

Archive for
MATH 227, Introduction to Linear Algebra, Fall 2025

Bilkent University, Laurence Barker, 27 January 2025.

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Course specification

MATH 227 Section 1, *Introduction to Linear Algebra*, Fall 2024

Laurence Barker, Bilkent University. Version: 5 November 2024

Course Aims: Firstly, to acquire practical knowledge and skill in some fundamental techniques of linear algebra. Secondly, to acquire a grasp of the underlying theory.

Course Description: This is an introductory course. It includes complete account of the theory, but the emphasis is on methods of calculation.

Prerequisites: MATH 106 Introduction to Calculus 2.

Instructor: Laurence Barker, Office SA 129, barker at fen dot bilkent dot edu dot tr.

Assistant: Serkan Doruk Hazinedar.

Required textbook: Howard Anton, Anton Rorres, “Elementary Linear Algebra” , 11th edition, Wiley 2015, ISBN 978-1-118-67745-2.

Recommended textbooks: Bernard Kolman, “Elementary Linear Algebra”, 9th edition, Prentice-Hall, 2014.

See also supplementary material on my homepage

<http://www.fen.bilkent.edu.tr/~barker/fenis2.html>

and on Moodle.

Classes: Tuesdays 09:30 - 10:20, Thursdays 13:30 - 15:20, room T 272.

Office Hours: Tuesdays 08:30 - 09:20, in my office, room SA-129.

Weekly Syllabus

The format below is, *Week number; Monday date; Subtopics*.

1: 16 Sept: Systems of linear equations, 1.1.

2: 23 Sept: Gaussian and Gauss–Jordan elimination, 1.2. Matrices, 1.3.

3: 30 Sept: Elementary matrices, matrix inversion of by row operations 1.4, 1.5. Existence and uniqueness of solutions, 1.6.

4: 7 Oct: Applications of linear systems, Leontief economic models, 1.8, 1.9, 10.8.

5: 14 Oct: Determinants, their algebraic properties, their evaluation by cofactor expansion, 2.1.

6: 21 Oct: Evaluation of determinants by row reduction, 2.2, 2.3.

7: 28 Oct, *with holiday on Monday and Tuesday*: Real vector spaces, 4.1.

8: 4 Nov: Subspaces, spanning, linear independence, basis, dimension 4.2, 4.3, 4.4.

9: 11 Nov: Change of basis, 4.6. Row and column spaces, rank and nullity, 4.6, 4.7.

10: 18 Nov: Linear transformations, 4.9. Eigenvalues and eigenvectors, 5.1.

11: 25 Nov: Diagonalization, 5.2

12: 2 Dec: Inner product spaces, 6.1, 6.2.

13: 9 Dec: Gram–Schmidt process, 6.3.

14: 16 Dec: Review. If time permits, Markov chains, 4.12, 10.5.

15: 23 Dec, *only Monday and Tuesday*: Review.

Assessment:

A minimum of 75% attendance is required.

- Quizzes: 20%
- Midterm: 40%. Monday 11 November, 17:30.
MA Z02, MA Z03, MA 101, MA 201, MA 202, MA 301.
- Final, 40%

FZ criteria: Less than 20% in Midterm.

Speaking: All speaking in class must address everyone in the room. Questions and comments are welcome: they make learning easier, and they make teaching easier too.

Class Announcements: All students, including any absentees from a class, will be deemed responsible for awareness of class announcements.

Quizzes, with solutions

Laurence Barker, version 17 December 2024

Quiz 1: Using augmented matrices and either Gaussian elimination or else Gauss–Jordan elimination, solve the linear system: $3x_1 + 2x_2 = 5$, $4x_1 + 3x_2 = 6$.

Solution: The augmented matrix is $\left[\begin{array}{cc|c} 3 & 2 & 5 \\ 4 & 3 & 6 \end{array} \right]$.

Dividing row 1 by 3 gives $\left[\begin{array}{cc|c} 1 & 2/3 & 5/3 \\ 4 & 3 & 6 \end{array} \right]$.

Subtracting 4 times row 1 from row 2 gives $\left[\begin{array}{cc|c} 1 & 2/3 & 5/3 \\ 0 & 1/3 & -2/3 \end{array} \right]$.

