Archive of documentation for

MATH 227, Introduction to Linear Algebra

Bilkent University, Fall 2018, Laurence Barker

version: 28 January 2019

Source file: arch227fall18.tex

Course specification

MATH 227 Sections 1 and 2, *Introduction to Linear Algebra*, Fall 2018

Laurence Barker, Bilkent University. Version: 28 December 2018.

Course Aims: To aquire practical knowledge and skill in some important techniques of linear algebra, and to aquire theoretical understanding and ability to critically assess methods and propositions in the area.

Course Description: This is an introductory course with an emphasis on methods of calculation, but with a theoretical grounding that is self-contained and complete. Some victory conditions, for the student, include: an understanding of the notion of a vector space as something more than just a system of coordinates; an ability to apply the method of diagonalization, a clear grasp of the theory behind that method.

Course Requirements: An official prerequisite in the University Catalogue is MATH 106 but, in practice, no knowledge of calculus is needed.

Instructor: Laurence Barker, Office SAZ 129, barker at fen dot bilkent dot edu dot tr.

Assistant: Hatice Mutlu, hatice dot mutlu at bilkent dot edu dot tr.

Main course text: Howard Anton, Chris Rorres, "Elementary Linear Algebra", 11th edition, Wiley, 2011, 2015. ISBN: 978-1-118-67745-2.

Warning: The Main course text is not the book with the same authors, same title, same edition number, same publisher but with a different ISBN. The main course text has the text "International Student Version" on the front cover, whereas the other book does not. The two books have different exercises.

See STARS for some other recommended texts.

For notes on diagonalization, see the file on my homepage: diagonalization.pdf .

Classes: Section 1: Wednesdays 13:40 - 15:30, Fridays 15:40 - 16:30, room V 04. Section 2: Wednesdays 09:40 - 10:30, Fridays 10:40 - 12:30, room V 02.

Office Hours: Wednesdays 08:40 - 09:30, Fridays 16:40 - 17:30, in my office, room SA-129.

Office hours is for *all* the students on the course, not just the proficient. If you are having difficulty with the course, then it is best to come to see me for advice. You have nothing to lose by doing so, since otherwise I will anyway find out how bad you are when I mark the exams.

Weekly Syllabus

The format below is, *Week number; Monday date; Subtopics and textbook section numbers.* The numbering $m.n$ indicates Chapter m Section n in the Anton–Rorres textbook.

1: 24 Sep: Systems of linear equations, 1.1. Sketch of Markov chain scenario and the method of diagonalization.

2: 1 Oct: Gaussian and Gauss–Jordan elemination 1.2. Matrics 1.3.

3: 8 Oct: Elementary matrices, inversion of matrices by row operations 1.4, 1.5. Existence and uniqueness of solutions, 1.6.

4: 15 Oct: Determinants, their algebraic properties, their evaluation by row reduction and by cofactor expansion, 2.1, 2.2, 2.3.

5: 22 Oct: Euclidian spaces, norm, dot product, distance, angle, 3.1, 3.2.

6: 29 Oct: Pearson correlation coefficient (special notes). Markov chains, 10.5. Review for Midterm 1.

7: 5 Nov: Real vector spaces, subspaces, 4.1, 4.2. (Midterm 1 on 5 November at 18:00.)

8: 12 Nov: Linear independence, spanning, bases, dimension, 4.3, 4.4, 4.5.

9: 19 Nov: Linear transformations, inverses, composition, 8.1, 8.2, 8.3.

10: 26 Nov: Change of basis, 4.6. Markov chains, 4.12.

11: 3 Dec: Row and column spaces, Rank-Nullity Formula, 4.7, 4.8. (Midterm 2 on 3 December at 18:00.)

12: 10 Dec: Complex vector spaces, eigenvalues and eigenvectors.

13: 17 Dec: Diagonalization, 5.2. Applications to Markov chains, 10.5.

14: 24 Dec: Inner product spaces, Gram–Schmidt orthogonalization, 6.1, 6.2, 6.3.

For both sections, the last class is on Friday 28 December.

Assessment:

- Quizzes and Participation 10%.
- Midterm I, 30%, Monday, 5th November, 18:00.
- Midterm II, 30%, Monday, 3 December, 18:00.
- Final, 30%, Wednesday, 2 January, 12:30.

75% attendance is compulsory.

- Midterm II Makeup, Wednesday, 26 December, 18:00.

Class Announcements: All students, including any absentees from a class, will be deemed responsible for awareness of class announcements.

Midterm 1 Exam Syllabus

Solving linear equations, [|]. Gaussian elimination, Gauss–Jordan elimination. 1.1, 1.2.

Inverting matrices by Gauss–Jordan method, [|]. 1.3, 1.4, 1.5.

Determinants and inverses by cofactor method, | |. 1.6, 2.1, 2.2, 2.3.

Geometry of $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^n$: $\|\|$. 3.1, 3.2, 3.3, 3.4.

Pearson correlation coefficient, $\bullet / \|\|\|\|$. (Discussed in lectures; also see notes below).

Midterm 2 Exam Syllabus

Real vector spaces, subspaces, 4.1, 4.2.

Linear independence, spanning, bases, dimension, 4.3, 4.4, 4.5.

Linear transformations, inverses, composition, 8.1, 8.2, 8.3.

Change of basis for vectors, 4.6.

Introduction to Markov chains, 4.12.

Final Exam Syllabus

4.6: Change of basis.
4.7 - 4.8: Rank-Nullity Formula,
5.1, 5.2: Eigenvalues, eigenvectors, diagonalization (including applications to Markov systems as discussed in lectures and the file diagonalization.pdf).
5.3: Complex vector spaces.
8.1: Linear maps.
8.3: Composition, inversion, matrix representation of linear maps.

