm 1 Exam Syllabus

Solving linear equations, $[\ |]$. Gaussian elimination, Gauss–Jordan elimination. 1.1, 1.2.

Inverting matrices by Gauss–Jordan method, $[\ |\ ]$. 1.3, 1.4, 1.5.

Determinants and inverses by cofactor method, $|\ \ |$. 1.6, 2.1, 2.2, 2.3 (but without Cramer's rule).

Euclidian vector spaces, $\|\ \|$. Vectors, 3.1. Norm, dot product, distance, Theorems 3.2.2, 3.2.4, 3.2.5. Orthogonality, Theorem 3.3.2. Pearson correlation coefficient (discussed in lectures and presented in notes on next page).

Some notes on the Pearson correlation coefficient

The following theory, and some examples, are given in class on Tuesday 5 March. I supply this summary because the topic is not in the textbook.

Let x_1, ..., x_n be sampled values of some variable, and let y_1, ..., y_n be the corresponding sampled values of some other variable. The **Pearson correlation coefficient** ρ of the two samples is calculated as follows:

Step 1: Calculate the average values

$$\overline{x} = (x_1 + ... + x_n)/n \ , \qquad \overline{y} = (y_1 + ... + y_n)/n \ .$$

Step 2: Calculate the centred vectors

$$\widetilde{x} = (\widetilde{x}_1, ..., \widetilde{x}_n) = (x_1 - \overline{x}, ..., x_n = \overline{x}) \ , \qquad \widetilde{y} = (\widetilde{y}_1, ..., \widetilde{y}_n) = (y_1 - \overline{y}, ..., y_n - \overline{y}) \ .$$

Step 3: Calculate $\rho = \dfrac{\widetilde{x}_\bullet \widetilde{y}}{\| \widetilde{x} \| \| \widetilde{y} \|}$ where

$$\widetilde{x}_\bullet \widetilde{y} = \widetilde{x}_1 \widetilde{y}_1 + ... + \widetilde{x}_n \widetilde{y}_n \ , \qquad \| \widetilde{x} \| = \sqrt{\widetilde{x}_1^2 + ... + \widetilde{x}_n^2} \ , \qquad \| \widetilde{y} \| = \sqrt{\widetilde{y}_1^2 + ... + \widetilde{y}_n^2} \ .$$

Let us give two illustrative examples.

Problem A: Snails A, B, C weigh 3, 3, 6 kilograms, respectively. The maximum speeds of A, B, C are 3, 4, 8 meters per second, respectively. Let ρ be the correlation coefficient for these samples of weights and speeds. Show that $\rho > 97/100$.

Solution: The given data is $(x_1, x_2, x_3) = (3, 3, 6)$ and $(y_1, y_2, y_3) = (3, 4, 8)$. The averages are $\overline{x} = (3 + 3 + 6)/3 = 4$ and $\overline{y} = (3 + 4 + 8)/3 = 5$. The centred vectors are $\widetilde{x} = (-1, -1, 2)$ and $\widetilde{y} = (-2, -1, 3)$. So $\| \widetilde{x} \|^2 = 1 + 1 + 4 = 6$ and $\| \widetilde{y} \|^2 = 4 + 1 + 9 = 14$. So

$$\rho = \frac{2 + 1 + 6}{\sqrt{6}\sqrt{14}} = \frac{9}{2\sqrt{21}} \ .$$

It follows that $\rho^2 = 81/84 > 96/100$, hence $\rho > 97/100$. $\square$

2: Problem B. Let X be a variable with sample values 3, 3, 3, 1, 5 and let Y be a variable with corresponding sample values 1, 3, 4, 3, 9. Calculate the Pearson correlation coefficient for the samples.

