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MATH 227, Introduction to Linear Algebra

Bilkent University, Spring 2017, Laurence Barker

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Course specification

MATH 227, Introduction to Linear Algebra, Spring 2017

Laurence Barker, Bilkent University. Version: 27 April 2017.

Course Aims: To aquire practical knowledge and skill in some important techniques of linear algebra, and to aquire theoretical understanding and ability to critically assess methods and propositions in the area.

Course Description: This is an introductory course with an emphasis on methods of calculation, but with a theoretical grounding that is self-contained and complete. Some victory conditions, for the student, include: an understanding of the notion of a vector space as something more than just a system of coordinates; an ability to apply the method of diagonalization, a clear grasp of the theory behind that method.

Course Requirements: An official prerequisite in the University Catalogue is MATH 106 but, in practice, no knowledge of calculus is needed.

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Main course text: Howard Anton, Chris Rorres, "Elementary Linear Algebra", 11th edition, Wiley, 2011, international student edition 2015. See STARS for some other recommended texts.

Classes: All classes in room V 03. Section 1: Tuesdays 09:40 - 10:30, Thursdays 10:40 - 12:30. Section 2: Tuesdays 13:40 - 14:30, Thursdays 15:40 - 17:30.

Office Hours: Tuesdays, 08:40 - 09:30, my office SA-129; Tuesdays 14:40 - 15:30, starting in the classroom, moving to my office upon diminishment of apparent demand.

If you are having difficulty with the course, then you must come to see me. One major cause of difficulty is having done insufficient work earlier in the semester, then finding that one cannot understand anything much. That is perfectly normal, in fact, exam scripts show that it always happens to at least a few students. Seeking help and trying to catch up is better than just accepting defeat.

Syllabus: The format below is, *Week number; Monday date; Subtopics and textbook section numbers.* The numbering $m.n$ indicates Chapter m Section n in the Anton–Rorres textbook.

1: 6 Feb: Systems of linear equations, 1.1. Sketch of Markov chain scenario and the method of diagonalization.

2: 13 Feb: Gaussian and Gauss–Jordan elemination 1.2. Matrics 1.3.

3: 20 Feb: Elementary matrices, inversion of matrices by row operations 1.4, 1.5. Existence and uniqueness of solutions, 1.6.

4: 27 Feb: Determinants, their algebraic properties, their evaluation by row reduction and by cofactor expansion, 2.1, 2.2, 2.3.

5: 6 Mar: Euclidian spaces, norm, dot product, distance, angle, 3.1, 3.2.

6: 13 Mar: Pearson correlation coefficient (special notes). Markov chains, 10.5.

7: 20 Mar: Real vector spaces, subspaces, 4.1, 4.2.

8: 27 Mar: Linear independence, spanning, bases, dimension, 4.3, 4.4, 4.5.

9: 3 Apr: Linear transformations, composition, 8.1, 8.3.

10: 10 Apr: Change of basis, 4.6.

11: 17 Apr: Row and column spaces, Rank-Nullity Formula, 4.7, 4.8.

12: 24 Apr: Complex vector spaces, eigenvalues and eigenvectors.

13: 1 May: Diagonalization, 5.2. Applications to Markov chains, 10.5.

14: 8 May: Inner product spaces, Gram–Schmidt orthogonalization, 6.1, 6.2, 6.3.

Our last class is on Thursday 11th May.

Assessment:

- Quizzes, Homework and Participation 20%.
- Midterm I, 20%, Thursday 9th March.
- Midterm II, 20%, Friday 28th April.
- Final, 40%, Thursday, 25 May.

75% attendance is compulsory. Attendance will be assessed through quiz returns.

Class Announcements: All students, including any absentees from a class, will be deemed responsible for awareness of class announcements.

Midterm 1 Exam Syllabus

Solving linear equations, $[\ |]$. Gaussian elimination, Gauss–Jordan elimination. 1.1, 1.2.

Inverting matrices by Gauss–Jordan method, $[\ |\ ]$. 1.3, 1.4, 1.5.

Determinants and inverses by cofactor method, $|\ |$. 1.6, 2.1, 2.2, 2.3.

Practice Midterm 1 Exam

Time allowed: 110 minutes. Please put your name on EVERY sheet of your manuscript. The use of telephones, calculators or other electronic devices is prohibited. The use of red pens or very faint pencils is prohibited too. You may take the question sheet home.

Remember to justify your answers, except in any cases where your answers are obvious.

1: 15 marks. Using Gaussian elimination, solve the equations

$$x + 2y + 3z = 4\,, \qquad 5x + 6y + 7z = 8\,, \qquad 9x + 10y + 19z = 20\,.$$

2: 20 marks. Consider the matrix $A = \begin{bmatrix} 0 & 8 & 15 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$.

(a) Using the Gauss–Jordan method, find A^{-1}.
(b) Using your answer to part (a), solve the equations

$$8y + 15z = 6x + 5y + 4z = 3x + 2y + z = 1\,.$$

3: 25 marks. **(a)** Using the method of cofactors, find A^{-1} where A is as in Question 2.

(b) Using the method of cofactors, find the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

(c) Is the matrix $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ invertible? If so, what is the inverse?

4: 20 marks. In this question, you may use the fact that, given an $n \times n$ matrix A, then A is invertible if and only if, for any n-dimensional vector y, there exists an n-dimensional x satisfying $y = Ax$. Let B be another $n \times n$ matrix.
(a) Without using the theory of determinants, show that AB is invertible if and only if A is invertible and B is invertible. (There are no marks for any argument using determinants.)
(b) Supposing that A and B are invertible, express $(AB)^{-1}$ in terms of A^{-1} and B^{-1}.
(c) Let D be a 3×4 matrix (that is, with 3 rows and 4 columns). Explain why there exists a non-zero vector x such that $Dx = 0$.
(d) For D as in part (c), explain why there does not exist a 4×3 matrix C such that CD is invertible.

5: 20 marks. Consider the matrix $A_s = \begin{bmatrix} s & 1 & 1 \\ 1 & s & 1 \\ 1 & 1 & s \end{bmatrix}$.
(a) Find two different numbers α and β such that $\det(A_\alpha) = 0$ and $\det(A_\beta) = 0$.
(b) Show that if $\det(A_s) = 0$, then $s = \alpha$ or $s = \beta$.
(c) Show that, given any x, y, z, then there exist a, b, c, d, e, f such that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} d \\ e \\ f \end{bmatrix}, \qquad A_\alpha \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0\,, \qquad A_\beta \begin{bmatrix} c \\ d \\ e \end{bmatrix} = 0\,.$$

Solutions to Practice Midterm 1 Exam

A student with a satisfactory level of competence, on target for a grade C, might produce a script with: Solution 1, right method, wrong answer, with just one small arithmetical mistake in the calculations (13 marks); Solution 2, fully correct; Solution 3, parts (a) and (b), roughly the correct method, but with two systematic mistakes, forgetting to divide by the determinant and forgetting to take the transpose matrices; Solution 3, part (c), just an answer "yes" or "no" (zero marks either way) with no explanation and no proposed inverse; Solution 4, nothing worth marks, except for an unclear but partially correct explanation in part (c); Solution 5: nothing worth marks, except for working out that $\det(A_s) = s(s^2-1)+2-2s$ in part (a).

Not having given this particular course before, I do not feel able to indicate a criterion for grade A or $A-$.

1: The system of equations is expressed by $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 19 & 20 \end{array}\right]$.

The row operations $r_2' = r_2 - 5r_1$ and $r_3' = r_3 - 9r_1$ give $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -8 & -16 \end{array}\right]$.

Then $r_2' = -r_2/4$ and $r_3' = -r_3/8$ give $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 \end{array}\right]$.

Two more row operations now quickly give $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]$.

We have reduced the system to the equations

$$x + 2y + 3z = 4 \;, \qquad y + 2z = 3 \;, \qquad z = 1 \;.$$

We have $y = 3 - 2z = 3 - 2 = 1$ and $x = 4 - 2y - 3z = 4 - 2 - 3 = -1$. We have obtained the answer $(x, y, z) = (-1, 1, 1)$.

Comment 1: Always, when a question demands a clear answer, you should clearly state the answer as the first line or as the last line of your solution. If the reader has to hunt through your reasoning or calculations to find the answer, then maybe you know a calculation routine but do not know what the answer is. Besides, that forces the reader to complete the work by carrying out a hunting expedition. So, if you if you hide the answer in the middle or just leave the answer implicit, you may lose marks.