Note: Due to disruption in week 14 due to heavy snow, material on inner product spaces in Chapter 6 is not on the syllabus for the Final Exam.

Some notes on the Pearson correlation coefficient

The following theory, and some examples, were given in class. I supply this summary because the topic is not in the textbook.

Let x_1, ..., x_n be sampled values of some variable, and let y_1, ..., y_n be the corresponding sampled values of some other variable. The **Pearson correlation coefficient** ρ of the two samples is calculated as follows:

Step 1: Calculate the average values

$$\overline{x} = (x_1 + ... + x_n)/n \ , \qquad \overline{y} = (y_1 + ... + y_n)/n \ .$$

Step 2: Calculate the centred vectors

$$\widetilde{x} = (\widetilde{x}_1, ..., \widetilde{x}_n) = (x_1 - \overline{x}, ..., x_n = \overline{x}) \ , \qquad \widetilde{y} = (\widetilde{y}_1, ..., \widetilde{y}_n) = (y_1 - \overline{y}, ..., y_n = \overline{y}) \ .$$

Step 3: Calculate $\rho = \dfrac{\widetilde{x}_\bullet \widetilde{y}}{\| \widetilde{x} \| \| \widetilde{y} \|}$ where

$$\widetilde{x}_\bullet \widetilde{y} = \widetilde{x}_1 \widetilde{y}_1 + ... + \widetilde{x}_n \widetilde{y}_n \ , \qquad \| \widetilde{x} \| = \sqrt{\widetilde{x}_1^2 + ... + \widetilde{x}_n^2} \ , \qquad \| \widetilde{y} \| = \sqrt{\widetilde{y}_1^2 + ... + \widetilde{y}_n^2} \ .$$

Let us give a small example problem. For another example, see Question 2 in the Practice Midterm 2.

Problem: Snails A, B, C weigh 3, 3, 6 kilograms, respectively. The maximum speeds of A, B, C are 3, 4, 8 meters per second, respectively. Let ρ be the correlation coefficient for these samples of weights and speeds. Show that $\rho > 97/100$.

Solution: The given data is $(x_1, x_2, x_3) = (3, 3, 6)$ and $(y_1, y_2, y_3) = (3, 4, 8)$. The averages are $\overline{x} = (3 + 3 + 6)/3 = 4$ and $\overline{y} = (3 + 4 + 8)/3 = 5$. The centred vectors are $\widetilde{x} = (-1, -1, 2)$ and $\widetilde{y} = (-2, -1, 3)$. So $\| \widetilde{x} \|^2 = 1 + 1 + 4 = 6$ and $\| \widetilde{y} \|^2 = 4 + 1 + 9 = 14$. So

$$\rho = \frac{2 + 1 + 6}{\sqrt{6}\sqrt{14}} = \frac{9}{2\sqrt{21}} \ .$$

It follows that $\rho^2 = 81/84 > 96/100$, hence $\rho > 97/100$. $\square$

Suggested Exercises and Quiz Solutions

MATH 227 Sections 1 and 2, *Introduction to Linear Algebra*, Fall 2018

Laurence Barker, Mathematics Department, Bilkent University,
version: 28 December 2018.

Office Hours: Wednesdays, 08:40 – 09:30, Fridays, 16:40 - 17:30. Room SA 129.

Exercises

Most of the questions are taken from the primary textbook: Howard Anton, Chris Rorres, "Elementary Linear Algebra", 11th Edition, 2011 (reprinted 2015).

End of Chapter 1, page 76 onwards, numbers 19 - 37, 51, 52, 102, 125, 127.
End of Chapter 2, page 111, routine: 16 - 20, theoretical: 34, 35, 63, 64.
End of Chapter 3, page 164, routine: 85 - 104, theoretical: 131, 132.

For Midterm 1 preparation, from the exams in the file arch227spr17.pdf:

- All questions from Practice Midterm 1,
- Question 2 from Practice Midterm 2,
- All questions from Midterm 1, 9 March 2017,
- Question 1 from Midterm 2, 28 April 2017.

End of Chapter 4, page 276, routine: 84 - 89, harder or theoretical: 90 - 94.

For Midterm 2 preparation, from the exams in the file arch227spr17.pdf:

- Questions 1, 3, 4, 5 from Practice Midterm 2,
- Questions 2, 3, 4, 5 from Midterm 2, 28 April 2017.

End of Chapter 5, page 325, standard: 3 - 8, 41 - 44, 52 - 54. harder: 34, 55, 56.

An introduction to complex vector spaces through some diagonalization problems:

Question 1: diagonalize the matrix $\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$.

Question 2: For a real number θ, let $R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$. Diagonalize R_θ. Hence evaluate $(R_\theta)^n$ for any integer n. Now give an easier explanation for your evaluation of $(R_\theta)^n$.

For Final preparation: all questions in the Final Exam in the file arch227spr17.pdf.

Quizzes

Quiz 1, Section 1: By Gauss–Jordan elimination, invert $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 1 \\ 3 & 3 & 1 \end{bmatrix}$.

Quiz 1, Section 2: By Gauss–Jordan elimination, invert $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$.

Quiz 2, Section 1: Find the distance between the point $(3, 4, 4)$ and the plane consisting of the points (x, y, z) such that $2x + 4y - 3z = 6$.

Quiz 2, Section 2: Find the distance between the point $(5, 5, 2)$ and the plane consisting of the points (x, y, z) such that $x + y + 3z = 10$.