Multiplying row 2 by 3 gives $\left[\begin{array}{cc|c} 1 & 2/3 & 5/3 \\ 0 & 1 & -2 \end{array} \right]$.

To complete the solution by Gaussian elimination, we note that we have reduced the linear system to $x + 2y/3 = 5/3$ and $y = -2$. Then $x = 5/3 - 2y/3 = 5/3 + 4/3 = 3$. In conclusion, the solution is $(x, y) = (3, -2)$.

Alternative: To complete the solution by Gauss–Jordan elimination, we subtract $2/3$ times row 2 from row 1, yielding

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array} \right] .$$

We can now read off the solution as $(x, y) = (3, -2)$.

Quiz 2: By inverting a 2×2 matrix, and not by any other method, solve the linear system $x + y = 5$, $2x + 5y = 9$.

Solution: The linear system is $\begin{bmatrix} 1 & 1 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$. We have $\begin{bmatrix} 1 & 1 \\ 2 & 5 \end{bmatrix}^{-1} = \frac{1}{3} \begin{bmatrix} 5 & -1 \\ -2 & 1 \end{bmatrix}$. So

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 5 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 9 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 25 - 9 \\ -10 + 9 \end{bmatrix} = \begin{bmatrix} 16/3 \\ -1/3 \end{bmatrix} .$$

Quiz 3: Using row operations, evaluate $\det(A)$ where $A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}$.

Solution: Subtracting row 1 from the other two rows gives $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ 0 & 2 & 12 \end{bmatrix}$.

Subtracting 2 times row 2 from row 3 gives $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}$.

Neither of those two row operations changes the determinant. So $\det(A)$ is the product of the diagonal entries of the upper triangular matrix. That is, $\det(A) = 2$.

Quiz 4: Using minors and cofactors, evaluate $\det(A)$ and A^{-1} , where $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 5 \\ 4 & 1 & 0 \end{bmatrix}$.

Solution: The matrix of minors for A is

$$\begin{bmatrix} 4.0 - 1.5 & 3.0 - 4.5 & 3.1 - 4.4 \\ 2.0 - 1.0 & 1.0 - 4.0 & 1.1 - 4.2 \\ 2.5 - 4.0 & 1.5 - 3.0 & 1.4 - 3.2 \end{bmatrix} = \begin{bmatrix} -5 & -20 & 10 \\ 0 & 0 & -7 \\ 10 & 5 & -2 \end{bmatrix}.$$

Introducing alternating $\pm$ signs to obtain the cofactors, then taking the transpose, we obtain

$$\text{adj}(A) = \begin{bmatrix} -5 & 0 & 10 \\ 20 & 0 & -5 \\ -13 & 7 & -2 \end{bmatrix}.$$

The $(2, 2)$ entry of the matrix $A \cdot \text{adj}(A) = \det(A)I$ is $3.0 + 4.0 + 5.7 = 35$. In conclusion, A is invertible and

$$\det(A) = 35, \quad A^{-1} = \frac{1}{35} \begin{bmatrix} -5 & 0 & 10 \\ 20 & 0 & -5 \\ -13 & 7 & -2 \end{bmatrix}.$$

Quiz 5: Let x, y, z be real numbers that are mutually distinct, we mean, no two of them are equal. Let

$$A = \begin{bmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{bmatrix}.$$

Show that A is invertible. (Hint: We just need to show that $\det(A) \neq 0$.)

Solution: Applying various row and column operations,

$$\begin{aligned} \det(A) &= \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix} = (y-x)(z-x) \begin{vmatrix} 1 & x+y \\ 1 & x+z \end{vmatrix} \\ &= (y-x)(z-x) \begin{vmatrix} 1 & y \\ 1 & z \end{vmatrix} = (y-x)(z-x)(z-y) \end{aligned}$$

which is plainly nonzero. Therefore, A is invertible.

Comment: The matrix A , here, is the 3×3 case of the **Vandemonde matrix**. The formula for the determinant in the $n \times n$ case is similar and can be found by reducing to the $(n-1) \times (n-1)$ case by the method indicated.

Quiz 6: Evaluate the rank of the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 5 & 9 \\ 1 & 5 & 9 & 17 \end{bmatrix}$.