Solution: The coordinates of the vectors $x = (3, 3, 3, 1, 5)$ and $y = (1, 3, 4, 3, 9)$ have average values $\overline{x} = (3 + 3 + 3 + 1 + 5)/5 = 15/5 = 3$ and $\overline{y} = (1 + 3 + 4 + 3 + 9)/5 = 30/5 = 4$. The centred vectors are $\widetilde{x} = (0, 0, 0, -2, 2)$ and $\widetilde{y} = (-3, -1, 0, -1, 5)$. Therefore

$$\rho = \frac{\widetilde{x}.\widetilde{y}}{\| \widetilde{x} \| . \| \widetilde{y} \|} = \frac{0 + 0 + 0 + 2 + 10}{\sqrt{(0 + 0 + 0 + 4 + 4)(9 + 1 + 0 + 1 + 25)}} = \frac{12}{\sqrt{8 \times 36}} = 1/\sqrt{2} \ .$$

Suggested Exercises and Quiz Solutions

MATH 227, *Introduction to Linear Algebra*, Spring 2019

Laurence Barker, Mathematics Department, Bilkent University,
version: 21 June 2019.

Office Hours: Fridays, 08:40 – 09:30, Room SA 129.

Exercises

Most of the questions are taken from the primary textbook: Howard Anton, Chris Rorres, "Elementary Linear Algebra", 11th Edition, 2011 (reprinted 2015).

End of Chapter 1, page 76 onwards, numbers 19 - 37, 51, 52, 102, 125, 127.
End of Chapter 2, page 111, routine: 16 - 20, theoretical: 34, 35, 63, 64.
End of Chapter 3, page 164, routine 85 - 104, theoretical: 131, 132.

For Midterm 1 preparation: See file midterm1revision227spr19.pdf on my homepage.

End of Chapter 4, page 271. On vector spaces and subspaces, 3 - 15, 21 - 27; on linear independence, 43 - 46; on bases, 67 - 80; on dimension, 84 - 100.

For Midterm 2 preparation, from the exams in the file arch227spr17.pdf:
- Questions 1, 3, 4, 5 from Practice Midterm 2,
- Questions 2, 3, 4, 5 from Midterm 2, 28 April 2017.

End of Chapter 5, page 325, standard: 3 - 8, 41 - 44, 52 - 54; harder: 34, 55, 56.

Quizzes

Quiz 1: Solve, by Gaussian elimination, the equations

$$x + 2y + 3z = 7 \;, \qquad x + 3y - z = 6 \;, \qquad y + 2z = 4 \;.$$

Quiz 2: Evaluate $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$.

Quiz 3: Calculate $x.y$ and $\|x\|$ and $\|y\|$ where $x = (1, 2, 3, 4, 5)$ and $y = (1, 1, 1, 1, 1)$.

Quiz 4: Is $\{(1, 1), (1, 2)\}$ a spanning set for $\mathbb{R}^2$?

Quiz 5: Consider the subset $S = \{(1, 2, 4), (4, 2, 1), (1, 1, 2)\}$ of $\mathbb{R}^3$. Is S linearly independent?

Quiz 6: Let $S = \{(1, 1, 1), (10, 11, 12), (20, 19, 18)\}$.
(a) Find a subset T of S such that T is a basis for span(S).
(b) For each $s \in S$ with $s \notin T$, express s as a linear combination of elements of T.

Quiz 7: Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}$. **(a)** Find a basis for Im(A). **(b)** Find rank(A) and null(A).

Quiz 8: Diagonalize the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$.

Solutions to Quizzes

Solution 1: We code the problem as $\left[\begin{array}{rrr|r} 1 & 2 & 3 & 7 \\ 1 & 3 & -1 & 6 \\ 0 & 1 & 2 & 4 \end{array}\right]$. Subtracting row 1 from row 2 gives $\left[\begin{array}{rrr|r} 1 & 2 & 3 & 7 \\ 0 & 1 & -4 & -1 \\ 0 & 1 & 2 & 4 \end{array}\right]$. Subtracting row 2 from row 3 gives $\left[\begin{array}{rrr|r} 1 & 2 & 3 & 7 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 6 & 5 \end{array}\right]$. So $z = 5/6$. We have $y = -1 + 4z$ and $x = 7 - 2y - 3z$. So $y = -1 + 10/3 = 7/3$ and $x = 7 - 14/3 - 15/6 = (42 - 28 - 15)/6 = -1/6$. In conclusion, $(x, y, z) = (-1/6, 7/3, 5/6)$.