Quiz 3, Section 1: Find a basis for the vector space of solutions to the simultaneous equations $x + y + z = 3x + 4y + 5z = 5x + 6y + 7z = 0$. What is the dimension of that vector space?

Quiz 3, Section 2: Find a basis for the vector space of solutions to the simultanous equations $3x + 2y + z = x + 2y + 3z = x + y + z = 0$. What is the dimension of that vector space?

Quiz 4, Section 1: Let S be the subspace of $\mathrm{Mat}_3(\mathbb{R})$ consisting of those matrices A such that $A = A^T$. Find a basis for S. What is the dimension of S?

Quiz 4, Section 2: Let P_6 denote the real vector space of polynomial functions of degree at most 6. Let S be the subspace of P_6 consisting of those f in P_6 such that $f(x) = f(-x)$ for all x. Find a basis for S. What is the dimension of S?

Quiz 5, Section 1: Find all the values λ such that $\det(A - \lambda I) = 0$ where $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

Quiz 5, Section 2: Find all the values z such that there exist nonzero (x, y) satsifying

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = z \begin{bmatrix} x \\ y \end{bmatrix} .$$

Quiz 6, Section 1: Find an invertible matrix P and a diagonal matrix D such that

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = PDP^{-1} .$$

Quiz 6, Section 2: Find an invertible matrix P and a diagonal matrix D such that

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 3 & 3 \end{bmatrix} = PDP^{-1} .$$

Solutions to Quizzes

Solution Q1 S1: We code the problem as $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 1 & 0 \\ 3 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$.

Adding multiples of 1st row, $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & -2 & -3 & 1 & 0 \\ 0 & 0 & -2 & -3 & 0 & 1 \end{array}\right]$.

Multiplying rows by factors, $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -3/2 & -1/2 & 0 \\ 0 & 0 & 1 & -3/2 & 0 & -1/2 \end{array}\right]$.

Adding multiples of 3rd row, $\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1/2 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & -1/2 & 1/2 \\ 0 & 0 & 1 & 3/2 & 0 & -1/2 \end{array}\right]$.

Adding multiples of 2nd, $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & 1/2 & 0 \\ 0 & 1 & 0 & 0 & -1/2 & 1/2 \\ 0 & 0 & 1 & 3/2 & 0 & -1/2 \end{array}\right]$.

So the inverse is $\frac{1}{2}\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 3 & 0 & -1 \end{bmatrix}$.

Solution Q1 S2: The problem codes as $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$.

Subtracting the 1st row from the 3rd, $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right]$.

Subtracting the 2nd from the 3rd, $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$.

Subtracting the 3rd from the others, $\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & 1 & -1 \\ 0 & 1 & 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$.

Subtracting the 2nd from the 1st, $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$.

So the answer is $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix}$.

Solution Q2 S1: The distance is $\dfrac{|2.3 + 4.4 - 3.4 - 6|}{\sqrt{2^2 + 4^2 + 3^2}} = 4/\sqrt{29}$.

Solution Q2 S2: The distance is $\dfrac{|1.5 + 1.5 + 3.2 - 10|}{\sqrt{1^2 + 1^2 + 3^2}} = 6/\sqrt{11}$.

Solution Q3 S1: Applying row operations to the matrix

$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{bmatrix}$, we obtain first $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$, then $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. Solving, we have $y = -2z$ and $x = -y - z = z$. Thus, $(x, y, z) = z(1, -2, 1)$. So the solution space has basis $\{(1, -2, 1)\}$ and dimension 1.

Solution Q3 S2: Applying row operations to $\begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$, we obtain $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$, then $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix}$, then $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. Solving as in Q3 S1, we obtain $(x, y, z) = z(1, -2, 1)$. As before, the solution space has basis $\{(1, -2, 1)\}$ and dimension 1.

Solution Q4 S1: A basis for S is

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

Evidently, $\dim(S) = 6$.

Solution Q4 S2: A basis for S is $\{f_0, f_2, f_4, f_6\}$ where $f_m(x) = x^m$. In particular, $\dim(S) = 4$.

Solution Q5 S1: The characteristic equation is

$$0 = (3 - \lambda)^2 - 1 = \lambda^2 - 6\lambda + 8 = (\lambda - 2)(\lambda - 4) .$$

The solutions are $\lambda \in \{2, 4\}$.

Solution Q5 S2: The characteristic equation is

$$0 = (1 - \lambda)^2 - 1 = \lambda^2 - 2\lambda = \lambda(\lambda - 2) .$$

The solutions are $\lambda \in \{0, 2\}$.

Solution Q6 S1: By inspection (or as in Quiz 5 Section 2), the eigenvalues are 0 and 2. The corresponding eigenvectors are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. So we can put $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$.

Solution Q6 S2: By inspection, the eigenvalues are 1, 2, 3. The corresponding eigenvectors are $\begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Hence, $P = \begin{bmatrix} 2 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

MATH 227: Introduction to Linear Algebra. Fall 2018. Midterm 1

LJB, 5 November 2018, Bilkent University.

Time allowed: 2 hours. Please put your name on EVERY sheet of your manuscript. The use of telephones, calculators or other electronic devices is prohibited. The use of red pens or very faint pencils is prohibited too. You may take the question sheet home.

Remember to justify your answers, except in any cases where your answers are obvious.

1: 15 marks. Using Gaussian elimination, solve the simultaneous equations

$$x + y + z = 4 \, , \qquad x + 2y + 4z = 5 \, , \qquad x + 4y + 16z = 6 \, .$$

2: 40 marks. Consider the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 16 \end{bmatrix}$.