2: The system can be expressed as $\left[\begin{array}{ccc|ccc} 0 & 8 & 15 & 1 & 0 & 0 \\ 6 & 5 & 4 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$. We obtain $\left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 0 & 0 & 1 \\ 6 & 5 & 4 & 0 & 1 & 0 \\ 0 & 8 & 15 & 1 & 0 & 0 \end{array}\right]$ and $\left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & -2 \\ 0 & 8 & 15 & 1 & 0 & 0 \end{array}\right]$ by interchanging two rows, then applying $r_2' = r_2 - 2r_1$. Then $r_3' = r_3 - 8r_2$ followed by $r_3' = -r_3$ yield $\left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & -2 \\ 0 & 0 & 1 & -1 & 8 & -16 \end{array}\right]$. We get $\left[\begin{array}{ccc|ccc} 3 & 2 & 0 & 1 & -8 & 17 \\ 0 & 1 & 0 & 2 & -15 & 30 \\ 0 & 0 & 1 & -1 & 8 & -16 \end{array}\right]$ by subtracting row 3 from row 1 and twice row 3 from row 2.

Finally, $r_1' = r_1 - 2r_2$ and then $r_1' = r_1/3$ give $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 22/3 & -43/3 \\ 0 & 1 & 0 & 2 & -15 & 30 \\ 0 & 0 & 1 & -1 & 8 & -16 \end{array}\right]$. So the answer is $A^{-1} = \begin{bmatrix} -1 & 22/3 & -43/3 \\ 2 & -15 & 30 \\ -1 & 8 & -16 \end{bmatrix}$.

Part (b). We have $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 + 22/3 - 43/3 \\ 2 - 15 + 30 \\ -1 + 8 - 16 \end{bmatrix} = \begin{bmatrix} -8 \\ 17 \\ -9 \end{bmatrix}$.

Comment 2: If you miss out the last sentence of part (a), then you deserve to lose one mark. After all, it is possible that someone may know how to do the calculation routine but does not know how to read off what A^{-1} actually is.

But I think the above solution to part (b) is okay. It gives the answer clearly in one line.

3: Part (a). The cofactor matrix is $\begin{bmatrix} 5-8 & 12-6 & 12-15 \\ 30-8 & 0-45 & 24-0 \\ 32-75 & 90-0 & 0-48 \end{bmatrix} = \begin{bmatrix} -3 & 6 & -3 \\ 22 & -45 & 24 \\ -43 & 90 & -48 \end{bmatrix}$.

For instance, we calculated the (1, 2) entry from $(-1)^{1+2} \begin{vmatrix} 6 & 4 \\ 3 & 1 \end{vmatrix} = 3 \times 4 - 6 \times 1 = 12 - 6$. So $\det(A) = 0 \times (-3) + 8 \times 6 + 15 \times (-3) = 0 + 48 - 45 = 3$. Taking the transpose of the cofactor matrix and dividing by the determinant, we obtain

$$A^{-1} = \frac{1}{3} \begin{bmatrix} -3 & 22 & -43 \\ 6 & -45 & 90 \\ -3 & 24 & -48 \end{bmatrix} = \begin{bmatrix} -1 & 22/3 & -43/3 \\ 2 & -15 & 30 \\ -1 & 8 & -16 \end{bmatrix}.$$

Part (b). The cofactor matrix is $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$. The determinant of the given matrix, via the top row, is $1 \times 1 + 1 \times 0 + 1 \times 0 = 1$. So the transpose of the cofactor matrix is

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Part (c). Yes, the given matrix is invertible. Indeed, by inspection, it has inverse

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Comment 3: Note that there is no need in part (c) to use any standard method. One can simply guess the answer from the pattern in part (b). As soon as we have made the right guess, it is easy to see that it is correct. There is no logical rule saying that the answer has to be worked out in some routine systematic way.

4: Part (a). Suppose A and B are invertible. Then

$$AB(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I\ .$$

So AB is invertible with inverse $B^{-1}A^{-1}$.

Conversely, suppose AB is invertible. To show that A is invertible, we must show that, given x, there is a y such that $x = Ay$. Since AB is invertible, there exists a z such that $x = ABz$. Put $y = Bz$. Then $x = Ay$. We have deduced that A is invertible.

We have $B(AB)^{-1}A = BB^{-1}A^{-1}A = I$. So B is invertible and $B^{-1} = (AB)^{-1}A$.

Part (b). As we saw in part (a), $(AB)^{-1} = B^{-1}A^{-1}$.

Part (c). Write

$$D = \begin{bmatrix} d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} \\ d_{2,1} & d_{2,2} & d_{2,3} & d_{2,4} \\ d_{3,1} & d_{3,2} & d_{3,3} & d_{3,4} \end{bmatrix}, \qquad D' = \begin{bmatrix} d_{1,1} & d_{1,2} & d_{1,3} \\ d_{2,1} & d_{2,2} & d_{2,3} \\ d_{3,1} & d_{3,2} & d_{3,3} \end{bmatrix}.$$

If D' is invertible, then there exists a vector (x_1, x_2, x_3) such that

$$D' \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} d_{1,4} \\ d_{2,4} \\ d_{3,4} \end{bmatrix}.$$

In that case, we put $x = (x_1, x_2, x_3, 1)$. On the other hand, if D' is not invertible, then there exists a vector (ξ_1, ξ_2, ξ_3) such that

$$D' \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} d_{1,4} \\ d_{2,4} \\ d_{3,4} \end{bmatrix}$$

in which case, we put $x = (\xi_1, \xi_2, \xi_3, 0)$. Either way, $Dx = 0$.

Part (d). If such a C were to exist, then we would have $0 = (CD)^{-1}CDx = x \neq 0$, which is impossible.

Comment 4.1: A marking dilemma: when explaining something, we break it up into obvious steps. So, if something is already obvious, no explanation is needed.

But how do we judge what is obvious and what is not obvious? If you listen to an expert mathematician trying to explain something to another expert, you will quickly see that this is a grey area. Frequently, one will assert that something is obvious, the other will demand an explanation, whereupon the first will attempt to do so, sometimes taking a long time over it.

Is it *obvious* that, when A and B are invertible, then $(AB)^{-1} = B^{-1}A^{-1}$? I am inclined to think that, at this stage in the course, it is not obvious, and that the quick argument in the first paragraph of part (a) is preferable. But this might be debatable. So if a script were to read "If A and B are invertible then $(AB)^{-1} = B^{-1}A^{1}$", then I would give full marks for that part.

But significantly harder deductions, say "If AB is invertible and A is invertible then B is invertible", full marks does require an explanation.

Comment 4.2: A quick way of doing part (c) would be to say "By considering row echelon form, we see that the equation $Dx = 0$ has solutions such that at least one of the coodinates can take any value." That argument — supplied, incidentally, by a student during the Office Hours before the Midterm — is not so easy to understand because it requires the reader to have some insight. But it is a good succinct argument, and I would give it full marks.

Another quick way of doing part (c) uses the Rank-Nullity Formula, which we will discuss later in the course.

5: Part (a). We have $\det(A_s) = s \begin{vmatrix} s & 1 \\ 1 & s \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & s \end{vmatrix} + \begin{vmatrix} 1 & s \\ 1 & 1 \end{vmatrix} = s(s^2-1) - (s-1) + (1-s)$. But $s^2 - 1 = (s+1)(s-1)$, so

$$\det(A_s) = (s-1)\big(s(s+1) - 2s + 2\big) = (s-1)(s^2 - s + 2) = (s-1)^2(s+2) \ .$$

Evidently, we can put $\alpha = 1$ and $\beta = -2$.

Part (b). This is clear from the equality $\det(A_s) = (s-1)^2(s+2)$.

Part (c). The condition on a, b, c is

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = A_1 \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b+c \\ a+b+c \\ a+b+c \end{bmatrix}$$

in other words, $a + b + c = 0$. The condition on d, e, f is

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = A_{-2} \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} -2d+e+f \\ d-2e+f \\ d+e-2f \end{bmatrix}$$

in other words, $-2d + e + f = d - 2e + f = d + e - 2f = 0$, which hold when $d = e = f$. So, for each given vector (x, y, z), it suffices to find a, b, c, d such that

$$x = a + d \ , \qquad y = b + d \ , \qquad z = c + d \ , \qquad a + b + c = 0 \ .$$

Adding the first three equations, then using the fourth, we obtain $d = (x+y+z)/3$, whereupon the first three equations yield $a = x - d = (2x - y - z)/3$ and similarly, $b = (-x + 2y - z)/3$ and $c = (-x - y + 2z)/3$. In conclusion, we have found a, b, c, d, e, f as required, where

$$a = (2x - y - z)/3 \ , \quad b = (-x + 2y - z)/3 \ , \quad c = (-x - y + 2z)/3 \ , \quad d = (x + y + z)/3 \ .$$

Comment 5.1: In the above, how did we find the solution $d = e = f$? As far as the deductions are concerned, there is no need to say. The argument is deductively sound as it stands. In a deductive argument, we do not have to explain how we thought up our clever ideas, "I was watching a wasp flying in a figure eight pattern, and suddenly I remembered once observing a somersaulting rabbit...".

