Archive for

MATH 225, Linear Algebra and Differential Equations

Spring 2026

Bilkent University, compilation date: 13 October 2026.

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Course specification

MATH 225, *Linear Algebra and Differential Equations*, Spring 2026

Sections 1 and 4: Laurence Barker, Office SA 129, barker at fen dot bilkent dot edu dot tr.

Assistants:
Ziya Kağan Akman, Section 1,
Zeyd Bahadir Kırçu, Section 4.

The main course documentation is on the course Moodle and STARS pages. Supplementary material can be found on homepage: http://www.fen.bilkent.edu.tr/ barker/fenis2.html .

Required textbook:

• C. Henry Edwards, David E. Penney, David T. Calvis, *Differential Equations and Linear Algebra*, 4th Edition, Global Edition (Pearson, 2021).

Classes:

Section 1: Tuesdays 15:30 - 17:20, Fridays 10:30 - 12:20, room FC-Z23C.

Section 4: Tuesdays 10:30 - 12:20, Thursdays 15:30 - 17:20, room A-Z27.

Weekly Syllabus

The format below is, *Week number; Monday date; Subtopics.*

1: 19 Jan, *no class Tuesday.* Linear systems, matrices and Gaussian elimination, reduced row echelon matrices: 3.1, 3.2, 3.3.

2: 26 Jan. Matrix operations, inverses of matrices: 3.4, 3.5.

3: 2 Feb. Determinants, vectors and vector spaces: 3.6, 4.7, 4.1.

4: 9 Feb. Subspaces, linear combinations, independence of vectors: 4.2, 4.3.

5: 16 Feb. Bases and dimensions for vector spaces: 4.4.

6: 23 Feb. Row and column spaces: 4.5

7: 2 Mar. Orthogonal vectors in $\mathbb{R}^n$, eigenvalues and eigenvectors: 4.6, 6.1

8: 9 Mar. Diagonalisation of matrices and applications: 6.2, 6.3.

9: 16 Mar. *Whole week holiday.*

10: 23 Mar. Introduction to differential equations, general and particular solutions, separable equations: 1.1-1.4.

11: 30 Mar. Linear first order equations, substitution methods, exact equations: 1.5, 1.6.

12: 6 Apr. Systems of differential equations: 7.1-7.3

13: 13 Apr. Second order linear ODEs equations, general solutions: 5.1, 5.2.

14: 20 Apr. *No class Thursday.* Homogeneous ODEs with constant coefficients: 5.3.

15: 27 Apr. *No class Friday.* Nonhomogenous ODEs, method of variation of parameters: 5.5

16: 4 May. Nonhomogenous ODEs, method of undetermined coefficients 5.5.

A minimum of 75% attendance is required.

Assessment:

- 5 Quizzes 5%
- 5 Homeworks 15%.
- Midterm 40%.
- Final, 40%.

FZ criteria: Any student whose Midterm exam score is less than 20 out of 100 receives an FZ and cannot take the Final exam.

Midterm details: Saturday, 14 March, 09:30 - 11:30,
B-Z01, B-Z02, B-Z04, B-Z05, B-Z08, B-106, B-204, B-206.

Speaking: All speaking in class must address everyone in the room. Questions and comments are welcome: they make learning easier, and they make teaching easier too.

Class Announcements: All students, including any absentees from a class, will be deemed responsible for awareness of class announcements.

Quizzes, with solutions

MATH 225, *Linear algebra and differential equations*

Spring 2026, Sections 1 and 4

Laurence Barker, version: 8 May 2026

Duration: 10 minutes. *So the quizzes are very easy, just quick tests of basic comprehension.* Quiz $m.n$ is the Section m quiz number n.

Quiz questions

Quiz 1.1: *6 February.* By Gaussian elimination, find the unique solution to the linear system

$$x + y + z = 6 \ , \qquad 3x - y - z = -2 \ , \qquad 4x + 2y - z = 5 \ .$$

Quiz 4.1: *5 February.* By Gaussian elimination, find the unique solution to the linear system

$$x + 2y + 3z = 10 \ , \qquad 3x + 8y + 17z = 42 \ , \qquad 2x + 6y + 15z = 33 \ .$$

Quiz 1.2: *20 February.* Let $A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.
Find a basis for the image of A and evaluate $\mathrm{rank}(A)$ and $\mathrm{null}(A)$.

Quiz 4.2: *19 February.* Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.
Find a basis for the null space of A and evaluate $\mathrm{rank}(A)$ and $\mathrm{null}(A)$.