(a) Using the Gauss–Jordan method, calculate the inverse A^{-1}.

(b) Using any method, calculate the determinant $\det(A)$.

(c) Using part (b) and the method of cofactors, again calculate A^{-1}.

3: 15 marks. Let $\underline{x} = (1, 4, 7, 5, 9, 7, 13, 10)$ and $\underline{y} = (4, 0, 5, 7, 5, 7, 8, 12)$ be sample values for two paired variables. Calculate the Pearson correlation coefficient of the sample values.

4: 15 marks. Find the distances between:

(a) the point $(1, 2, 3)$ and the plane consisting of the points (x, y, z) such that $2x+y+2z = 19$.

(b) the point $(1, 2, 3)$ and the line consisting of the points (x, y, z) such that $2x + y + 2z = 19$ and $2x + 2y + z = 21$.

5: 15 marks. Let n be a positive integer. An $n \times n$ matrix M is called a **pseudostochastic matrix** provided the sum of the entries in each column is 1.

(a) Let M and N be pseudostochastic $n \times n$ matrices. Show that MN is pseudostochastic.

(b) Let $\underline{x} = (x_1, ..., x_n)$ and $\underline{y} = (y_1, ..., y_n)$ be n-dimensional vectors such that $\underline{y} = M\underline{x}$. Express $y_1 + ... + y_n$ in terms of the n numbers x_1, ..., x_n. (Remember to justify your answer.)

Midterm 1 Solutions

1: The system is $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 2 & 4 & 5 \\ 1 & 4 & 16 & 6 \end{array}\right]$.

Subtracting row 1 from the other two rows, $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 1 \\ 0 & 3 & 15 & 2 \end{array}\right]$.

Subtracting 3 times row 2 from row 3 gives $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 6 & -1 \end{array}\right]$.

So $z = -1/6$. Hence $y = 1 - 3z = 3/2$ and $x = 4 - y - z = 24/6 - 9/6 + 1/6 = 8/3$. In conclusion, $(x, y, z) = (8/3, 3/2, -1/6)$.

2: Part (a). We code the problem as $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 1 & 0 \\ 1 & 4 & 16 & 0 & 0 & 1 \end{array}\right]$. The operations in part (a) give $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 3 & 15 & -1 & 0 & 1 \end{array}\right]$, then $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 6 & 2 & -3 & 1 \end{array}\right]$.

Multiplying row 3 by a factor, $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1/3 & -1/2 & 1/6 \end{array}\right]$.

Adding multiples of row 3, $\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 2/3 & 1/2 & -1/6 \\ 0 & 1 & 0 & -2 & 5/2 & -1/2 \\ 0 & 0 & 1 & 1/3 & -1/2 & 1/6 \end{array}\right]$.

Subtracting row 2 from row 1, $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8/3 & -2 & -1/3 \\ 0 & 1 & 0 & -2 & 5/2 & -1/2 \\ 0 & 0 & 1 & 1/3 & -1/2 & 1/6 \end{array}\right]$.

Therefore, $A^{-1} = \dfrac{1}{6}\begin{bmatrix} 16 & -12 & 2 \\ -12 & 15 & -3 \\ 2 & -3 & 1 \end{bmatrix}$.

Part (b). In view of the upper triangular matrix in part (a), we have $\det(A) = 6$.

Part (c). The matrix of minors is

$$\begin{bmatrix} 2.16 - 4.4 & 1.16 - 1.4 & 1.4 - 1.2 \\ 1.16 - 1.4 & 1.16 - 1.1 & 1.4 - 1.1 \\ 1.4 - 1.2 & 1.4 - 1.1 & 1.2 - 1.1 \end{bmatrix} = \begin{bmatrix} 16 & 12 & 2 \\ 12 & 15 & 3 \\ 2 & 3 & 1 \end{bmatrix}.$$

The matrix of cofactors is $\begin{bmatrix} 16 & -12 & 2 \\ -12 & 15 & -3 \\ 2 & -3 & 1 \end{bmatrix}$. Taking the transpose and dividing by the determinant calculated in part (b), we recover A^{-1} as in part (a).

3: The means are $\overline{x} = (1 + 4 + 7 + 5 + 9 + 7 + 13 + 10)/8 = 56/8 = 7$ and $\overline{y} = (4 = 0 + 5 + 7 + 5 + 7 + 8 + 12)/8 = 48/8 = 6$. The centred vectors are

$$\widetilde{x} = (-6, -3, 0, -2, 2, 0, 6, 3)\ , \qquad \widetilde{y} = (-2, -6, -1, 1, -1, 1, 2, 6)\ .$$

We have $\widetilde{x}.\widetilde{y} = 12 + 18 + 0 - 2 - 2 + 0 + 12 + 18 = 2.28 = 56$. Also,

$$\| \widetilde{x} \|^2 = 36+9+4+4+36+9 = 2.49 = 98\,, \quad \| \widetilde{y} \|^2 = 4+36+1+1+1+1+36+4 = 2.42 = 84\,.$$

The correlation coefficient is $\rho = \dfrac{\widetilde{x}.\widetilde{y}}{\| \widetilde{x} \| \,.\, \| \widetilde{y} \|} = \dfrac{56}{\sqrt{2.49}.\sqrt{2.42}} = \dfrac{56}{2.7.\sqrt{42}} = 4/\sqrt{42}$.

4: Part (a). The distance is $\dfrac{|2.1 + 1.2 + 2.3 - 19|}{\sqrt{2^2 + 1^2 + 2^2}} = \dfrac{|-9|}{\sqrt{9}} = 3$.