We solved for d, e, f simply by applying the usual method: $\left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$,
whence $r_1' = r_3$ and $r_3' = r_1$ gives $\left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ -2 & 1 & 1 & 0 \end{array}\right]$,
whereupon $r_2' = r_2 - r_1$ and $r_3' = r_3 + 2r_1$ give $\left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right]$,
which easily reduces to echelon form $\left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$,
in other words, $d + e - 2f = 0$ and $e - f = 0$. We find that $d = e = f = 0$.

Comment 5.2: Behind this question, there is a standard routine called *diagonalization*, which we will be discussing towards the end of the course.

Syllabus and Notes for Midterm 2

- Pearson correlation coefficient (notes below).
- Real vector spaces and subspaces, textbook sections 4.1, 4.2.
- Linear independence, spanning, bases, dimension, sections 4.3, 4.4, 4.5.
- Markov chains, 4.12.
- Linear transformations, in first part of section 8.1.

Some notes on the Pearson correlation coefficient

The following theory, and some examples, were given in class. I supply this summary because the topic is not in the textbook.

Let x_1, ..., x_n be sampled values of some variable, and let y_1, ..., y_n be the corresponding sampled values of some other variable. The **Pearson correlation coefficient** ρ of the two samples is calculated as follows:

Step 1: Calculate the average values

$$\overline{x} = (x_1 + ... + x_n)/n \ , \qquad \overline{y} = (y_1 + ... + y_n)/n \ .$$

Step 2: Calculate the centred vectors

$$\widetilde{x} = (\widetilde{x}_1, ..., \widetilde{x}_n) = (x_1 - \overline{x}, ..., x_n = \overline{x}) \ , \qquad \widetilde{x} = (\widetilde{x}_1, ..., \widetilde{x}_n) = (x_1 - \overline{x}, ..., x_n = \overline{x}) \ .$$

Step 3: Calculate $\rho = \dfrac{\widetilde{x}_\bullet \widetilde{y}}{\| \widetilde{x} \| \| \widetilde{y} \|}$ where

$$\widetilde{x}_\bullet \widetilde{y} = \widetilde{x}_1 \widetilde{y}_1 + ... + \widetilde{x}_n \widetilde{y}_n \ , \qquad \| \widetilde{x} \| = \sqrt{\widetilde{x}_1^2 + ... + \widetilde{x}_n^2} \ .$$

Let us give a small example problem. For another example, see Question 2 in the Practice Midterm 2.

Problem: Snails A, B, C weigh 3, 3, 6 kilograms, respectively. The maximum speeds of A, B, C are 3, 4, 8 meters per second, respectively. Let ρ be the correlation coefficient for these samples of weights and speeds. Show that $\rho > 97/100$.

Solution: The given data is $(x_1, x_2, x_3) = (3, 3, 6)$ and $(y_1, y_2, y_3) = (3, 4, 8)$. The averages are $\overline{x} = (3 + 3 + 6)/3 = 4$ and $\overline{y} = (3 + 4 + 8)/3 = 5$. The centred vectors are $\widetilde{x} = (-1, -1, 2)$ and $\widetilde{y} = (-2, -1, 3)$. So $\| \widetilde{x} \|^2 = 1 + 1 + 4 = 6$ and $\| \widetilde{y} \|^2 = 4 + 1 + 9 = 14$. So

$$\rho = \frac{2 + 1 + 6}{\sqrt{6}\sqrt{14}} = \frac{9}{2\sqrt{21}} \ .$$

It follows that $\rho^2 = 81/84 > 96/100$, hence $\rho > 97/100$. $\square$

Practice Midterm 2 Exam

Time allowed: 110 minutes. Please put your name on EVERY sheet of your manuscript. The use of telephones, calculators or other electronic devices is prohibited. The use of red pens or very faint pencils is prohibited too. You may take the question sheet home.

1: 25 marks. Let $S = \{(1,0,0,0), (1,1,1,1), (1,2,4,8), (1,3,9,27)\}$ in $\mathbb{R}^4$.

(a) Show that S is linearly independent by directly showing that the vectors in S do not satisfy any non-trivial linear relation.

(b) For which vectors v in $\mathbb{R}^4$ do there exist real numbers a, b, c, d such that

$$v = a(1,0,0,0) + b(1,1,1,1) + c(1,2,4,8) + d(1,3,9,27)\ ?$$

2: 20 marks. Let X be a variable with sample values 3, 3, 3, 1, 5 and let Y be a variable with corresponding sample values 1, 3, 4, 3, 9. Calculate the Pearson correlation coefficient for the samples.

3: 25 marks. A machine has a rotating wheel. Every time a engineer hits the machine with a hammer, the wheel has a 1/2 probability of not moving, and a 1/2 probability of rotating 1/4 of a revolution clockwise,

(a) If the engineer hits the machine 5 times, what is the probability of the wheel turning through an angle of 180 degrees?

(b) If the engineer hits the machine 6 times, what is the probability of the wheel turning through a total angle of 180 degrees, not including the case of a turn through a total angle of 540 degrees?

4: 20 marks. Let U and V be vector spaces over $\mathbb{R}$ and let α be a linear map $U \to V$.

(a) Suppose there exists a function $\beta : V \to U$ such that $\beta(\alpha(u)) = u$ and $\alpha(\beta(v)) = v$ for all $u \in U$ and $v \in V$. Show that β is a linear map.

(b) Now suppose only that there exists a function $\gamma : V \to U$ such that $\gamma(\alpha(u)) = u$ for all $u \in U$. Give a single clear example to show that γ does not need to be a linear map. (Warning: do not give a vague general discussion of how something might go wrong. Just give one particular example where such γ is clearly not linear.)

5: 10 marks. Let U be a subspace of a finite-dimensional vector space V over $\mathbb{R}$. Show that there exists a basis B of V such that B contains a basis of U.

Solutions to Practice Midterm 2 Exam

1: Part (a). Write $\alpha(1,0,0,0)+\beta(1,1,1,1)+\gamma(1,2,4,8)+\delta(1,3,9,27)=0$. Then

$$\alpha+\beta+\gamma+\delta = \beta+2\gamma+3\delta = \beta+4\gamma+9\delta = \beta+8\gamma+27\delta = 0 \ .$$

Performing row operations on the matrix of this system of equations, we obtain

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 9 \\ 0 & 1 & 8 & 27 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 6 & 24 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

The row operations were: subtraction of a higher row from a lower row; multiplication of a row by a non-zero factor. From the echelon form of the last matrix, we see that the only solution is $\alpha=\beta=\gamma=\delta=0$. So S is independent.

Part (b). Since S is independent and $|S| = 4 = \dim(\mathbb{R}^4)$, it follows that S is a basis. In particular, every vector $v \in \mathbb{R}^4$ can be written in the specified form.

2: The coordinates of the vectors $x = (3,3,3,1,5)$ and $y = (1,3,4,3,9)$ have average values $\overline{x} = (3+3+3+1+5)/5 = 15/5 = 3$ and $\overline{y} = (1+3+4+3+9)/5 = 30/5 = 4$. The centred vectors are $\widetilde{x} = (0,0,0,-2,2)$ and $\widetilde{y} = (-3,-1,0,-1,5)$. Therefore

$$\rho = \frac{\widetilde{x}.\widetilde{y}}{\|\widetilde{x}\|.\|\widetilde{y}\|} = \frac{0+0+0+2+10}{\sqrt{(0+0+0+4+4)(9+1+0+1+25)}} = \frac{12}{\sqrt{8\times 36}} = 1/\sqrt{2} \ .$$

3: Let states 1, 2, 3, 4 be the states where the total angle of rotation of the wheel is an integer multiple of 2π plus 0, $\pi/2$, π, $3\pi/2$ due clockwise. The Markov matrix M satisfies

$$M = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} , \quad M^2 = \frac{1}{4}\begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix} , \quad M^4 = \frac{1}{16}\begin{bmatrix} 2 & 4 & 6 & 4 \\ 4 & 2 & 4 & 6 \\ 6 & 4 & 2 & 4 \\ 4 & 6 & 4 & 2 \end{bmatrix} .$$

Part (a). The answer is the $(3,1)$-entry of the matrix $M^5 = M^4.M$, that is, $\dfrac{6+4}{16\times 2} = \dfrac{5}{16}$.