Solution 2: The answer is $4 - 6 = -2$.

Solution 3: We have $x.y = 1 + 2 + 3 + 4 + 5 = 15$. Also $\| x \|^2 = 1 + 4 + 9 + 16 + 25 = 55$ and $\| y \|^2 = 1 + 1 + 1 + 1 + 1 = 5$. So $\| x \| = \sqrt{55}$ and $\| y \| = \sqrt{5}$.

Solution 4: Yes, because the set has size 2 and is plainly independent.

Solution 5: The set S is linerly independent because

$$\begin{vmatrix} 1 & 1 & 2 \\ 4 & 2 & 1 \\ 1 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 2 & 4 \end{vmatrix} - \begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} + 2 \begin{vmatrix} 4 & 2 \\ 1 & 2 \end{vmatrix} = 6 - 15 + 2.6 = 3 \neq 0 \, .$$

Solution 6: Part (a). We can put $T = \{(10, 11, 12), (20, 19, 18)\}$.

Part (b). We have $(1, 1, 1) = \big((10, 11, 12) + (20, 19, 18)\big)/30$.

Comment: We can also take T to be either one of the other two subsets of S with size 2.

Solution 7: Part (a). We have $\mathrm{Im}(A) = \mathrm{span}\{(1, 1, 1), (1, 2, 3), (2, 3, 4)\}$. Since the third of those vectors is the sum of the other two, the set $\{(1, 1, 1), (2, 3, 4)\}$ is a basis for $\mathrm{Im}(A)$.

Part (b). By the above, $\mathrm{rank}(A) = 2$. By rank-nullity, $\mathrm{null}(A) = 1$.

Solution 8: By inspection, the eigenvalues are 1 and 2, with respective eigenvectors $(1, 0)$ and $(1, 1)$. Taking the columns of P to be the eigenvectors and the diagonal entries of D to be the corresponding eigenvalues, we have $A = PDP^{-1}$ where

$$P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} , \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} .$$

MATH 227: Introduction to Linear Algebra. Spring 2019. Midterm 1

LJB, 13 March 2019, Bilkent University.

Time allowed: 2 hours. Please put your name on EVERY sheet of your manuscript. The use of telephones, calculators or other electronic devices is prohibited. The use of red pens or very faint pencils is prohibited too. You may take the question sheet home.

Remember to justify your answers, except in any cases where your answers are obvious.

1: 20 marks. By Gaussian elimination, solve the equations

$$3x + 2y + z = 10 , \qquad 6x + 5y + 4z = 28 , \qquad 2x + y + z = 7 .$$

2: 20 marks. Let $A = \begin{bmatrix} 3 & 2 & 1 \\ 6 & 5 & 4 \\ 2 & 1 & 1 \end{bmatrix}$. Calculate A^{-1} using the Gauss–Jordan method.

3: 20 marks. Let A be as in the previous question.
(a) Evaluate $\det(A)$.
(b) Calculate A^{-1} using the method of cofactors.

4: 20 marks. Seven stars, numbered 1 to 7, have luminosities 5, 2, 4, 3, 2, 4, 1, respectively, and their spectra have red-shifts 9, 8, 9, 10, 11, 11, 12, respectively. Calculate the Pearson correlation coefficient for the two sampled variables.

5: 20 marks. (a) Find a linear equation involving r, s, t such that your equation is satisfied if and only if the equation $\begin{bmatrix} 3 & 2 & 1 \\ 6 & 5 & 4 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} f \\ g \\ h \end{bmatrix} = \begin{bmatrix} r \\ s \\ t \end{bmatrix}$ has infinitely many solutions in f, g, h.
(b) Find a vector $a \in \mathbb{R}^3$ such that a is orthogonal to all the vectors (r, s, t) satisfying your equation in part (a).
(c) Find a real number λ such that $(1, 1, 1) = \lambda a + (r, s, t)$ where (r, s, t) satisfies your equation in part (a) and a is the vector you found in part (b).