Part (b). Subtracting the equation of one plane from the equation of the other yields $y - z = 2$. Hence $2x = 19 - y - 2z = -3z + 17$. So the point (x, y, z) lies on the line if and only if

$$(x, y, z) = (-3z/2 + 17/2, z + 2, z)\,.$$

The distance D between (x, y, z) and $(1, 2, 3)$ is given by

$$D^2 = \| (x, y, z) - (1, 2, 3) \| = (x - 1)^2 + (y - 2)^2 + (z - 3)^2\,.$$

Completing the square,

$$4D^2 = (-3z + 15)^2 + 4z^2 + 4(z - 3)^2 = 17z^2 - 114z + 261 = 17(z - 57/17)^2 + 261 - 57^2/17\,.$$

The required distance d is the minimum value of D. We have

$$17.261 - 57^2 = 17 + 17.4.65 - 4.14.57 - 57 = 4(17.65 - 14.57 - 10)\,.$$

So $17d^2 = 3.65 + 14(65 - 57) - 10 = 195 + 14.8 - 10 = 185 + 112 = 297$. So $d = \sqrt{297/17}$.

Alternative for part (b): Let π_1 and π_2 be the planes given by the equations $2x+y+2z-19 = 0$ and $2x + 2y + z - 21 = 0$, respectively. Let $P = (1, 2, 3)$. Let P_1 and P_2, respectively, be the closest point on π_1 and π_2 to P. In part (a), we saw that the distance between P and P_1 is 3. The distance between P and P_2 is

$$\frac{|2.1 + 2.2 + 1.3 - 21|}{\sqrt{2^2 + 2^2 + 1^2}} = \frac{|-12|}{3} = 4\,.$$

The angle θ between P_1 and P_2 at P is given by

$$\cos(\theta) = \frac{(2, 1, 2).(2, 2, 1)}{\| (2, 1, 2) \| \,.\, \| (2, 2, 1) \|} = \frac{4 + 2 + 2}{3.3} = \frac{8}{9}\,.$$

Let Q be the closest point on the line to P. The points P, P_1, P_2, Q all lie on a plane. We can coordinatize the plane by putting $P = (0, 0)$ and $P_1 = (p, q)$ and $P_2 = (4, 0)$ where $p = 3\cos(\theta)$ and $\| (p, q) \| = 3$. Then $q^2 = 3^2 - p^2 = 9 - 64/9 = (81 - 64)/9 = 17/9$. So $q = \sqrt{17}/3$. Now Q is the intersection of the line whose points (x, y) satisfy $x = 4$ and the line whose points (x, y) satisfy $(p, q).(x, y) = \| (p, q) \|^2$. So $Q = (x, y)$ where $x = 4$ and $8x/3 + \sqrt{17}y/3 = 9$, in other words, $y = (27 - 32)/\sqrt{17} = -5/\sqrt{17}$. Finally, the distance d between P and Q is given by $d^2 = x^2 + y^2 = 16 + 25/17 = (272 + 25)/17 = 297/17$. So $d = \sqrt{297/17}$.

5: Part (a). Let $M_{i,j}$ denote the (i, j)-entry of M, and similarly for N and MN. We have $\sum_i M_{i,j} = \sum_j N_{j,k} = 1$ and $(MN)_{i,k} = \sum_j M_{i,j}N_{j,k}$, hence $\sum_i (MN)_{i,k} = \sum_{i,j} M_{i,j}N_{j,k} = 1$.

Part (b). We have $y_i = \sum_j M_{i,j}x_j$, so $\sum_i y_i = \sum_{i,j} M_{i,j}x_j = \sum_j x_j$.

MATH 227: Introduction to Linear Algebra. Fall 2018. Midterm 2

LJB, 3 December 2018, Bilkent University.

Time allowed: 2 hours. Please put your name on EVERY sheet of your manuscript. The use of telephones, calculators or other electronic devices is prohibited. The use of red pens or very faint pencils is prohibited too. You may take the question sheet home.

Remember to justify your answers, except in any cases where your answers are obvious.

1: 25 marks. For each of the following real vector spaces V, find a basis for V and evaluate $\dim(V)$:

(a) $V = \mathbb{R}^5$.

(b) $V = \text{span}\{(1,1,1,1,1), (1,3,5,7,11), (3,5,7,9,13), (3,5,7,11,13), (7,13,19,25,37)\}$ as a subspace of $\mathbb{R}^5$.

(c) V is the vector space whose vectors are the linear transformations $\mathbb{R}^2 \to \mathbb{R}^2$.

2: 25 marks. For each of the following subsets U of a real vector space V, show that U is a subspace of V, find a basis for U and evaluate $\dim(U)$.

(a) $V = \text{Mat}_2(\mathbb{R})$ (the real vector spaces whose vectors are the 2×2 matrices with entries in $\mathbb{R}$), and U is the set of matrices A in V such that $A \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} A$.

(b) V is as in part (c) of Question 1 and U is the set of linear transformations $\theta : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\theta((1,0)) \in \text{span}\{(1,0)\}$.

3: 25 marks. Consider a Markov process with three states. The transition probability from State 1 to State 2 is equal to the transition probability from State 1 to State 3. If the system is in State 1 or 3 at time t, then the probability of remaining in that state at time $t+1$ is $1/2$. If it is in State 2 at time t, then the probability of remaining in that state at time $t+1$ is $1/4$. The system cannot move from State 2 or State 3 to State 1. Suppose the system is in State 1 at time $t = 0$. What is the probability of the system being in State 3 at time $t = 6$? (For full marks, express your answer in a simple form.)