Part (b). The probability of ending in state 3 after 6 hits is the $(3,1)$-entry of the matrix $M^6 = M^4.M^2$. The value of that entry is $(6+8+2)/(16\times 4) = 16/64$. But the probability of arriving in state 3 after a rotation through 3π is $1/64$. So the answer is $(16-1)/64 = 15/64$.

Comment: Those who are familiar with enumeration using binomial coefficients will see a more direct way of doing this problem: in particular, the answer to part (b) is $\dfrac{1}{2^6}\dbinom{6}{2}$.

4: Part (a). For all $\lambda, \lambda' \in \mathbb{R}$ and $v, v' \in V$, we have

$$\beta(\lambda v + \lambda' v') = \beta(\lambda\alpha(\beta(v)) + \lambda'\alpha(\beta(v'))) = \beta(\alpha(\lambda\beta(v) + \lambda'\beta(v'))) = \lambda\beta(v) + \lambda'\beta(v') \ .$$

Part (b). Put $U = \mathbb{R} = \mathbb{R}^1$ and $V = \mathbb{R}^2$. Put $\alpha(x) = (x,0)$ and $\gamma(x,y) = x + y^2$.

5: Let $\{u_1, ..., u_m\}$ be a basis for U. Let $\{v_1, ..., v_n\}$ be a basis for V. By the Replacement Lemma, the second basis can be renumbered such that the set $B = \{u_1, ..., u_m, v_{m+1}, ..., v_n\}$ spans V. But $|B| = n = \dim(V)$, hence B is a basis for V. Furthermore, B contains the given basis of U.

Homeworks and Quizzes

MATH 227, *Introduction to Linear Algebra*, Spring 2017

Laurence Barker, Mathematics Department, Bilkent University,
version: 12 May 2017.

Office Hours: Tuesdays, 08:40 – 09:30, 14:40 - 15:30.

Homeworks

Most of the questions are taken from the primary textbook: Howard Anton, Chris Rorres, “Elementary Linear Algebra, 11th Edition, 2011 (reprinted 2015).

Homework 1 due Thursday, 23 February.

From Chapter 1 Exercises, page 75.

1.4: For each of the four parts of this question, determine whether the system of equations is linear. When it is linear, determine whether a solution exists.
(a) $-2x + 4y + z = 2$, $3x - 2/y = 0$.
(b) $x = 4$, $2x = 8$.
(c) $4x - y + 2z = -1$, $-x + (\ln 2)y - 3z = 0$.
(d) $3z + x = -4$, $y + 5z = 1$, $6x + 2z = 3$, $-x - y - z = 4$.

1.6: Write down a system of linear equations consisting of three equations in three unknowns with: **(a)** no solutions; **(b)** exactly one solution; **(c)** infinitely many solutions.

1.24: Solve the following linear system by Gaussian elimination:

$$2x_1 + 2x_2 + 2x_3 = 4 , \qquad -2x_1 + 5x_2 + 2x_3 = 1 , \qquad 8x_1 + x_2 + 4x_3 = 11 .$$

1.26: Solve the following linear system by Gaussian elimination:

$$-2y + 3x = 3 , \qquad 3x + 6y - 3z = -2 , \qquad 6x + 6y + 3z = 4 .$$

1.52: Suppose that $x = x' \cos(\theta) - y' \sin(\theta)$ and $y = x' \sin(\theta) + y' \cos(\theta)$. Express x' and y' in terms of x and y,
(a) using Gauss–Jordan elimination,
(b) optionally, for zero further marks, by any other reasoning.

Homework 2 due Tuesday, 7 March. (Midterm 1 is on Thursday 9th March.)

From Chapter 1 and Chapter 2 Exercises.

1.142: Find the inverse, if it exists, of the matrix $\begin{bmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{bmatrix}$.

1.150: Find the inverse, if it exists, of the matrix $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 3 & 0 \\ 2 & 1 & 4 & -3 \end{bmatrix}$.

2.20: Evaluate, by a cofactor expansion along a row or column of your choice, the determinant shown on the right.

$$\begin{vmatrix} 2 & 2 & 1 & 0 \\ -1 & 0 & 3 & 0 \\ 4 & 9 & 3 & 1 \\ 0 & -1 & 5 & 7 \end{vmatrix} .$$

2.58: Without doing any calculation, give a simple one-sentence explanation why, for all α, β, γ, δ, we have

$$\begin{vmatrix} \sin\alpha & \cos\alpha & \sin(\alpha+\delta) \\ \sin\beta & \cos\beta & \sin(\beta+\delta) \\ \sin\gamma & \cos\gamma & \sin(\gamma+\delta) \end{vmatrix} = 0 .$$

Homework 3 due Thursday, 27 April.

1 (4.68 in textbook): Which of the following sets of vectors are bases for $\mathbb{R}^3$? (Remember to justify your answers.)
(a) $\{(1,0,0),(2,2,0),(3,3,3)\}$.
(b) $\{(3,1,-4),(2,5,6),(1,4,8)\}$.
(c) $\{(2,-3,1),(4,1,1),(0,-7,1)\}$.
(d) $\{(1,6,4),(2,4,-1),(-1,2,5)\}$.

2 (4.78 in textbook, modified): Consider the matrices

$$A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} , \quad A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} , \quad A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} , \quad A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} , \quad A_4 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} .$$

(a) Express A as a linear combination of A_1, A_2, A_3, A_4.
(b) Recall, $\mathrm{Mat}_2(\mathbb{R})$ denotes the vector space whose vectors are the 2×2 matrices over $\mathbb{R}$. Show that $\{A_1, A_2, A_3, A_4\}$ is a basis for $\mathrm{Mat}_2(\mathbb{R})$.

3: Recall, the axiomatic conditions for the addition operation of a vector space V over $\mathbb{R}$ are as follows:

Associativity: For all $u, v, w \in V$, we have $(u + v) + w = u + (v + w)$.
Commutativity: For all $u, v \in V$, we have $u + v = v + u$.
Existence of Zero: There exists a vector $0 \in V$, called the **zero vector** of V, such that, for all $u \in V$, we have $0 + u = u$.
Existence of Negative: For each $v \in V$, there exists a vector $-v \in V$, called the **negative** of v, such that $v + (-v) = 0$.

You may make use of the fact, proved in class, that V has only one zero vector 0.

(a) Show that, for each vector $v \in V$, there is only one negative $-v$ of v.
(b) Show that, given vectors $u, v, w \in V$ satisfying $u + v = u + w$, then $v = w$.

4: Let ρ be the Pearson correlation coefficient for a set of samples of variables X and Y, the sample values for X being x_1, ..., x_n and the corresponding sample values for Y being y_1, ..., y_n. Suppose there exist real numbers a, b, c such that $ax_i + by_i + c = 0$ for all i in the range $1 \le i \le n$. Show that $|\rho| = 1$.

Section 1 Quizzes

Quiz 1: Consider the equations $x + ty = 3$ and $2x + 3y = 4$.
(a) Suppose $t = 2$. Find x and y.
(b) For which value of t is there no solution?

Quiz 2: By any method, invert $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

Quiz 3: Express $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ in the form $\begin{bmatrix} u \\ v \end{bmatrix}$.

Quiz 4: Let $\begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. Express $u^2 + v^2$ in terms of x and y.

Quiz 5: For variables X and Y, find the correlation coefficient ρ when the sample values for X are $x_1 = 3$, $x_2 = 4$, $x_3 = 5$ and the sample values for Y are $y_1 = 1$, $y_2 = 1$, $y_3 = 4$.

Quiz 6: Consider the set $S = \{(1, -1, 0), (-2, 2, 0), (3, 5, 7)\}$ in $\mathbb{R}^3$. Is S independent? Justify your answer in only one line.

Quiz 7: Each year, the market is either bullish or bearish. The probability of the market changing state, from one year to the next, is 1/3. In year 0, the market is bullish. What is the probability of the market being bullish in year 2?