Quiz 1.3: *3 March.* Let V be the inner product space of functions $f : \mathbb{R} \to \mathbb{R}$ that have the form $f(x) = ax^2 + bx^4$, the inner product given by

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x) \, \mathrm{d}x \ .$$

Let $g_1(x) = x^2$ and $g_2(x) = x^4$. Evaluate $h_1(x)$ and $h_2(x)$ where $\{h_1, h_2\}$ is the orthogonal basis obtained by the Gram–Schmidt process from $\{g_1, g_2\}$.

Quiz 4.3: *5 March.* Let V be the real vector space of functions $f : \mathbb{R} \to \mathbb{R}$ that have the form $f(x) = a + bx$. We make V become an inner product space with inner product given by

$$\langle f, g \rangle = \int_{0}^{1} f(x)g(x) \, \mathrm{d}x \ .$$

Let $p_1(x) = 1$ and $p_2(x) = x$. Evaluate $r_1(x)$ and $r_2(x)$ where $\{r_1, r_2\}$ is the orthonormal basis obtained from $\{p_1, p_2\}$ by the Gram–Schmidt process.

Quiz 1.4: *17 April.* Using the method of integrating factors, solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y = \cos(e^x) \ .$$

Your answer should include an arbitrary constant.

Quiz 4.4: *16 April.* Using an integrating factor, solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 6x^2 y = x^2 \ .$$

Your answer should include an arbitrary constant.

Quiz 1.5: *5 May.* **(a)** Find all the solutions to the equation $y''' - 4y'' + 5y' - 2y = 0$. (The roots of the auxilliary cubic are 1 and 2.)

(b) How many solutions are there satisfying $y'(0) = y(0) = 0$?

Quiz 4.5: *5 May.* **(a)** Find all the solutions to the equation $y^{(4)} + y = 0$.

(b) How many solutions are there satisfying $y'''(0) = y''(0) = y'(0) = y(0) = 0$?

Quiz solutions

Solution 1.1: We apply Gaussian elimination. The augmented matrix is $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 3 & -1 & -9 & -2 \\ 4 & 2 & -1 & 5 \end{array}\right]$.

Subtracting multiples of row 1 from rows 2 and 3 gives $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -4 & -4 & -20 \\ 0 & -2 & -5 & -19 \end{array}\right]$.

Multiplying rows 2 and 3 by nonzero factors, $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & 5 \\ 0 & 2 & 5 & 19 \end{array}\right]$.

Subtracting twice row 2 from row 3, we obtain $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 3 & 9 \end{array}\right]$.

Completing the reduction to row echelon form, $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right]$.

So $x + y + z = 6$ and $y + z = 5$ and $z = 3$. Finally, $(x, y, z) = (1, 2, 3)$.

Solution 4.1: The augmented matrix is $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 3 & 8 & 17 & 42 \\ 2 & 6 & 15 & 33 \end{array}\right]$.

Subtracting multiples of the first row from the others, we have $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & 2 & 8 & 12 \\ 0 & 2 & 9 & 13 \end{array}\right]$.

Subtracting row 2 from row 3, then dividing row 2 by 2, yields $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 1 & 3 \end{array}\right]$.

The linear system has been reduced to

$$x + 2y + 3z = 10\,, \qquad y + 4z = 6\,, \qquad z = 3\,.$$

The unique solution is $(x, y, z) = (3, 2, 1)$.

Solution 1.2: Plainly, a basis for the image is $\{(1, 0, 0), (0, 1, 0)\}$. Hence $\mathrm{rank}(A) = 2$. By the rank-nullity formula, $\mathrm{null}(A) = 2$.

Solution 4.2: Plainly, a basis for the null space is $\{(0, 1, -1, 0)\}$. Hence $\mathrm{null}(A) = 1$. By the rank-nullity formula, $\mathrm{rank}(A) = 3$.

Solution 1.3: We have $h_1 = g_1$ and

$$h_2 = g_2 - \frac{\langle h_1, g_2 \rangle}{\|h_1\|^2}\, h_1\,.$$

So $h_1(x) = x^2$. We have

$$\langle h_1, g_2 \rangle = \int_{-1}^{1} h_1(x)\, g_2(x)\, \mathrm{d}x = \int_{-1}^{1} x^6\, \mathrm{d}x = \left.\frac{x^7}{7}\right|_{-1}^{1} = 2/7\,,$$

$$\|h_1\|^2 = \int_{-1}^{1} h_1(x)^2 \, \mathrm{d}x = \frac{x^5}{5} \, |_{-1} \, |^1 = 2/5 \; .$$

In conclusion, $h_1(x) = x^2$ and $h_2(x) = x^4 = 5x^2/7$.