Midterm 1 Solutions

1: The system is $\left[\begin{array}{ccc|c} 3 & 2 & 1 & 10 \\ 6 & 5 & 4 & 28 \\ 2 & 1 & 1 & 7 \end{array}\right]$.

Rearranging the rows and doubling one of them, $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 7 \\ 6 & 5 & 4 & 28 \\ 6 & 4 & 2 & 20 \end{array}\right]$.

Subtracting thrice row 1 from the other two, $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 7 \\ 0 & 2 & 1 & 7 \\ 0 & 1 & -1 & -1 \end{array}\right]$.

Interchanging two rows, $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 7 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & 1 & 7 \end{array}\right]$.

Subtracting twice row 2 from row 3, then dividing row 3 by 3 gives $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 7 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$.

So $z = 3$. We have $y - z = -1$ and $2x + y + z = 7$. So $y = 2$ and $x = 1$. Thus, in conclusion, $(x, y, z) = (1, 2, 3)$.

2: We code the problem as $\left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 0 & 0 \\ 6 & 5 & 4 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$.

Repeating the operations in question 1, two at a time, we obtain $\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 & 1 & -3 \\ 0 & 1 & -1 & 2 & 0 & -3 \end{array}\right]$,

and then $\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 2 & 0 & -3 \\ 0 & 0 & 1 & -4/3 & 1/3 & 1 \end{array}\right]$.

Adding multiples of row 3 to the other rows, $\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 4/3 & -1/3 & 0 \\ 0 & 1 & 0 & 2/3 & 1/3 & -2 \\ 0 & 0 & 1 & -4/3 & 1/3 & 1 \end{array}\right]$.

Subtracting row 2 from row 1, $\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 2/3 & -2/3 & 2 \\ 0 & 1 & 0 & 2/3 & 1/3 & -2 \\ 0 & 0 & 1 & -4/3 & 1/3 & 1 \end{array}\right]$.

Finally, halving row 1, $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & -1/3 & 1 \\ 0 & 1 & 0 & 2/3 & 1/3 & -2 \\ 0 & 0 & 1 & -4/3 & 1/3 & 1 \end{array}\right]$. Therefore, $A^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & -6 \\ -4 & 1 & 3 \end{bmatrix}$.

3: Part (a). We have $\det(A) = 3\begin{vmatrix} 5 & 4 \\ 1 & 1 \end{vmatrix} - 2\begin{vmatrix} 6 & 4 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 6 & 5 \\ 2 & 1 \end{vmatrix} = 3 + 4 - 4 = 3$.

Part (b). The matrix of minors is $\begin{bmatrix} 5.1 - 4.1 & 6.1 - 4.2 & 6.1 - 5.2 \\ 2.1 - 1.1 & 3.1 - 1.2 & 3.1 - 2.2 \\ 2.4 - 1.5 & 3.4 - 1.6 & 3.5 - 2.6 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -4 \\ 1 & 1 & -1 \\ 3 & 6 & 3 \end{bmatrix}$.

That of cofactors, $\begin{bmatrix} 1 & 2 & -4 \\ -1 & 1 & 1 \\ 3 & -6 & 3 \end{bmatrix}$. Taking the transpose and dividing by 3, we recover the answer to Question 2.