Solution: Subtracting column 1 from the other columns gives $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 3 & 7 \\ 1 & 2 & 4 & 8 \\ 1 & 4 & 8 & 16 \end{bmatrix}$.

Subtracting multiples of column 2 from columns 3 and 4 gives $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 3 \\ 1 & 2 & 0 & 0 \\ 1 & 4 & 0 & 0 \end{bmatrix}$.

Subtracting 3 times column 3 from column 4 gives the zero column. So the image of A has a basis consisting of the vectors $(1, 1, 1, 1)$ and $(0, 1, 2, 4)$ and $(0, 1, 0, 0)$. Perforce, $\text{rank}(A) = 3$.

Quiz 7: By finding a basis for the null space, not by any other method, evaluate the nullity of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}.$$

Solution: Since rows 3 and 4 are duplicates, we can remove them without changing the null space. After that, we subtract row 1 from row 2, yielding

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \end{bmatrix}.$$

Dividing row 2 by -2 , then subtracting the new row 2 from row 1 yields

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

A vector $v = (a, b, c, d)$ belongs to $\text{Null}(A)$ provided $a + c = b + d = 0$. That is equivalent to the condition that $v = a(1, 0, 1, 0) + b(0, 1, 0, 1)$. So $\text{Null}(A)$ has basis $\{(1, 0, 1, 0), (0, 1, 0, 1)\}$. In particular, $\dim(\text{Null}(A)) = 2$.

Quiz 8: Find a basis for the null space, in $\mathbb{C}^2$, of the matrix $A = \begin{bmatrix} 1 & 1 \\ i & i \end{bmatrix}$.

Solution: Obviously, one basis for $\text{Null}(A)$ is $\{(1, -1)\}$.

Quiz 9: Let $u_1 = (1, 1, 1)$ and $u_2 = (1, 1, 0)$. Find an orthogonal basis for the subspace $\text{span}\{u_1, u_2\}$ of $\mathbb{R}^3$.

Solution: We put $v_1 = u_1$ and

$$v_2 = u_2 - \frac{\langle v_1, u_2 \rangle}{\|v_1\|^2} v_1 = (1, 1, 0) - \frac{2}{3}(1, 1, 1) = (1/3, 1/3, -2/3).$$

In conclusion, such a basis is $\{v_1, v_2\} = \{(1, 1, 0), (1/3, 1/3, -2/3)\}$.

Quiz 10: Find the eigenvalues of the matrix $A = \begin{bmatrix} 9 & 501 \\ 0 & 17 \end{bmatrix}$.

Solution: The characteristic equation is

$$0 = \begin{vmatrix} 9 - \lambda & 501 \\ 0 & 17 - \lambda \end{vmatrix} = (9 - \lambda)(17 - \lambda) .$$

So the eigenvalues are 9 and 17.

Quiz 11: Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$. Find P and diagonal D such that $A = PDP^{-1}$.

Solution: The characteristic equation is

$$0 = \begin{vmatrix} 2 - \lambda & 1 \\ 0 & 3 - \lambda \end{vmatrix} = (2 - \lambda)(3 - \lambda) .$$

The eigenvalues are 2 and 3, with eigenvectors $(1, 0)$ and $(1, 1)$, respectively. So we can put

$$P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} , \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} .$$

Quiz 12: Diagonalize $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. That is, find P and diagonal D such that $A = PDP^{-1}$.

Solution: The characteristic equation is

$$0 = \begin{vmatrix} 1 - \lambda & -1 \\ -1 & 1 - \lambda \end{vmatrix} = (1 - \lambda)^2 - 1 = \lambda(\lambda - 2) .$$

The eigenvalue 2 has corresponding eigenvector $(1, 1)$. The eigenvalue 0 has corresponding eigenvector $(1, -1)$. So we can put

$$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} , \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} .$$



Midterm

Examiners: Laurence Barker (Section 1), Bülent Ünal (Section 2).

Exam time and date: 17:30 - 19:30, duration two hours, 11 November 2024.

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- The exam consists of 4 questions of equal weight.
- Show your working. Correct answers with insufficient justification will not earn full credit.
- The use of electronic devices, red pens, or very faint pencils is prohibited.

1: Evaluate the determinant of the matrix $\begin{bmatrix} 1 & -2 & 4 & -8 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{bmatrix}$.