Quiz 8: Give an example of a set $S = \{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}$ such that both of the following conditions hold: some subset of S is a basis for $\mathbb{R}^2$; some subset of S with size 2 is not a basis for $\mathbb{R}^2$.

Quiz 9: Find the eigenvalues and their multiplicities for the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

Section 2 Quizzes

Quiz 1: Find a value of a such that the system of equations

$$x + 2y + 3z = 2\ , \qquad 2x + 2y + 4z = 3\ , \qquad 3x + 4y + az$$

has infinitely many solutions.

Quiz 2: By any method, invert $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

Quiz 3: Find nonzero real $\begin{bmatrix} x \\ y \end{bmatrix}$ and real t such that $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} x \\ y \end{bmatrix}$.

Quiz 4: Let $\begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1/\sqrt{3} \\ 1/\sqrt{3} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. Express $u^2 + v^2$ in terms of x and y.

Quiz 5: Find a vector $v = (v_1, v_2)$ such that $\| v \| = 1$ and the angle θ between v and the vector $(1, 0)$ satisfies $\cos(\theta) = \sqrt{3}/2$.

Quiz 6: Consider the set $S = \{(5,6,0),(7,0,8),(12,6,8)\}$ in $\mathbb{R}^3$. Is S independent?

Quiz 7: How many subsets of the set

$$\{(1,1,1,1),(1,-1,0,0),(1,0,-1,0),(1,0,0,-1),(3,-1,-1,-1)\}$$

are bases for $\mathbb{R}^4$?

Quiz 8: Three rooms of a maze are arranged in a triangle, with three doors between them [as depicted in class]. Each minute a mouse has a 1/3 chance of moving clockwise, a 1/3 chance of staying put, a 1/3 chance of moving anticlockwise. What is the probability that, after 1024 minutes, the mouse is in a room different from the starting room?

Quiz 9: Find the eigenvalues and their multiplicities for the matrix $\begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}$.

MATH 227: Introduction to Linear Algebra. Midterm 1

LJB, 9 March 2017, Bilkent University.

Time allowed: 2 hours. Please put your name on EVERY sheet of your manuscript. The use of telephones, calculators or other electronic devices is prohibited. The use of red pens or very faint pencils is prohibited too. You may take the question sheet home.

Remember to justify your answers, except in any cases where your answers are obvious.

1: 15 marks. Using Gaussian elimination, solve the equations

$$2x + y + 4z = 3x + 5y + 7z = 6x + 9y + 8z = 1 \ .$$

2: 25 marks. Consider the matrix $A = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix}$.

(a) Using the Gauss–Jordan method, find A^{-1}.

(b) Check your answer to Question 1 using the matrix A^{-1}.

3: 25 marks. Let A be as in Question 2.

(a) Evaluate $\det(A)$.

(b) Check your answer to the first part of Question 2 by calculating A^{-1} using the cofactor method.

4: 20 marks. Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix such that, for all vectors $\begin{bmatrix} x \\ y \end{bmatrix}$, we have

$$(ax + by)^2 + (cx + dy)^2 = x^2 + y^2 \ .$$

(a) Show that $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \pm 1$, we mean to say, the determinant is 1 or -1.

(b) Simplify the expression $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$.

5: 15 marks. (Recall, the kind of numbers we have been working with, thus far in the course, are called the real numbers. Of course, they can be viewed as the points on the real number line. In this question, we continue to work with the real numbers, just as usual. The question does not involve any other kind of number, such as the imaginary numbers.)

(a) Let A be a 3×3 matrix (with entries in the real numbers, as usual). Show that there exists some t (a real number) and some nonzero vector x (whose coordinates are real numbers) such that $Ax = tx$.

(b) Does the conclusion still hold when we replace A with a 4×4 matrix?

Solutions to Midterm 1

1: The system is $\left[\begin{array}{ccc|c} 2 & 1 & 4 & 1 \\ 3 & 5 & 7 & 1 \\ 6 & 9 & 8 & 1 \end{array}\right]$. Operation $r_2' = 2r_2$ gives $\left[\begin{array}{ccc|c} 2 & 1 & 4 & 1 \\ 6 & 10 & 14 & 2 \\ 6 & 9 & 8 & 1 \end{array}\right]$. Then $r_2' = r_2 - 3r_1$ and $r_3' = r_3 - 3r_1$ give $\left[\begin{array}{ccc|c} 2 & 1 & 4 & 1 \\ 0 & 7 & 2 & -1 \\ 0 & 6 & -4 & -2 \end{array}\right]$. Then $r_2' = r_2 - r_3$ gives $\left[\begin{array}{ccc|c} 2 & 1 & 4 & 1 \\ 0 & 1 & 6 & 1 \\ 0 & 6 & -4 & -2 \end{array}\right]$. Applying the operation $r_3' = r_3 - 6r_2$ gives $\left[\begin{array}{ccc|c} 2 & 1 & 4 & 1 \\ 0 & 1 & 6 & 1 \\ 0 & 0 & -40 & -8 \end{array}\right]$.

So $z = 1/5$ and $y = 1 - 6z = 1 - 6/5 = -1/5$ and $2x = 1 - y - 4z = 1 + 1/5 - 4/5 = 2/5$, hence $x = 1/5$. In conclusion, $(x, y, z) = (1, -1, 1)/5$.

2: Part (a). Starting from $\left[\begin{array}{ccc|ccc} 2 & 1 & 4 & 1 & 0 & 0 \\ 3 & 5 & 7 & 0 & 1 & 0 \\ 6 & 9 & 8 & 0 & 0 & 1 \end{array}\right]$, the row operations in Question 1 yield

$\left[\begin{array}{ccc|ccc} 2 & 1 & 4 & 1 & 0 & 0 \\ 0 & 7 & 2 & -3 & 2 & 0 \\ 0 & 6 & -4 & -3 & 0 & 1 \end{array}\right]$, then $\left[\begin{array}{ccc|ccc} 2 & 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 6 & 0 & 2 & -1 \\ 0 & 6 & -4 & -3 & 0 & 1 \end{array}\right]$

then $\left[\begin{array}{ccc|ccc} 2 & 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 6 & 0 & 2 & -1 \\ 0 & 0 & -40 & -3 & -12 & 7 \end{array}\right]$. The operation $r_3' = -r_3/40$ yields

$\left[\begin{array}{ccc|ccc} 2 & 1 & 4 & 40/40 & 0 & 0 \\ 0 & 1 & 6 & 0 & 80/40 & -40/40 \\ 0 & 0 & 1 & 3/40 & 12/40 & -7/40 \end{array}\right]$. Then $r_1' = r_1 - 4r_3$ and $r_2' = r_2 - 6r_3$ give $\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 28/40 & -48/40 & 28/40 \\ 0 & 1 & 0 & -18/40 & 8/40 & 2/40 \\ 0 & 0 & 1 & 3/40 & 12/40 & -7/40 \end{array}\right]$, whence

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 23/40 & -28/40 & 13/40 \\ 0 & 1 & 0 & -18/40 & 8/40 & 2/40 \\ 0 & 0 & 1 & 3/40 & 12/40 & -7/40 \end{array}\right]$ using $r_1' = (r_1 - r_2)/2$.

In conclusion, $A^{-1} = \frac{1}{40}\begin{bmatrix} 23 & -28 & 13 \\ -18 & 8 & 2 \\ 3 & 12 & -7 \end{bmatrix}$.

Part (b), $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 23 - 28 + 13 \\ -18 + 8 + 2 \\ 3 + 12 - 7 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

3: Part (a). We have $\det(A) = \begin{vmatrix} 2 & 1 & 4 \\ 3 & 5 & 7 \\ 6 & 9 & 8 \end{vmatrix} = 2\begin{vmatrix} 5 & 7 \\ 9 & 8 \end{vmatrix} - \begin{vmatrix} 3 & 7 \\ 6 & 8 \end{vmatrix} + 4\begin{vmatrix} 3 & 5 \\ 6 & 9 \end{vmatrix}$

$= 2(40 - 63) - (24 - 42) + 4(27 - 30) = -46 + 18 - 12 = -40.$

Part (b). The cofactor matrix is $\begin{bmatrix} 40 - 63 & 42 - 24 & 27 - 30 \\ 36 - 8 & 16 - 24 & 6 - 18 \\ 7 - 20 & 12 - 14 & 10 - 3 \end{bmatrix} = \begin{bmatrix} -23 & 18 & -3 \\ 28 & -8 & -12 \\ -13 & -2 & 7 \end{bmatrix}$. Taking the transpose and dividing by $\det(A) = -40$ yields the value calculated for A^{-1} in Question 1.