4: Let $x = (5, 2, 4, 3, 2, 4, 1)$ and $y = (9, 8, 9, 10, 11, 11, 12)$. The means are

$$\overline{x} = (5+2+4+3+2+4+1)/7 = 21/7 = 4\,,\ \ \overline{y} = (9+8+9+10+11+11+12)/7 = 70/7 = 10\,.$$

The centred vectors are $\widetilde{x} = (2, -1, 1, 0, -1, 1, -2)$ and $\widetilde{y} = (-1, -2, -1, 0, 1, 1, 2)$ We have $\widetilde{x}.\widetilde{y} = -2 + 2 - 1 + 0 - 1 + 1 - 4 = -5$. Also, $\|\widetilde{x}\|^2 = 4 + 1 + 1 + 1 + 1 + 4 = 12$ and, similarly,

$\| \widetilde{y} \|^2 = 12$. The correlation coefficient is $\rho = \dfrac{\widetilde{x}.\widetilde{y}}{\| \widetilde{x} \| . \| \widetilde{y} \|} = \dfrac{-5}{\sqrt{12}.\sqrt{12}} = -5/12$.

5: Part (a). To solve the system of equations for given r, s, t, we code the problem as

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & r \\ 6 & 5 & 4 & s \\ 1 & 1 & 1 & t \end{array}\right] .$$

Subtracting multiples of row 3 from rows 1 and 2, we obtain

$$\left[\begin{array}{ccc|c} 0 & -1 & -2 & r-3t \\ 0 & -1 & -2 & s-3t \\ 1 & 1 & 1 & t \end{array}\right] .$$

Plainly, the system has infinitely many solutions if and only if $r - 3t = s - 6t$, in other words,

$$r - s + 3t = 0 .$$

Part (b). We can put $a = (1, -1, 3)$.

Part (c). The vector λa is the orthogonal projection of $(1, 1, 1)$ to the line through 0 and a. Therefore,

$$\lambda = \frac{(1,1,1).a}{\| a \|^2} = \frac{1-1+3}{1+1+9} = 3/11 .$$

MATH 227: Introduction to Linear Algebra. Spring 2019. Midterm 2

LJB, 30 April 2019, Bilkent University.

Time allowed: 2 hours. Please put your name on EVERY sheet of your manuscript. The use of telephones, calculators or other electronic devices is prohibited. The use of red pens or very faint pencils is prohibited too. You may take the question sheet home.

Remember to justify your answers, except in any cases where your answers are obvious.

1: 30 marks. Consider the matrix $A = \begin{bmatrix} 1 & 1 & 3 & 0 & 3 \\ 3 & 4 & 11 & 1 & 10 \\ 5 & 9 & 23 & 4 & 19 \end{bmatrix}$.

(a) Find a basis for the image of A.
(b) Evaluate null(A).
(c) Evaluate rank(A).

2: 30 marks. Let $S = \{(2, 2, -4), (1, -2, 1), (3, 1, -4), (-1, 7, 6)\}$.

(a) How many of the subsets of S are linearly independent?
(b) How many of the subsets of S are spanning sets for span(S)?
(c) How many of the subsets of S are bases for span(S)?

3: 15 marks. Find the eigenvalues of the matrix $B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

4: 25 marks. Suppose A is an $n \times m$ matrix and $y \in \mathbb{R}^n$. Let X be the set of vectors x in $\mathbb{R}^m$ such that $y = Ax$. In terms of y, give a necessary and sufficient condition for X to be a subspace of $\mathbb{R}^m$. (That is, give a condition on y such that the condition holds if and only if X is a subspace of $\mathbb{R}^m$.) Justify your answer in full detail.

Midterm 2 Solutions

1: Part (a). Adding multiples of the first column to the others, we obtain the matrix below on the left. Adding multiples of the second column to the third, fourth and fifth, we obtain the matrix on the right.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 2 & 1 & 1 \\ 5 & 4 & 8 & 4 & 4 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 5 & 4 & 0 & 0 & 0 \end{bmatrix}.$$

So a basis for Im(A) is $\{(1, 3, 5), (0, 1, 4)\}$.

Part (b). From part (c) below together with the rank-nullity formula, null$(A) = 5 -$ rank$(A) = 3$.

Part (c). From part (a), rank$(A) = 2$.

2: Part (a). Consider the 16 subsets of S. Plainly, the 1 subset of size 0, the 4 subsets of size 1 and the 6 subsets of size 2 are independent. It is also clear that the 1 subset of size 4 is not independent. It remains to consider the subsets of size 3.