2: For which real number a does the linear system

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

have infinitely many solutions? For that value of a , find all the solutions.

3: (a) Let $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 9 & 19 \\ 3 & 12 & 22 \end{bmatrix}$. By any method, evaluate A^{-1} .

(b) Using part (a), not by any other method, solve the linear system

$$x + 4y + 7z = 2x + 9y + 19z = 3x + 12y + 22z = 1.$$

4: For any positive integer n , we write P_n to denote the vector space of polynomial functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that f has degree at most n . Which of the following subsets of P_n are subspaces of P_n ? (Do not forget to justify your answers.)

- (a)** The subset U consisting of those functions f in P_n such that $f(x) = f(1-x)$ for all $x \in \mathbb{R}$.
- (b)** The subset V consisting of those functions f in P_n such that $(f(x))^2 = (f(-x))^2$ for all $x \in \mathbb{R}$.
- (c)** The subset W consisting of those functions f in P_n such that $-1 < f(x) < 1$ for all $x \in \mathbb{R}$.

Solutions to Midterm

1: Let Δ denote the specified determinant. Twice subtracting multiples of a top row from the other rows, we have

$$\Delta = \begin{vmatrix} 1 & -2 & 4 & -8 \\ 0 & 1 & -3 & 7 \\ 0 & 3 & -3 & 9 \\ 0 & 4 & 0 & 16 \end{vmatrix} = \begin{vmatrix} 1 & -3 & 7 & \\ 3 & -3 & 9 & \\ 4 & 0 & 16 & \end{vmatrix} = \begin{vmatrix} 1 & -3 & 7 & \\ 0 & 6 & -12 & \\ 0 & 12 & -12 & \end{vmatrix} = \begin{vmatrix} 6 & -12 & \\ 12 & -12 & \end{vmatrix}.$$

Taking out a factor of 6 from the first row and 12 from the second, we obtain

$$\Delta = 6.12 \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = 6.12(-1 + 2) = 6.12 = 72.$$

2: If there are infinitely many solutions then, subtracting the top row from the others,

$$0 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & a \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & a-1 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 2 & a-1 \end{vmatrix} = a-5.$$

Putting $a = 5$, we solve the linear system with augmented matrix $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 2 & 3 & 3 \\ 1 & 3 & 5 & 2 \end{array} \right]$.

Applying the same row operations as before, we obtain $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 4 & -2 \end{array} \right]$.

Subtracting 2 times row 2 from row 3 replaces all the entries in row 3 with 0. We have reduced the linear system to the equations $x + y + z = 4$ and $y + 2z = 1$. Introducing a parameter $t \in \mathbb{R}$, we can put $z = t$. Then $y = -1 - 2z = -1 - 2t$ and $x = 4 - y - z = 5 + t$. It is now clear that when $a = 5$, there are infinitely many solutions.

Thus, we have shown that there are infinitely many solutions if and only if $a = 5$, in which case the solutions are

$$(x, y, z) = (5 + t, -1 - 2t, t) = (5, -1, 0) + t(1, -2, 1)$$

where t is any real number.

Comment: The first sentence shows that $a = 5$ if and only if there are no solutions or infinitely many solutions. Only after finding solutions did it become clear that, when $a = 5$, there are indeed infinitely many solutions.

3: Part (a). We set up the problem as $\left[\begin{array}{ccc|ccc} 1 & 4 & 7 & 1 & 0 & 0 \\ 2 & 9 & 19 & 0 & 1 & 0 \\ 3 & 12 & 22 & 0 & 0 & 1 \end{array} \right]$.

Subtracting multiples of row 1 from the other two rows, $\left[\begin{array}{ccc|ccc} 1 & 4 & 7 & 1 & 0 & 0 \\ 0 & 1 & 5 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array} \right]$.

Subtracting multiples of row 3 from the other two rows, $\left[\begin{array}{ccc|ccc} 1 & 4 & 0 & 22 & 0 & -7 \\ 0 & 1 & 0 & 13 & 1 & -5 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array} \right]$.

Subtracting 4 times row 2 from row 1 gives $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -30 & -4 & 13 \\ 0 & 1 & 0 & 13 & 1 & -5 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array} \right]$.