Solution 4.3: Let $\{q_1, q_2\}$ be the orthogonal basis obtained from $\{p_1, p_2\}$. Then $r_1 = q_1/\|q_1\|$ and $r_2 = q_2/\|q_2\|$. We have $q_1 = p_1$. Now

$$\|q_1\|^2 = \int_0^1 \mathrm{d}x = 1 \; .$$

So $r_1(x) = 1$. We have

$$q_2 = p_2 - \frac{\langle q_1, p_2 \rangle}{\|q_1\|^2} \, q_1 \; .$$

Since $\langle q_1, p_2 \rangle = \displaystyle\int_0^1 x \, \mathrm{d}x = 1/2$ and $\|q_1\|^2 = \displaystyle\int_0^1 \mathrm{d}x = 1$, we have $q_2(x) = x - 1/2$. We have,

$$\|q_2\|^2 = \int_0^1 (x - 1/2)^2 \, \mathrm{d}x = \int_{-1/2}^{1/2} y^2 \, \mathrm{d}y = \left. \frac{y^3}{3} \right|_{-1/2}^{1/2} = 1/12 \; .$$

So $r_2(x) = 2\sqrt{3} q_2(x)$. In conclusion, $r_1(x) = 1$ and $r_2(x) = \sqrt{3}(2x - 1)$.

Solution 1.4: The integrating factor is

$$\exp\left(\int \mathrm{d}x \right) = e^x \; .$$

Multiplying the given equation by the integrating factor,

$$\frac{\mathrm{d}}{\mathrm{d}x}(e^x y) = e^x y' + e^x y = e^x \cos(e^x) \; .$$

Maying a substitution $u = e^x$, we have

$$e^x y = \int e^x \cos(e^x) \, \mathrm{d}x = \int \cos(u) \, \mathrm{d}u = \sin(u) + C = \cos(e^x) + C$$

where C is an arbitrary constant. In conclusion, $y(x) = e^{-x}(\sin(e^x) + C)$.

Solution 4.4: Multiplying by the integrating factor $\exp\left(\displaystyle\int 6x^2 \, \mathrm{d}x \right) = \exp(2x^3)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}(\exp(2x^3) y) = \exp(2x^3) y' + 6x^2 \exp(2x^3) y = x^2 \exp(2x^3) \; .$$

Making a substitution $u = 2x^3$ and noting that $\mathrm{d}u/\mathrm{d}x = 6x^2$, we obtain

$$e^u y = \exp(2x^3) y = \int x^2 \exp(2x^3) \, \mathrm{d}x = \frac{1}{6} \int e^u \, \mathrm{d}u = e^u/6 + C$$

where C is an arbitrary constant. So $y = 1/6 + Ce^{-u}$, that is, $y(x) = 1/6 + C \exp(-2x^3)$.

Solution 1.5: Part (a). The auxilliary polynomial is $t^3 - 4t^2 + 5t - 2 = (t-1)^2(t-2)$, so the general solution is

$$y = c_1 e^x + c_2 x e^x + c_3 e^{2x}$$

where c_1, c_2, c_3 are arbitrary.

Part (b). Since $y''(0)$ is undetermined, there are infinitely many solutions satisfying the given initial conditions.

Solution 4.5: Part (a). The auxilliary polynomial is $t^4 + 1$, whose roots are distinct and of the form $(\pm 1 \pm i)/\sqrt{2}$. So the general solution is

$$y = e^{x/\sqrt{2}}\big(c_1 \cos(x/\sqrt{2}) + c_2 \sin(x/\sqrt{2})\big) + e^{-x/\sqrt{2}}\big(c_3 \cos(x/\sqrt{2}) + c_4 \sin(x/\sqrt{2})\big)$$

where c_1, c_2, c_3, c_4 are arbitrary.

Part (b). By standard existence and uniqueness theorems, there is a unique solution to the given initial value problem.

Midterm, with solutions

09:30 - 11:30, 14 March 2026.

The exam paper had a different format, with space for writing solutions. Because of that difference, the rules itemized below have been subsequently edited and are just for general illustration.