Let $a = (2, 2, -4)$, $b = (1, -2, 1)$, $c = (3, 1, -4)$, $d = (-1, 7, 6)$. For a, b, c but not d, the sums of the coordinates are 0. Therefore $\{a, b, c\}$ is not independent, while the other 3 subsets of size 3 must be independent because d cannot be a linear combination of a, b, c. So the number of independent subsets is $1 + 4 + 6 + 3 = 14$.

Part (b). By part (a), span(S) has dimension 3. So the subsets spanning that subspace are $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$ and S. The number of those subsets is 4.

Part (c). By parts (a) and (b), only the subsets $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$ are bases for span(S). The number of those subsets is 3.

Comment 1: Another way of dealing with the subsets of size 3 is to consider the determinants of the associated matrices. The determinant of the matrix with rows a, b, c is

$$\begin{vmatrix} 2 & 2 & -4 \\ 1 & -2 & 1 \\ 3 & 1 & -4 \end{vmatrix} = 2\begin{vmatrix} -2 & 1 \\ 1 & -4 \end{vmatrix} - 2\begin{vmatrix} -1 & 1 \\ 1 & -4 \end{vmatrix} - 4\begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} = 14 + 14 - 28 = 0\,.$$

So $\{a, b, c\}$ is not independent. For the other 3 subsets of size 3, the determinant is non-zero. So those other 3 subsets are independent.

Comment 2: Yet another approach to part (a) is to consider the solutions to the equation $\alpha a + \beta b + \gamma c + \delta d = 0$. The matrix for that equation is shown on the left. Adding multiples of the first row to the other two rows, we obtain the matrix in the middle. Adding row 2 to row 3 gives the matrix on the right.

$$\begin{bmatrix} 2 & 1 & 3 & -1 \\ 2 & -2 & 1 & 7 \\ -4 & 1 & -4 & 6 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 & 3 & -1 \\ 0 & -3 & -2 & 8 \\ 0 & 3 & 2 & 4 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 & 3 & -1 \\ 0 & -3 & -2 & 8 \\ 0 & 0 & 0 & 12 \end{bmatrix}.$$

We see that $\delta = 0$, but γ is arbitrary, while α and β are determined by γ. Therefore, the three subsets of S with size 3 that include d must be independent, whereas the subset $\{a, b, c\}$ is not independent.

3: We have $\det(B - \lambda I) = (1 - \lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda + 1)(\lambda - 3)$. So the eigenvalues are 3 and -1.

4: The necessary and sufficient condition is that $y = 0$. Indeed, suppose that $y = 0$. Given $x, x' \in X$ and $\lambda, \lambda' \in \mathbb{R}$, then $A(\lambda x + \lambda' x') = \lambda Ax + \lambda' Ax' = 0$. We deduce that X is a subspace. So $\lambda x + \lambda' x' \in X$. Conversely, if $y \neq 0$, then $0 \notin X$, so X is not a subspace.

Comment 3: If $y = 0$ then X is the null space of A.

MATH 227: Introduction to Linear Algebra. Spring 2019. Final

LJB, 31 May 2019, Bilkent University.

Time allowed: 2 hours. Please put your name on EVERY sheet of your manuscript. The use of telephones, calculators or other electronic devices is prohibited. The use of red pens or very faint pencils is prohibited too. You may take the question sheet home.

Remember to justify your answers, except in any cases where your answers are obvious.

1: 15 marks. Consider the matrix $A = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) Does there exist a vector $y \in \mathbb{R}^3$ such that there are no solutions $x \in \mathbb{R}^3$ to the equation $Ax = y$? If so, give an example of such a vector y.

(b) Does there exist a vector $y \in \mathbb{R}^3$ such that there is exactly 1 solution $x \in \mathbb{R}^3$ to the equation $Ax = y$? If so, give an example of such a vector y.