In conclusion, $A^{-1} = \begin{bmatrix} -30 & -4 & 13 \\ 13 & 1 & -5 \\ -3 & 0 & 1 \end{bmatrix}$.

Part (b). We have $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -30 - 4 + 13 \\ 13 + 1 - 5 \\ -3 + 0 + 1 \end{bmatrix} = \begin{bmatrix} -21 \\ 9 \\ -2 \end{bmatrix}$.

Comment: The alert will have noticed that upon subtracting multiples of row 1 from the other rows, the matrix is already rendered in a form that ensures no large numbers will appear when we apply the Gauss–Jordan method. Nevertheless, the method of minors and cofactors can be used instead. It is as follows.

The cofactor matrix is $\begin{bmatrix} 9.22 - 12.19 & 2.22 - 3.19 & 2.12 - 3.9 \\ 4.22 - 7.12 & 22 - 3.7 & 12 - 3.4 \\ 4.19 - 7.9 & 19 - 2.7 & 9 - 2.4 \end{bmatrix} = \begin{bmatrix} -30 & -13 & -3 \\ 4 & 1 & 0 \\ 13 & 5 & 1 \end{bmatrix}$.

Putting in alternating $\pm$ signs and taking the transpose, $\text{adj}(A) = \begin{bmatrix} -30 & -4 & 13 \\ 13 & 1 & -5 \\ -3 & 0 & 1 \end{bmatrix}$.

By considering the $(1, 1)$ entry of $\text{adj}(A)A$, we have $\det(A) = -30 - 8 + 39 = 1$. It follows that $A^{-1} = \text{adj}(A)/\det(A) = \text{adj}(A)$. Thus, we have obtained the same value of A^{-1} as before.

4: Part (a). Let $f, g \in U$ and $\lambda \in \mathbb{R}$. For all $x \in \mathbb{R}$, we have

$$(f + g)(x) = f(x) + g(x) = f(1 - x) + g(1 - x) = (f + g)(1 - x).$$

So $f + g \in U$ and we have shown that U is closed under addition. For all x , we also have

$$(\lambda f)(x) = \lambda(f(x)) = \lambda(f(1 - x)) = (\lambda f)(1 - x).$$

So $\lambda f \in U$ and we have shown that U is closed under scalar multiplication. In conclusion, U is a subspace of P_n .

Part (b). Let f and g be the elements of V such that $f(x) = 1$ and $g(x) = x$. Then

$$(f + g)(0)^2 = 1 \neq 4 = (f + g)(1 - 0)^2.$$

So $f + g \notin V$. In conclusion, V is not a subspace of P_n .

Part (c). Let $f_0 \in P_n$ such that $f_0(x) = 0$ for all x . Plainly, $f_0 \in W$. Let $f \in P_n$ such that $f \neq f_0$. Then there are real numbers $a_0, a_1, \dots, a_m$ with $a_m \neq 0$ and $m \leq n$ such that

$$f(x) = a_0 + a_1x + a_2x^2 + \dots + a_mx^m.$$

When the absolute value of x becomes large enough, the absolute value of the final term a_mx^m overwhelms the absolute values of the other terms of the sum, whereupon the absolute value

of $f(x)$ becomes very large and perforce greater than 1. So $f \notin W$. We have shown that $W = \{f_0\}$. It is now clear that W is closed under addition and scalar multiplication. In conclusion, W is a subspace of P_n .

Comment: In the style of more advanced mathematics, the key point in part (c) can be expressed more concisely as: for sufficiently large $|x|$, we have $|f(x)| > |a_m x^m|/2 > 1$. That kind of style is merely elegant here, not at all necessary, but it does become very useful in more sophisticated contexts.



Final

Examiners: Laurence Barker (Section 1), Bülent Ünal (Section 2).

Exam time and date: 09:00 - 11:00, duration two hours, 26 December 2024.

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- The exam consists of 4 questions of equal weight.
- Show your working. Correct answers with insufficient justification will not earn full credit.
- The use of electronic devices, red pens, or very faint pencils is prohibited.

1: Let $M = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{bmatrix}$.

- (a) Find a basis for the null space of M .
(b) Evaluate the rank and nullity of M .

2: Let $A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ -1 & -1 & 0 \end{bmatrix}$. Find P and diagonal D such that $A = PDP^{-1}$.