• This exam consists of 2 pages, with 4 questions each carrying 25 points. Some of the questions may be divided in subquestions, for which the corresponding number of points awarded is specified. All the questions are mandatory.

• Read each question carefully. Write your answer neatly.

• Show all your working. Correct answers without sufficient explanation will not get full credit.

• Electronic devices and cheat-sheets are prohibited.

Question 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

where a_0, a_1, a_2, a_3, a_4 are real numbers. Let u and v be real numbers. Suppose

$$f(0) = v\ , \qquad f(1) = u\ , \qquad f(2) = v\ , \qquad f(3) = 3u - 2v\ .$$

Find all the possible values of $(a_0, a_1, a_2, a_3, a_4)$ in terms of u and v.

Question 2. Let $A_x = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x & 1 & 1 & 1 \\ x & x & 1 & 1 \\ x & x & x & 1 \end{bmatrix}$.

(a) For which values of x is A_x invertible?

(b) For those values of x, evaluate the inverse A_x^{-1}.

(c) For all values of x, evaluate the matrix $\mathrm{adj}(A)$ (the transpose of the matrix of cofactors).

Question 3. Let $M = \begin{bmatrix} 1 & 2 & 4 & 2 & 1 \\ 1 & 3 & 9 & 3 & 1 \\ a & b & c & d & e \end{bmatrix}$ where a, b, c, d, e are real numbers such that the equation $Mx = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ does not have a solution.

(a) Evaluate the rank and nullity, $\mathrm{rank}(M)$ and $\mathrm{null}(M)$.

(b) Find a basis $\{e_1, e_2, e_3\}$ for the null space $\mathrm{Null}(A)$ such that e_1 and e_2 and e_3 have the form

$$e_1 = (x_1, x_2, 1, 0, 0)\ , \qquad e_2 = (y_1, y_2, 0, 1, 0)\ , \qquad e_3 = (z_1, z_2, 0, 0, 1)\ .$$

Question 4.(10+10+5 points) Let $V = \mathrm{span}\{v_1, v_2\} \subset \mathbb{R}^5$ where $v_1 = (-1, 2, 0, 0, 2)$ and $v_2 = (2, 0, 2, 0, -1)$.

(a) Find a basis $\{w_1, w_2, w_3\}$ for the orthogonal complement $V^\perp$ where

$$w_1 = (2, 1, a, 0, 0) \ , \qquad w_2 = (b, c, 0, 1, 0) \ , \qquad w_3 = (-a, 0, d, 0, 1) \ .$$

Determine the values of a, b, c, d.

(b) Using your basis in part (a), apply the Gram-Schmidt process to obtain an orthogonal basis for $V^\perp$.

(c) Does the vector $z = (1, 1, 0, 0, 1)$ belong to $V^\perp$? Does it belong to V?

Solutions to Midterm

Solution 1: Plainly, $a_0 = v$. Eliminating a_0, we obtain the linear system

$$a_1 + a_2 + a_3 + a_4 = u - v \ , \quad 2a_1 + 4a_2 + 8a_3 + 16a_4 = 0 \ , \quad 3a_1 + 9a_2 + 27a_3 + 81a_4 = 3u - 3v.$$

Subtracting multiples of the first equation from the others, we obtain augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & u-v \\ 0 & 2 & 6 & 14 & -2u+2v \\ 0 & 6 & 24 & 78 & 0 \end{array}\right] .$$

Multiplying rows by suitable factors to make all the leading entries 1, then subtracting row 2 from row 3, we obtain

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & u-v \\ 0 & 1 & 3 & 7 & -u+v \\ 0 & 0 & 1 & 6 & u-v \end{array}\right] .$$

Subtracting multiples of row 3 from the others,

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & -5 & 0 \\ 0 & 1 & 0 & -11 & -4u+4v \\ 0 & 0 & 1 & 6 & u-v \end{array}\right] .$$

Subtracting row 2 from row 3, we arrive at

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 6 & 4u-4v \\ 0 & 1 & 0 & -11 & -u+v \\ 0 & 0 & 1 & 6 & u-v \end{array}\right] .$$

Put $a_4 = s$, a parameter. Then

$$a_3 = -6s + u - v \ , \quad a_2 = 11s - 4u + 4v \ , \quad a_1 = -6s + 4u - 4v \ .$$

Recalling that $a_0 = v$, we conclude that

$$(a_0, a_1, a_2, a_3, a_4) = s(0, -6, 11, -6, 1) + u(0, 4, -4, 1, 0) + v(1, -4, 4, -1, 0)$$

where s is any real number.