4: Part (a). Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\Delta = \det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$. We are to show that $\Delta^2 = 1$. Write

$\begin{bmatrix} u \\ v \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$. Thus, $u = ax + by$ and $v = cx + dy$. The given assumption on A is that $u^2 + v^2 = x^2 + y^2$. Letting (x, y) take the values $(1, 0)$, $(0, 1)$, $(1, 1)$, respectively, then (u, v) takes the values (a, c), (b, d), $(a + b, c + d)$. By the given assumption,

$$1 = a^2 + c^2 = b^2 + d^2 \ , \qquad 2 = (a + b)^2 + (c + d)^2 \ .$$

Expanding the right-hand equation, then using the left-hand equations, we obtain

$$ab + cd = 0 \ .$$

Squaring, $0 = (ab + cd)^2 = a^2b^2 + c^2d^2 + 2abcd$. Meanwhile, $\Delta = ad - bc$. Therefore

$$\Delta^2 = a^2d^2 + b^2c^2 - 2abcd = a^2d^2 + b^2c^2 + a^2b^2 + c^2d^2 = (a^2 + c^2)(b^2 + d^2) = 1 \ .$$

Part (b). We have $\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} u \\ v \end{bmatrix} = \dfrac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \dfrac{1}{\Delta} \begin{bmatrix} du - bv \\ av - cu \end{bmatrix}$. Letting (u, v) take the values $(1, 0)$, $(0, 1)$, $(1, 1)$, respectively, then $\Delta(x, y)$ takes the values $(d, -c)$, $(-b, a)$, $(d - b, a - c)$. Arguing as before, again using two equations to modify a third, we have

$$1 = a^2 + b^2 = c^2 + d^2 \ , \qquad ac + bd = 0 \ .$$

Therefore $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Comment: The given assumption on A is that action by A preserves distances between points on the plane. The only matrices with that effect are rotations about the point $(0, 0)$ and reflections across a line passing through $(0, 0)$. An alternative solution, more conceptual, is based on that observation.

We can recover the observation algebraically as follows. Since $a^2 + c^2 = 1$, we have $0 \leq a^2 \leq 1$, hence $-1 \leq a \leq 1$. So there exists some θ such that $a = \cos(\theta)$. It follows that $c = \pm \sin(\theta)$. Replacing θ with $-\theta$ if necessary, we can choose θ such that $c = \sin(\theta)$. It is now not hard to show that

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \text{ or } A = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & -\cos(\theta) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} .$$

In the former of those two cases, A is a rotation, in the latter, a reflection.

5: Part (a). The equation $Ax = tx$ can be rewritten as $(A - tI)x = 0$. Given t, then there exists a nonzero such x if and only if $A - tI$ is non-invertible, in other words, $\det(A - tI) = 0$. Since A is a 3×3 matrix, there exist β, γ, δ, depending only on A, such that

$$\det(A - tI) = -t^3 + \beta t^2 + \gamma t + \delta \ .$$

That expression is plainly positive for some value $t = a < 0$ and negative for some value $t = b > 0$. By continuity, there exists some c in the range $a < c < b$ such that $\det(A - tI) = 0$.

Part (b). No, the conclusion fails when $A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & \cos(\theta) & -\sin(\theta) \\ 0 & 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$.

MATH 227: Introduction to Linear Algebra. Midterm 2

LJB, 28 April 2017, Bilkent University.

Time allowed: 2 hours. Please put your name on EVERY sheet of your manuscript. The use of telephones, calculators or other electronic devices is prohibited. The use of red pens or very faint pencils is prohibited too. You may take the question sheet home.

Remember to justify your answers, except in any cases where your answers are obvious.

1: 20 marks. A variable X has sampled values 5, 6, 8, 9, 9, 12, 14. A variable Y has corresponding sampled values 6, 7, 9, 9, 11, 14, 14. Calculate the Pearson correlation coefficient ρ of the two samples.

2: 20 marks. Let $S = \{(1,2,3,4),(5,6,7,8),(9,10,11,12),(13,14,15,16)\}$ in $\mathbb{R}^4$.
(a) Show that S is not linearly independent.
(b) What is the dimension of the subspace $\text{span}(S)$?
(c) Find a basis B of $\text{span}(S)$ such that B is contained in S.

3: 20 marks. A cat is either in the house or in the garden. There is a cat-door between the house and the garden. In each interval of 5 minutes, there is a 3/4 probability of the cat passing through the cat-door an even number of times (thus ending where it started), a 1/4 probability of it passing through the cat-door an odd number of times (thus not ending where it started).
(a) Write down the Markov matrix for the system (relating the state of the cat at a given time and the state of the cat 5 minutes later).
(b) Suppose the cat is in the house. What is the probability of the cat being in the garden 30 minutes later?

4: 20 marks. Let $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ and $E = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.
(a) Calculate the matrices E^2 and EAE.
(b) Using part (a), find formulas, in terms of any given positive integer n, for the four entries of the matrix A^n.

5: 20 marks. Let U, V, W be finite-dimensional vector spaces over $\mathbb{R}$. Let $\alpha : U \to V$ and $\beta : V \to W$ be linear maps. Let $\gamma : U \to W$ be the function such that $\gamma(u) = \beta(\alpha(u))$ for all $u \in U$.
(a) Show that γ is a linear map.
(b) Let $\mathcal{U} = \{u_1, ..., u_a\}$ and $\mathcal{V} = \{v_1, ..., v_b\}$ and $\mathcal{W} = \{w_1, ..., w_c\}$ be bases for U and V and W, respectively. Let A and B be the matrices representing α and β with respect to those bases. It may help to remember that the (s,r)-entry $a_{s,r}$ of A is given by

$$\alpha(u_r) = \sum_{s=1}^{b} a_{s,r}\, v_s \ .$$

Find the matrix representing γ with respect to the bases $\mathcal{U}$ and $\mathcal{W}$. (Be careful to justify your answer clearly.)

Solutions to Midterm 2

1: The sample vectors are $x = (5, 6, 8, 9, 9, 12, 14)$ and $y = (6, 7, 9, 9, 11, 14, 14)$. The averages are $\overline{x} = (5+6+8+9+9+12+14)/7 = 9$ and $\overline{y} = (16+9+1+1+1+16+16)/7 = 10$. The centred vectors are $\widetilde{x} = (-4, -3, -1, 0, 0, 3, 5)$ and $\widetilde{y} = (-4, -3, -1, -1, 1, 4, 4)$. We have

$$\| \widetilde{x} \|^2 = 16+9+1+0+0+9+25 = 60 \ , \qquad \| \widetilde{y} \|^2 = 16+9+1+1+1+16+16 = 60 \ .$$

So $\| \widetilde{x} \| . \| \widetilde{y} \| = 60$. Meanwhile, $\widetilde{x}.\widetilde{y} = 16+9+1+0+0+12+20 = 58$. Therefore,

$$\rho = \widetilde{x}.\widetilde{y} / \ \| \widetilde{x} \| . \| \widetilde{y} \| = 58/60 = 29/30 \ .$$

2: Part (a). The set S is not independent because we have linear relations

$$4(1,1,1,1) = (13,14,15,16) - (9,10,11,12) = (9,10,11,12) - (5,6,7,8) = (5,6,7,8) - (1,2,3,4).$$

Part (b). Those relations show that $\{(1,2,3,4), (1,1,1,1)\}$ is a basis for $\text{span}(S)$. In particular, $\dim(\text{span}(S)) = 2$.

Part (c). The set $B = \{(1,2,3,4), (5,6,7,8)\}$ is a linearly independent set of size 2 in the 2-dimensional subspace $\dim(\text{span}(S))$, so B is a basis for that subspace.

3: Part (a). The Markov matrix is $M = \frac{1}{4} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. Part (b). We have

$$M^2 = \frac{1}{4^2} \begin{bmatrix} 9+1 & 3+3 \\ 3+3 & 9+1 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \ , \qquad M^4 = \frac{1}{8^2} \begin{bmatrix} 25+9 & 15+15 \\ 15+15 & 25+9 \end{bmatrix} = \frac{1}{32} \begin{bmatrix} 17 & 15 \\ 15 & 17 \end{bmatrix} \ .$$

The required probability is the $(2,1)$-entry of $M^6 = M^2 M^4$, which is

$$(3.17 + 5.15)/8.32 = 126/256 = 63/128 \ .$$

4: Part (a). We have $E^2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. Also $EA = \begin{bmatrix} 4 & 4 \\ 2 & -2 \end{bmatrix}$, hence $EAE = \frac{1}{4} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.