4: 25 marks. Let X and Y be subspaces of a finite-dimensional real vector space V.

(a) We define $X \cap Y$ to be the set of vectors v such that $v \in X$ and $v \in Y$. Show that $X \cap Y$ is a subspace of V.

(b) We define $X + Y$ to be the set of vectors having the form $x + y$ where $x \in X$ and $y \in Y$. Show that $X + Y$ is a subspace of V.

(c) Show that there exists a basis $\mathcal{B}$ for V such that $\mathcal{B}$ contains bases for all four subspaces X, Y, $X \cap Y$, $X + Y$.

(d) Find a formula expressing $\dim(X + Y)$ in terms of $\dim(X)$, $\dim(Y)$, $\dim(X \cap Y)$. (Hint: to prove your answer, part (c) may be helpful.)

Midterm 2 Solutions

1: Part (a). The standard basis

$$\{(1,0,0,0,0),(0,1,0,0,0),(0,0,1,0,0),(0,0,0,1,0),(0,0,0,0,1)\}$$

is a basis for V. In particular, $\dim(V) = 5$.

Part (b). To examine the linear relations between the given vectors,

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 5 & 7 & 11 \\ 3 & 5 & 7 & 9 & 13 \\ 3 & 5 & 7 & 11 & 13 \\ 7 & 13 & 19 & 25 & 37 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 6 & 10 \\ 0 & 2 & 4 & 6 & 10 \\ 0 & 2 & 4 & 8 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 5 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

To obtain the second matrix, we added multiples of the first row to the other rows. To obtan the third, we added multiples of the second row to the third, fourth, fifth rows, then divided two rows by 2 and interchanged two rows. Evidently, $\{(1,1,1,1,1),(0,1,2,3,5),(0,0,0,1,0)\}$ is a basis for V and and $\dim(V) = 3$.

Part (c). Let $\alpha_{1,1}$, $\alpha_{1,2}$, $\alpha_{2,1}$, $\alpha_{2,2}$, respectively, be the linear transformations $\mathbb{R}^2 \to \mathbb{R}^2$ represented by the matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ with respect to the standard basis $\{(1,0),(0,1)\}$. Then $\{\alpha_{1,1}, \alpha_{1,2}, \alpha_{2,1}, \alpha_{2,2}\}$ is a basis for V. Hence, $\dim(V) = 4$.

2: Part (a). Given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V$, then $A \in U$ if and only if $\begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ b & d \end{bmatrix}$, that is, $c = 0$ and $a = d$. Hence $U = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : a, b \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$. Since U is the span of a set, U is a subspace. Plainly, U has basis $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$ and $\dim(U) = 2$.

Part (b). Given $\theta \in V$, then $\theta \in U$ if and only if θ is represented by a matrix having the form $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ with $a, b, c \in \mathbb{R}$. So, in the notation of part (c) of Question 1, $U = \operatorname{span}\{\alpha_{1,1}, \alpha_{1,2}, \alpha_{2,2}\}$. In particular, U is a subspace of V. Furthermore, U has basis $\{\alpha_{1,1}, \alpha_{1,2}, \alpha_{2,2}\}$ and $\dim(U) = 3$.

3: The Markov matrix for the process is $M = N/4$ where $N = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 2 \end{bmatrix}$.

We have $N^2 = \begin{bmatrix} 4 & 0 & 0 \\ 2+1+2 & 1+6 & 2+4 \\ 2+3+2 & 3+6 & 6+4 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 5 & 7 & 6 \\ 7 & 9 & 10 \end{bmatrix}$

and $N^4 = \begin{bmatrix} 16 & * & * \\ 20+35+42 & * & * \\ 28+45+70 & * & * \end{bmatrix} = \begin{bmatrix} 16 & * & * \\ 97 & * & * \\ 143 & * & * \end{bmatrix}$

where $*$ indicates an unspecified entry. The $(3,1)$ entry of N^6 is

$$16.7 + 97.9 + 143.10 = 112 + 873 + 1430 = 2415\ .$$

The answer is the $(3,1)$ entry of M, which is $2415/4^6 = 2415/4096$.

4: Part (a). Let $v_1, v_2 \in X \cap Y$ and $\lambda_1, \lambda_2 \in \mathbb{R}$. Write $v = \lambda_1 v_1 + \lambda_2 v_2$. We are to show that $v \in X \cap Y$. We have $v \in X$ because X is a subspace. Similarly, $v \in Y$. Therefore, $v \in X \cap Y$.

Part (b). Let $v_1, v_2 \in X + Y$ and λ_1, λ_2 as before. Write $v_1 = x_1 + y_1$ and $v_2 = x_2 + y_2$ with $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Then

$$\lambda_1 v_1 + \lambda_2 v_2 = (\lambda_1 x_1 + \lambda_2 x_2) + (\lambda_2 y_1 + \lambda_2 y_2) \in X + Y \; .$$

Part (c). We make use of the fact that, given a subspace K of a finite-dimensional real vector space L, then any basis $\{s_1, ..., s_m\}$ of K extends to a basis $\{s_1, ..., s_n\}$ of L, with $m \leq n$.