(c) Does there exist a vector $y \in \mathbb{R}^3$ such that there are infinitely many solutions $x \in \mathbb{R}^3$ to the equation $Ax = y$? If so, give an example of such a vector y.

2: 35 marks. Let A be as in Question 1.

(a) Find the eigenvalues of A.

(b) For each of the eigenvalues, find a corresponding eigenvector.

(c) Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

(d) Without doing very long calculations, evaluate A^{50}.

3: 30 marks. A motorbike has two states: in state 1, there is a rider and a passenger; in state 2, there is only a rider, no passenger. For each kilometer travelled, the transition probability from state 1 to state 2 is 1/2, and the transition probability from state 2 to state 2 is 1.

(a) Write down the Markov matrix M for the system.

(b) Find an invertible matrix Q and a diagonal matrix E such that $M = QEQ^{-1}$.

(c) Using part (b) and a consideration of the matrix E^n, evaluate M^n, where n is any given positive integer.

(d) Explain how, by thinking about the context of application, you could have evaluated M^n without doing any calculation.

4: 20 marks: Let $u_1 = (3, 4)$ and $u_2 = (2, 1)$. Apply the Gram–Schmidt process to the basis $\{u_1, u_2\}$ of $\mathbb{R}^2$ to find an orthonormal basis $\{v_1, v_2\}$, (that is, a basis $\{v_1, v_2\}$ such that $\langle v_1 \,|\, v_1 \rangle = \langle v_2 \,|\, v_2 \rangle = 1$ and $\langle v_1 \,|\, v_2 \rangle = 0$).

Solutions to Final

1: Plainly, $\det(A) = 0$. So the answers to (a), (b), (c) are, in order: yes, no, yes. For part (a), put $y = (0, 0, 1)$. For part (c), put $y = (0, 0, 0)$.

Comment: This easy question is just to check that the candidate has a basic grasp of the significance of invertibility of a square matrix.

2: Part (a). By inspection, the characteristic equation is $0 = \det(A - \lambda I) = (-1 - \lambda)\lambda(1 - \lambda)$. So the eigenvalues are -1 and 0 and 1.

Part (b). By inspection, the eigenvalues -1 and 0 and 1 are associated with eigenvectors $(1, 0, 0)$ and $(1, 1, 0)$ and $(1, 1, 1)$, respectively.

Part (c). By parts (a) and (b), we can put $P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Part (d), $A^{50} = PD^{50}P^{-1} = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

3: Part (a), $M = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix}$.

Part (b). The characteristic equation is $0 = \det(M - \lambda I) = (1/2 - \lambda)(1 - \lambda)$. So the eigenvalues are $1/2$ and 1. For eigenvalue $1/2$, the eigenvectors (x, y) are given by $x = -y$, so we can put $(x, y) = (1, -1)$. For eigenvalue 1, the eigenvectors (x, y) are given by $x = 0$, so we can put $(x, y) = (0, 1)$. Those eigenvectors yield $Q = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ and $E = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$.

Part (c). We have

$$M^n = QD^nQ^{-1} = Q \begin{bmatrix} 1/2^n & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/2^n & 0 \\ -1/2^n & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/2^n & 0 \\ 1 - 1/2^n & 1 \end{bmatrix} .$$

Part (d). After n kilometers, the probability of having retained the passenger is $1/2^n$.

4: We have $\| u_1 \|^2 = 3^2 + 4^2 = 5^2$. So $v_1 = (3, 4)/5$. Put $w_2 = u_2 - \langle v_1 \,|\, u_2 \rangle v_1$, which is orthogonal to v_1. Then

$$w_2 = (2, 1) - \frac{3.2 + 4.1}{5^2}(3, 4) = (10, 5)/5 - (6, 8)/5 = (4, -3)/5 .$$

Normalizing, $v_2 = w_2 / \| w_2 \| = w_2 = (4, -3)/5$. In conclusion, $\{v_1, v_2\} = \{(3, 4)/5, (4, -3)/5\}$.