Part (b). We have $E(EAE)^n E = E^2 A E^2 ... E^2 A E^2$, the right-hand expression having $n+1$ occurances of the factor E^2. So, using part (a),

$$A^n = E(EAE)^n E / 2^{n+1} = \frac{4^n}{2^{n+1}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= 2^{n-1} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2^n & 2^n \\ 1 & -1 \end{bmatrix} = 2^{n-1} \begin{bmatrix} 2^n+1 & 2^n-1 \\ 2^n-1 & 2^n+1 \end{bmatrix} .$$

Comment: This gives a check on the calculation in Question 3. Indeed, since $M = A/4$, we have $M^n = \frac{1}{2^{n+1}} \begin{bmatrix} 2^n+1 & 2^n-1 \\ 2^n-1 & 2^n+1 \end{bmatrix}$ in general and $M^6 = \frac{1}{128} \begin{bmatrix} 64+1 & 64-1 \\ 64-1 & 64+1 \end{bmatrix}$ in particular.

5: Part (a). Given $\lambda, \lambda' \in \mathbb{R}$ and $u, u' \in U$, then the linearity of α and β yield

$$\gamma(\lambda u + \lambda' u') = \beta(\alpha(\lambda u + \lambda' u')) = \beta(\lambda \alpha(u) + \lambda' \alpha'(u')) = \lambda \beta(\alpha(u)) + \lambda' \beta(\alpha(u')) = \lambda \gamma(u) + \lambda' \gamma(u').$$

Part (b). The (t,s)-entry $b_{t,s}$ of B is given by $\beta(v_s) = \sum_t b_{t,s} w_t$. So

$$\gamma(u_r) = \beta(\alpha(u_r)) = \textstyle\sum_t \left(\sum_s b_{t,s} a_{s,r} \right) w_t \ .$$

The term in the bracket is the (t,r)-entry of BA. Therefore BA is the matrix representing γ with respect to the specified bases.

MATH 227: Introduction to Linear Algebra. Makeup for Midterm 2

LJB, 11 May 2017, Bilkent University.

Time allowed: 2 hours. Please put your name on EVERY sheet of your manuscript. The use of telephones, calculators or other electronic devices is prohibited. The use of red pens or very faint pencils is prohibited too. You may take the question sheet home.

Remember to justify your answers, except in any cases where your answers are obvious.

1: 20 marks. A variable X has sampled values $x_1, x_2, ...x_{15}$, a variable Y, sampled values $y_1, ..., y_{15}$. Suppose that $x_m = 1+m$ and $y_m = 1-m$ for each integer m in the range $1 \leq m \leq 15$. Calculate the Pearson correlation coefficient ρ of the two samples.

2: 20 marks. Let V be the subspace of $\mathbb{R}^5$ consisting of the vectors (a, b, c, d, e) such that

$$a + b + c + d + e = -2a - b + d + 2e = 0 \, .$$

Find a basis for V. What is the dimension of V?

3: 20 marks. On day 0, a vampire is sleeping in a coffin in the basement of a house. Each night, it wakes up. If it wakes up in the coffin, there is a 1/3 probability of it getting out of the coffin, in which case it will spend the next day sleeping on the basement floor. If it wakes up on the basement floor, then there is a 1/3 probability of it spending the next day sleeping in the coffin, a 1/3 probability of it spending the next day sleeping on the basement floor and a 1/3 probability of it getting out of the basement. What is the probability of it getting out of the basement before day 8?

4: 20 marks. **(a)** Let N be a matrix such that $N^3 = 0$. Let n be a positive integer. Express the matrix $(I + N)^n$ as a linear combination of the matrices I and N and N^2.

(b) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Find or guess a formula for A^n.

(c) Prove your formula for A^n using part (a).

5: 20 marks. Let $\mathrm{Mat}_2(\mathbb{R})$ denote the vector space of 2×2 matrices over $\mathbb{R}$. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an invertible 2×2 matrix over $\mathbb{R}$. Let θ be the function $\mathrm{Mat}_2(\mathbb{R}) \to \mathrm{Mat}_2(\mathbb{R})$ such that $\theta(M) = AMA^{-1}$.

(a) Show that θ is a linear map.

(b) Find the matrix representing θ with respect to the basis

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} .$$

MATH 227: Introduction to Linear Algebra. Final

LJB, 25 May 2017, Bilkent University.

Time allowed: 2 hours. Please put your name on EVERY sheet of your manuscript. The use of telephones, calculators or other electronic devices is prohibited. The use of red pens or very faint pencils is prohibited too. You may take the question sheet home.

Remember to justify your answers, except in any cases where your answers are obvious.

1: 20 marks. Find the rank and nullity of the matrix $A = \begin{bmatrix} 5 & 4 & 3 & 2 & 1 \\ 6 & 7 & 8 & 9 & 10 \\ 15 & 14 & 13 & 12 & 11 \\ 16 & 17 & 18 & 19 & 20 \end{bmatrix}$.

2: 40 marks. Consider the Markov matrix $M = \begin{bmatrix} 1/3 & 5/6 \\ 2/3 & 1/6 \end{bmatrix}$. Let $p(n)$ be the probability of the associated Markov system being in state 1 at time n. Let $q(n)$ be the probability of the system being in state 2 at time n. Thus, $\begin{bmatrix} p(n) \\ q(n) \end{bmatrix} = M^n \begin{bmatrix} p(0) \\ q(0) \end{bmatrix}$.
(a) Express $p(3)$ and $q(3)$ in terms of $p(0)$ and $q(0)$.
(b) Find the eigenvalues of M. For each eigenvalue, find a corresponding eigenvector.
(c) Find an invertible matrix P and a diagonal matrix D such that $M = PDP^{-1}$.
(d) Evaluate the matrix P^{-1}.
(e) Give formulas for $a(n), b(n), c(n), d(n)$ such that

$$p(n) = a(n)p(0) + b(n)q(0) \ , \qquad q(n) = c(n)p(0) + d(n)q(0) \ .$$

3: 15 marks. Let B be a 3×3 matrix. Let V be the space of 3-dimensional column vectors.
(a) Suppose that the characteristic equation $\det(B - \lambda I) = 0$ is

$$(1 - \lambda)(2 - \lambda)(3 - \lambda) = 0 \ .$$

Show that V has a basis $\{e_1, e_2, e_3\}$ whose elements e_1, e_2, e_3 are eigenvectors of B.
(b) Give just one numerical example of B such that the characteristic equation of B is

$$(1 - \lambda)^2(2 - \lambda) = 0$$

and V has a basis whose elements are eigenvectors of B.
(c) Give just one numerical example of B such that the characteristic equation is as in part (b) and V does not have a basis whose column vectors are eigenvectors of B.

4: 15 marks. We write $\mathrm{Mat}_5(\mathbb{R})$ to denote the real vector space of 5×5 real matrices. Let C be a 5×5 matrix. Let γ be the function from $\mathrm{Mat}_5(\mathbb{R})$ to $\mathrm{Mat}_5(\mathbb{R})$ such that $\gamma(X) = CX$ for any 5×5 real matrix X.
(a) Show that γ is a linear map (a linear transformation).
(b) Now suppose that $\mathrm{rank}(C) = 3$. Evaluate $\mathrm{rank}(\gamma)$.

5: 10 marks. Let n be a positive integer, N a non-zero $n \times n$ matrix and m an integer such that $m \geq 2$ and $N^{m-1} \neq N^m = 0$.
(a) Show that $m \leq n$.
(b) Now suppose that $m = n$. What can you say about the rank and nullity of N?

Solutions to Final

1: We perform row operations on A. Addiing row 1 to rows 2 and 4, we obtain the left-hand matrix below. Dividing row 2 by 11, then subtracting 21 times row 2 from row 4, we obtain the middle matrix. Subtracting row 1 from row 3, then subtracting 10 times row 2 from row 3, we obtain the right-hand matrix

$$\begin{bmatrix} 5 & 4 & 3 & 2 & 1 \\ 11 & 11 & 11 & 11 & 11 \\ 15 & 14 & 13 & 12 & 11 \\ 21 & 21 & 21 & 21 & 21 \end{bmatrix}, \qquad \begin{bmatrix} 5 & 4 & 3 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 15 & 14 & 13 & 12 & 11 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 5 & 4 & 3 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is now clear that the span of the rows has dimension $\mathrm{rank}(A) = 2$. Since $rank(A) + \mathrm{null}(A)$ is the number of columns, $\mathrm{null}(A) = 3$.