Let $\{e_1, ..., e_a\}$ be a basis for $X \cap Y$. Extend to a basis $\{e_1, ..., e_a, f_1, ..., f_b\}$ of X and to a basis $\{e_1, ..., e_a, g_1, ..., g_c\}$ of Y. We claim that the set $\mathcal{A} = \{e_1, ..., e_a, f_1, ..., f_b, g_1, ..., g_c\}$ is a basis for $X + Y$. Plainly, $\mathcal{A}$ spans $X + Y$. To show linear independence, consider a linear combination

$$\sum_i \lambda_i e_i + \sum_j \mu_j f_j + \sum_k \nu_k g_k = 0 \; .$$

Then $\sum_k \nu_k g_k \in X \cap Y$, so $\sum_k \nu_k g_k = \sum_i \lambda'_i e_i$ for some coefficients λ'_i. Therefore

$$\sum_i (\lambda_i + \lambda'_i) e_i = \sum_j \mu_j f_j \; .$$

By the linear independence of the above basis for Y, each $\mu_j = 0$. Similarly, each $\nu_k = 0$. The linear independence of the above basis for $X \cap Y$ now implies that each $\lambda_i = 0$. We have shown that $\mathcal{A}$ is linearly independent, and the claim is established.

Finally, we can extend the above basis for $X + Y$ to a basis

$$\mathcal{B} = \{e_1, ..., e_a, f_1, ..., f_b, g_1, ..., g_c, h_1, ..., h_d\}$$

for V. Plainly, $\mathcal{B}$ contains the above bases for X, Y, $X \cap Y$, $X + Y$.

Part (d). In the notation of part (c),

$$\dim(X + Y) = a + b + c = (a + b) + (a + c) - a = \dim(X) + \dim(Y) - \dim(X \cap Y) \; .$$

MATH 227: Introduction to Linear Algebra. Fall 2018.

Makeup for Midterm 2

LJB, 26 December 2018, Bilkent University.

Time allowed: 2 hours. Please put your name on EVERY sheet of your manuscript. The use of telephones, calculators or other electronic devices is prohibited. The use of red pens or very faint pencils is prohibited too. You may take the question sheet home.

Remember to justify your answers, except in any cases where your answers are obvious.

1: 25 marks. For each of the following real vector spaces, find a basis.

(a) The span, in $\mathbb{R}^5$, of the set

$$\{(1,1,1,1,1),(0,1,2,3,4),(0,0,1,4,9),(0,0,0,1,8),(0,0,0,0,1)\}\ .$$

(b) The space of functions from $\{1,2,3,4,5,6\}$ to $\mathbb{R}$.

(c) The space of functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x)$ has the form $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ where $a_0, a_1, a_2, a_3 \in \mathbb{R}$.

(d) The space of functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x)$ has the form in part (b) and $f(1) = f(2) = f(3) = 0$.

2: 25 marks. Let $x_0 = y_0 = z_0 = 1$ and

$$x_{n+1} = x_n + y_n + z_n\ , \qquad y_{n+1} = x_n + 2y_n + 3z_n\ , \qquad z_{n+1} = x_n + 3y_n + 4z_n\ .$$

Using matrix algebra, calculate x_8. (No marks will be awarded for using a different method.)

3: 25 marks. Consider subspaces $V_1 \leq V_2 \leq \mathbb{R}^3$ where $\dim(V_1) = 1$ and $\dim(V_2) = 2$. (The relation $\leq$ means *is a subspace of.*) Let W be the set of linear transformations α of $\mathbb{R}^3$ such that $\alpha(V_1) \leq V_1$ and $\alpha(V_2) \leq V_2$.

(a) Show that W is a vector space.

(b) Describe a basis for W in terms of a suitably chosen basis for $\mathbb{R}^3$.

(c) What is the dimension of W?

4: 25 marks. Show that, given any finite-dimensional real vector space W and any subspace X of W satisfying $0 < \dim(X) < \dim(W)$, there exist subspaces Y and Z of W such that $X + Y = X + Z = W$ and $Y \cap Z = \{0\}$. (Recall, for subspaces A and B of W, we define $A + B$ to be the subspace of W consisting of the vectors having the form $a + b$ with $a \in A$ and $b \in B$.)

MATH 227: Introduction to Linear Algebra. Fall 2018. Final

LJB, 2 January 2019, Bilkent University.

Time allowed: 2 hours. Please put your name on EVERY sheet of your manuscript. The use of telephones, calculators or other electronic devices is prohibited. The use of red pens or very faint pencils is prohibited too. You may take the question sheet home.

Remember to justify your answers, except in any cases where your answers are obvious.

1: 15 marks. Consider the matrix $M = \begin{bmatrix} 1 & 4 & 9 & 16 & 25 \\ 1 & 5 & 11 & 21 & 24 \\ 1 & 6 & 13 & 26 & 23 \\ 1 & 7 & 15 & 31 & 22 \end{bmatrix}$.

(a) Find a basis for the image of M.
(b) Without finding any basis for the null space of M, determine the nullity of M.

2: 55 marks. Consider the matrix $A = \begin{bmatrix} 5 & 2 & 1 \\ 1 & 4 & -1 \\ -1 & -2 & 3 \end{bmatrix}$.

(a) Show that $(1, -1, 1)$ is an eigenvector of A.

(b) Find the eigenvalues of A.

(c) Find a basis of $\mathbb{R}^3$ consisting of eigenvectors of A.

(d) Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

(e) Calculate the inverse matrix P^{-1}.

(f) Suppose that $x(0) = y(0) = z(0) = 1$ and

$$x(n+1) = 5x(n)+2y(n)+z(n),\ y(n+1) = x(n)+4y(n)-z(n),\ z(n+1) = -x(n)-2y(n)+3z(n).$$

Find a formula for $x(n)$.