3: Let V be the vector space consisting of the functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that there exist real numbers a, b, c satisfying $f(x) = ax^4 + bx^3 + cx^2$. We make V become an inner product space with inner product given by

$$(f, g) = \int_{-1}^1 f(x)g(x) \, dx .$$

Apply the Gram–Schmidt process to find an orthogonal basis for V starting from the basis $\{u_1, u_2, u_3\}$ where $u_1(x) = x^2$ and $u_2(x) = x^3$ and $u_3(x) = x^4$.

4: Let $H = \begin{bmatrix} 99 & 1 & 1 \\ 1 & 99 & 1 \\ 1 & 1 & 99 \end{bmatrix}$.

- (a) Which vectors (x, y, z) are eigenvectors of H ?
(b) Find an orthonormal basis of eigenvectors of H .

Solutions to Final

1: Part (a). Subtracting the first row from the others yields $\begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 2 & 0 & 2 \\ 0 & 3 & 3 & 9 \end{bmatrix}$.

Interchanging rows 2 and 3, then scaling them gives $\begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

Subtracting row 2 from row 3, then scaling row 3 gives $\begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$.

Subtracting row 3 from the other two rows gives $\begin{bmatrix} 1 & -1 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$.

Adding row 2 to row 1 gives $\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$.

So $(a, b, c, d) \in \text{Null}(M)$ if and only if $a = 2d$ and $b = -d$ and $c = -2d$. Therefore, a basis for $\text{Null}(M)$ is $\{(2, -1, -2, 1)\}$.

Part (b). By part (a), $\text{null}(M) = 1$. By rank-nullity, $\text{rank}(M) = 4 - \text{null}(M) = 3$.

2: The characteristic equation is

$$\begin{aligned} 0 = \det(A - \lambda I) &= \begin{vmatrix} 3 - \lambda & 1 & 0 \\ 1 & 3 - \lambda & 0 \\ -1 & -1 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} 3 - \lambda & 1 \\ 1 & 3 - \lambda \end{vmatrix} \\ &= -\lambda((3 - \lambda)^2 - 1) = -\lambda(\lambda^2 - 6\lambda + 8) = -\lambda(\lambda - 2)(\lambda - 4). \end{aligned}$$

So the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 2$ and $\lambda_3 = 4$.

An eigenvector with eigenvalue $\lambda_1 = 0$ is $f_1 = (0, 0, 1)$. An eigenvector $f_2 = (x, y, z)$ with eigenvalue $\lambda_2 = 2$ is given by

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ -1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

So we can put $f_2 = (1, -1, 0)$. An eigenvector $f_3 = (x, y, z)$ with eigenvalue $\lambda_3 = 4$ is given by

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ -1 & -1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

So we can put $f_3 = (2, 2, -1)$, whence $P = \begin{bmatrix} 0 & 1 & 2 \\ 0 & -1 & 2 \\ 1 & 0 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

3: We have

$$v_1(x) = u_1(x), \quad v_2(x) = u_2(x) - \frac{\langle v_1 | u_2 \rangle}{\|v_1\|^2} v_1(x), \quad v_3 = u_3(x) - \frac{\langle v_1 | u_3 \rangle}{\|v_1\|^2} v_1(x) - \frac{\langle v_2 | u_3 \rangle}{\|v_2\|^2} v_2(x).$$

Trivially, $v_1(x) = x^2$. For n odd, $\int_{-1}^1 x^n dx = 0$ while, for n even, $\int_{-1}^1 x^n dx = \frac{2}{n+1}$. We have

$$\langle v_1 | u_2 \rangle = \int_{-1}^1 x^5 dx = 0. \quad \text{Therefore, } v_2(x) = u_2(x) = x^3. \quad \text{We have } \langle v_2 | u_3 \rangle = \int_{-1}^1 x^7 dx = 0.$$