Alternative for 1: By inspection, the span of the rows has basis $\{(0,1,2,3,4), (1,1,1,1,1)\}$, so $\mathrm{rank}(A) = 2$. By the Rank-Nullity Formula, $\mathrm{null}(A) = 5 - 2 = 3$.

Comment: The most standard method is as in the first solution above: Use row operations to reduce to echelon form. The rank may become clear before getting all the way down to echelon form. In the above, to finish the process, we can multiply row 2 by 5 then subtract row 1 from row 2, then divide row 1 by 5, obtaining the echelon matrix

$$\begin{bmatrix} 1 & 4/5 & 3/5 & 2/5 & 1/5 \\ 0 & 1 & 2 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

There are many other alternatives. For instance, one can find $\mathrm{rank}(A)$ as the column rank. Or one can find all the solutions to $Ax = 0$, determine $\mathrm{null}(A)$ as the dimension of the space of solutions, then calculate $\mathrm{rank}(A)$ using rank-nullity.

2: Part (a). We have

$$M = \frac{1}{6}\begin{bmatrix} 2 & 5 \\ 4 & 1 \end{bmatrix}, \quad M^2 = \frac{1}{36}\begin{bmatrix} 24 & 15 \\ 12 & 21 \end{bmatrix} = \frac{1}{12}\begin{bmatrix} 8 & 5 \\ 4 & 7 \end{bmatrix}, \quad M^3 = \frac{1}{72}\begin{bmatrix} 36 & 45 \\ 36 & 27 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 4 & 5 \\ 4 & 3 \end{bmatrix}.$$

So $p(3) = (4p(0) + 5q(0))/8$ and $q(3) = (4p(0) + 3q(0))/8$.

Part (b). The eigenvalues λ_1 and λ_2 of M are the solutions λ to the characteristic equation

$$0 = \det(M - \lambda I) = \begin{vmatrix} 1/3 - \lambda & 5/6 \\ 2/3 & 1/6 - \lambda \end{vmatrix} = (\lambda - 1/3)(\lambda - 1/6) - 10/18 = \lambda^2 - \lambda/2 - 1/2\,.$$

Thus, $\lambda_1 = \frac{1}{2}\left(\frac{1}{2} + \sqrt{\frac{1}{4} + 2}\right) = \frac{1}{2}\left(\frac{1}{2} + \frac{3}{2}\right) = 1$ and $\lambda_2 = \frac{1}{2}\left(\frac{1}{2} - \sqrt{\frac{1}{4} + 2}\right) = -1/2$.

A non-zero vector $\begin{bmatrix} x \\ y \end{bmatrix}$ is an eigenvector with eigenvalue λ_1 if and only if $M\begin{bmatrix} x \\ y \end{bmatrix} = \lambda_1 \begin{bmatrix} x \\ y \end{bmatrix}$, in other words, $\begin{bmatrix} 2 & 5 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$. Equating the first coordinate, $2x + 5y = 6x$. Equivalently, equating the second coordinate, $4x + y = 6y$. One solution is $x = 5$ and $y = 4$. Thus, $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda_1 = 1$.

Similarly, $\begin{bmatrix} x \\ y \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda_2 = -1/2$ if and only if

$$\frac{1}{6}\begin{bmatrix} 2 & 5 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{-1}{2}\begin{bmatrix} x \\ y \end{bmatrix}, \quad \text{that is,} \quad \begin{matrix} 2x + 5y = -3x, \\ 4x + y = -3y. \end{matrix} \quad \text{hence } x + y = 0.$$

Thus, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda_2 = -1/2$.

Part (c). From part (b), we can put $P = \begin{bmatrix} 5 & 1 \\ 4 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}$.

Part (d). Since $\det(P) = -9$, we have $P^{-1} = \dfrac{1}{-9}\begin{bmatrix} -1 & -1 \\ -4 & 5 \end{bmatrix} = \dfrac{1}{9}\begin{bmatrix} 1 & 1 \\ 4 & -5 \end{bmatrix}$.

Part (e). We have

$$\begin{bmatrix} a(n) & b(n) \\ c(n) & d(n) \end{bmatrix} = M^n = PD^nP^{-1} = \frac{1}{9}\begin{bmatrix} 5 & 1 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & (-1/2)^n \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 5 & -5 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 5 & 1 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 4(-1/2)^n & -5(-1/2)^n \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 5 + 4(-1/2)^n & 5 - 5(-1/2)^n \\ 4 - 4(-1/2)^n & 4 + 5(-1/2)^n \end{bmatrix}.$$

3: Part (a). Since 1, 2, 3 are eigenvalues, there exist eigenvectors e_1, e_2, e_3 with eigenvalues 1, 2, 3, respectively. To show that $\{e_1, e_2, e_3\}$ is a basis, it is enough to show that $\{e_1, e_2, e_3\}$ is independent. Writing

(1) $$0 = \mu_1 e_1 + \mu_2 e_2 + \mu_3 e_3$$

we are to show that μ_1, μ_2, μ_3 are all zero. Using equation (1),

(2) $$0 = B0 = \mu_1 Be_1 + \mu_2 Be_2 + \mu_3 Be_3 = \mu_1 e_1 + 2\mu_2 e_2 + 3\mu_3 e_3.$$

From (1) and (2), we obtain

(3) $$\mu_2 e_2 + 2\mu_3 e_3 = 0.$$

Using equation (3),

(4) $$0 = B0 = \mu_2 Be_2 + 2\mu_3 Be_3 = 2\mu_2 e_2 + 6\mu_3 e_3.$$

From equations (3) and (4), $\mu_3 = \mu_2 = 0$ whence, from equation (1), $\mu_1 = 0$, as required.

Part (b). Put $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Plainly, $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis of eigenvectors.

Part (c). Now put $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Plainly, the characteristic equation is still as specified. Consider a vector $v = (x, y, z)$. We claim that, if v is in the span of the eigenvectors, then $y = 0$. To prove this, we may assume that v is an eigenvector. Let λ be the eigenvalue associated with v. By considering the first and second coordinates of the equation $Bv = \lambda v$, we obtain

$x + y = \lambda x$ and $y = \lambda y$. Dealing separately with the cases $\lambda = 1$ and $\lambda \neq 1$, we conclude that $y = 0$, as required.

4: Part (a). For all $a, a' \in \mathbb{R}$ and $X, X' \in \mathrm{Mat}_5(\mathbb{R})$, we have

$$\gamma(aX + a'X') = C(aX + a'X') = aCX + a'CX' = a\gamma(X) + a'\gamma(X') .$$

Part (b). For any $C \in \mathrm{Mat}_5(\mathbb{R})$, the image of γ is the subpace of $\mathrm{Mat}_5(\mathbb{R})$ consisting of those matrices Y such that each column of Y belongs to the image of C. Therefore,

$$\mathrm{rank}(\gamma) = 5\,\mathrm{rank}(C) .$$

Now supposing that $\mathrm{rank}(C) = 3$, it follows that $\mathrm{rank}(\gamma) = 15$.

5: As a convention, we define $N^0 = I$. For any integer t with $t \geq 0$, let $\mathcal{I}_t$ denote the image of N^t. Plainly, $\mathcal{I}_t \supseteq \mathcal{I}_{t+1}$. Suppose that $\mathcal{I}_t = \mathcal{I}_{t+1}$. Then, for any n-dimensional vector x, there exists an n-dimensional vector y such that $N^t y = N^{t+1} x$. Hence, $\mathcal{I}_t = \mathcal{I}_s$ for all integers s with $t \leq s$. Choosing s such that $t \leq s \geq m$, then $\mathcal{I}_s = 0$, in other words, $\mathcal{I}_t = 0$. We conclude that $t \geq m$. So $I_0 \supset I_1 \supset ... \supset I_{m-1} \supset I_m$ and

$$n = \dim(I_0) > \dim(I_1) > ... > \dim(I_{m-1}) > \dim(I_m) = 0 .$$

Therefore, $m \leq n$. Furthermore, if $m = n$, then the sequence of dimensions must be

$$n > n - 1 > n - 2 > ... > 2 > 1 > 0$$

and, in particular, $\mathrm{rank}(N) = \dim(I_1) = n - 1$ and $\mathrm{null}(N) = n - \mathrm{rank}(N) = 1$.