3: 15 marks. Let $\mathrm{Mat}_2(\mathbb{R})$ denote the real vector space of 2×2 matrices over $\mathbb{R}$. Let $\theta : \mathrm{Mat}_2(\mathbb{R}) \to \mathrm{Mat}_2(\mathbb{R})$ be the linear transformation such that

$$\theta\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

(a) What is the dimension of $\mathrm{Mat}_2(\mathbb{R})$?

(b) What is the rank of θ?

(c) What is the nullity of θ?

4: 15 marks. Does there exist a 2×2 matrix X over the complex numbers such that X is both non-invertible and non-diagonalizable? (If your answer is yes, write down such a matrix X and explain why it has the required properties. If your answer is no, prove it.)

Solutions to Final

1: Part (a). Subtracting multiples of the first column from the others, we obtain

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 5 & -1 \\ 1 & 2 & 4 & 10 & -1 \\ 1 & 3 & 6 & 15 & -3 \end{bmatrix} .$$

It is now clear that the image of M has basis $\{(1,1,1,1),(0,1,2,3)\}$.

Part (b). By part (a), $\mathrm{rank}(M) = 2$. By the rank-nullity formula, $\mathrm{null}(A) = 5 - 2 = 3$.

2: Part (a). Write $f = (1,-1,1)$. We have $Af = 4f$. So f is an eigenvector with eigenvalue 4.

Part (b). The eigenvalues λ_1, λ_2, λ_3 satisfy

$$0 = \begin{vmatrix} 5-\lambda & 2 & 1 \\ 1 & 4-\lambda & -1 \\ -1 & -2 & 3-\lambda \end{vmatrix} = (5-\lambda)\big((4-\lambda)(3-\lambda)-2\big) - 2(2-\lambda) + 2 - \lambda$$

$$= (5-\lambda)(\lambda^2 - 7\lambda + 10) + \lambda - 2 = -\lambda^3 + 12\lambda^2 - 44\lambda + 48 .$$

So $\lambda_1 + \lambda_2 + \lambda_3 = 12$ and $\lambda_1\lambda_2\lambda_3 = 48$. By part (a), one of the eigenvalues is 4. So the other two eigenvalues have sum 8 and product 12. It is now clear that we can put $\lambda_1 = 2$ and $\lambda_2 = 4$ and $\lambda_3 = 6$.

Part (c). We shall find a basis $\{f_1, f_2, f_3\}$ for $\mathbb{R}^3$ where each f_i is an eigenvector with eigenvalue λ_i. By part (a), we can put $f_2 = f$.

Write $f_1 = (x,y,z)$. In row notation, $0 = (A-\lambda_1)f_1 = (3x+2y+z, x+2y-z, -x-2y+z)$. If $x = -1$, then $\begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, whence $\begin{bmatrix} y \\ z \end{bmatrix} = \frac{1}{-4}\begin{bmatrix} -1 & -1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. So we can put $f_1 = (-1,1,1)$.

Now write $f_3 = (x,y,z)$. Then $0 = (A-\lambda_3)f_3 = (-x+2y+z, x-2y-z, -x-2y-3z)$. If $x = 1$, then $\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, whence $\begin{bmatrix} y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. So we can put $f_3 = (1,1,-1)$.

Part (d). By (b) and (c), we can put $P = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$.

Part (e). The cofactor matrix for P is $\begin{bmatrix} 0 & -2 & 2 \\ -2 & 0 & -2 \\ 2 & -2 & 0 \end{bmatrix}$. Taking alternating $\pm$ signs and then the transpose, we obtain the adjugate matrix, which we divide by $\det(P) = 0+2+2 = 4$ to obtain $P^{-1} = \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

Part (f). We have

$$A^n = PD^nP^{-1} = \frac{1}{2}\begin{bmatrix} -2^n & 4^n & 6^n \\ 2^n & -4^n & 6^n \\ 2^n & 4^n & -6^n \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 6^n + 4^n & 6^n - 2^n & 4^n - 2^n \\ * & * & * \\ * & * & * \end{bmatrix}$$

where $*$ indicates entries that we have no need to calculate. We have, in row notation,

$$(x(n), y(n), z(n)) = A^n(1, 1, 1) \, .$$

Therefore, $x(n) = \big((6^n + 4^n) + (6^n - 2^n) + (4^n - 2^n)\big)/2 = 6^n + 4^n - 2^n$.

Comment 1: The point of part (a) was to make it straightforward to find out the eigenvalues without having to solve a cubic equation.

Comment 2: For those who know how the trace and determinant can be expressed in terms of eigenvalues (mentioned briefly in lectures), the following alternative for part (b) is slightly easier: the eigenvalues λ_1, λ_2, λ_3 satisfy $\lambda_1 + \lambda_2 + \lambda_3 = \mathrm{tr}(A) = 5 + 4 + 3 = 12$ and $\lambda_1\lambda_2\lambda_3 = \det(A) = 5.10 - 2.2 + 1.2 = 48$. The rest of the argument proceeds as before.

3: Part (a). We have $\dim(\mathrm{Mat}_2(\mathbb{R})) = 4$.

Part (b). By considering the subspace given by $b = d = 0$ and the subspace given by $a = c = 0$, we see that $\mathrm{rank}(\theta) = 1 + 1 = 2$.

Part (c). By an argument similar to that in part (b) or, alternatively, by invoking the rank-nullity formula, we have $(\theta) = 2$.

4: Yes, such X does exist, for instance, $X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Plainly, X is non-invertible. The eigenvectors of X are precisely the vectors having the form $(x, 0)$ where $x \neq 0$. So $\mathbb{C}^2$ does not have a basis consisting of eigenvectors of X. Therefore, X is non-diagonalizable.