Also, $\langle v_1 | u_3 \rangle = \int_{-1}^1 x^6 dx = 2/7$ and $\|v_1\|^2 = \int_{-1}^1 x^4 dx = 2/5$. So $\frac{\langle v_1 | u_3 \rangle}{\|v_1\|^2} = 5/7$, whence $v_3(x) = x^4 - 5x^2/7$. In conclusion,

$$v_1(x) = x^2, \quad v_2(x) = x^3, \quad v_3(x) = x^4 - 5x^2/7.$$

4: Part (a). The eigenvectors of H are the same as the eigenvectors of

$$H_\mu = \begin{bmatrix} \mu & 1 & 1 \\ 1 & \mu & 1 \\ 1 & 1 & \mu \end{bmatrix}$$

where μ is any real number. We have

$$\text{Null}(H_1) = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}, \quad \text{Null}(H_{-2}) = \{(x, y, z) \in \mathbb{R}^3 : x = y = z\}$$

and $\text{Null}(H_\mu) = \{0, 0, 0\}$ for $\mu \notin \{1, -2\}$. Therefore (x, y, z) is an eigenvector of H if and only if (x, y, z) is nonzero and either $x + y + z = 0$ or else $x = y = z$.

Part (b). By part (a), one orthonormal basis of eigenvectors of H is

$$\{(1, -1, 0)/\sqrt{2}, (1, 1, -2)/\sqrt{6}, (1, 1, 1)/\sqrt{3}\}.$$

Comments: Questions 1 and 2 are entirely standard routines of numerical calculation, just testing basic competence in drilled procedures. Question 3 is a straightforward adaptation of standard course material on Legendre polynomials. The pedagogical purpose of that material is to illustrate the usefulness of the abstract notion of an inner product space.

In the context of this introductory course, Question 4 is hard because, in the standard drill for diagonalization, the student habitually finds only a basis of eigenvectors. That is probably why many examinees were able to find only finitely many eigenvectors. With a conceptual grasp of the notion of an eigenvector, one recognizes that, actually, any matrix has infinitely many eigenvectors. Part (b) of Question 4 can be done routinely by taking an obvious basis and applying the Gram–Schmidt process. But it is easier to simply contrive an orthonormal basis straight away.

If use of arbitrary complex numbers is admitted, a more elegant answer to (b) is

$$\{(1, \omega^r, \omega^{2r}) : r \in \mathbb{Z}\}$$

where $\omega = e^{2\pi i/3}$. This orthonormal basis is of fundamental importance in the theory of the discrete Fourier transform.

A weak pass corresponds to competence in the drilled numerical procedures, absent a conceptual understanding of notions such as that of an inner product space or an eigenvector. An A or A minus grade — or an erratic ability to sometimes make terrible mistakes with the routines yet succeed with the likes of Question 4 — indicates talent and suitability for more advanced mathematics courses.



Makeup Final

Examiners: Laurence Barker (Section 1), Bülent Ünal (Section 2).

Exam time and date: 09:00 - 11:00, duration two hours, 9 January 2025.

- The exam consists of 4 questions of equal weight.
- Show your working. Correct answers with insufficient justification will not earn full credit.
- The use of electronic devices, red pens, or very faint pencils is prohibited.

1: Let $K = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & 2 & 2 & 2 \end{bmatrix}$.

- (a) Find bases for the image (column space) and null space (kernel) of K .
(b) Evaluate the rank and nullity of K .

2: Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & -3 & 3 \end{bmatrix}$. Find P and diagonal D such that $A = PDP^{-1}$.

3: Let M be the vector space consisting of the 2×2 matrices of real numbers. We define the trace of a 2×2 matrix to be the sum of the diagonal entries,

$$\text{tr} \left(\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \right) = a_{1,1} + a_{2,2} .$$

We make M become an inner product space with inner product given by

$$(A, B) = \text{tr}(A^T B)$$

where A^T denotes the transpose of A . Apply the Gram–Schmidt process to find an orthonormal set in M , starting from the set $\{u_1, u_2, u_3\}$ where

$$u_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} , \quad u_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} , \quad u_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} .$$

4: Let S be the set consisting of the zero vector $(0, 0, 0)$ and all the eigenvectors of the matrix

$$B = \begin{bmatrix} a & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} .$$

For which values of a is S a subspace of $\mathbb{R}^3$?

Brief solutions to Makeup Final

1: Part (a) is straightforward, with many possible answers. For part (b), the rank and nullity are both 2.

2: We can put $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 3 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

3: A routine calculation not carried out by any candidate.

4: Answer: $a = 1$, to ensure that a linear combination of a 1-eigenvector and an a -eigenvector is, itself, an eigenvector.