Solution 2: Part (a). We set up the problem as $\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ x & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ x & x & 1 & 1 & 0 & 0 & 1 & 0 \\ x & x & x & 1 & 0 & 0 & 0 & 1 \end{array}\right]$.

Putting $y = x - 1$, subtractions of row 1 yield $\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & y & y & y & -x & 1 & 0 & 0 \\ 0 & 0 & y & y & -x & 0 & 1 & 0 \\ 0 & 0 & 0 & y & -x & 0 & 0 & 1 \end{array}\right]$.

It is now clear that $\det(A_x) = y^3$. Therefore, A is invertible if and only if $x \neq 1$.

Part (b). We define $z = y^{-1} = 1/(1-x)$. Scaling rows, $\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & -xz & z & 0 & 0 \\ 0 & 0 & 1 & 1 & -xz & 0 & z & 0 \\ 0 & 0 & 0 & 1 & -xz & 0 & 0 & z \end{array}\right]$.

Subtracting the bottom row from the others, $\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1+xz & 0 & 0 & -z \\ 0 & 1 & 1 & 0 & 0 & z & 0 & -z \\ 0 & 0 & 1 & 0 & 0 & 0 & z & -z \\ 0 & 0 & 0 & 1 & -xz & 0 & 0 & z \end{array}\right]$.

Subtracting row 3 from those above, $\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1+xz & 0 & -z & 0 \\ 0 & 1 & 0 & 0 & 0 & z & -z & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & z & -z \\ 0 & 0 & 0 & 1 & -xz & 0 & 0 & z \end{array}\right]$.

Subtracting row 2 from row 1, we deduce that $A_x^{-1} = \begin{bmatrix} 1+xz & -z & 0 & 0 \\ 0 & z & -z & 0 \\ 0 & 0 & z & -z \\ -xz & 0 & 0 & z \end{bmatrix}$.

Part (c). For $y \neq 0$, we have $\mathrm{adj}(A_x) = \det(A_x)A_x^{-1} = y^3 \begin{bmatrix} yz+xz & -z & 0 & 0 \\ 0 & z & -z & 0 \\ 0 & 0 & z & -z \\ -xz & 0 & 0 & z \end{bmatrix}$.

Simplifying using $yz = 1$, then appealing to continuity in the case where $y = 0$, we deduce that, for all x, we have

$$\mathrm{adj}(A_x) = (1-x)^2 \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -x & 0 & 0 & 1 \end{bmatrix} .$$

Solution 3: Part (a). Since the image of M is not the whole of $\mathbb{R}^3$, we have $\mathrm{rank}(M) \leq 2$. But the top 2 rows of M are linearly independent, so $\mathrm{rank}(M) \geq 2$. By the rank-nullity formula, $\mathrm{null}(A) = 3$.

Part (b). Since $\mathrm{rank}(A) = 2$, the bottom row of M is a linear combination of the other two rows. So the null space of A is also the null space of the matrix obtained from M by deleting the bottom row. Subtracting the top row of that matrix from the bottom row, we obtain

$$\begin{bmatrix} 1 & 2 & 4 & 2 & 1 \\ 0 & 1 & 5 & 1 & 0 \end{bmatrix} .$$

Now subtracting twice the bottom from the top, we obtain

$$\begin{bmatrix} 1 & 0 & -6 & 0 & 1 \\ 0 & 1 & 5 & 1 & 0 \end{bmatrix} .$$

We now see that

$$\mathrm{Null}(M) = \{(6r - t, -5r - s, r, s, t) : r, s, t \in \mathbb{R}\} .$$

So we can put $e_1 = (6, 0, 1, 0, 0)$ and $e_2 = (0, -1, 0, 1, 0)$ and $e_3 = (-1, 0, 0, 0, 1)$. In fact, those values of e_1, e_2, e_3 are the only possibilities.

Solution 4: (a) Let $X = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$. Then $X \in V^\perp$ if and only if X is orthogonal to both v_1 and v_2. Thus we require

$$\langle X, v_1 \rangle = 0 \quad \text{and} \quad \langle X, v_2 \rangle = 0.$$

These conditions become

$$-x_1 + 2x_2 + 2x_5 = 0,$$
$$2x_1 + 2x_3 - x_5 = 0.$$

Let $x_2 = s$, $x_4 = t$, $x_5 = u$ be free variables, we find

$$X = (2s+2u,\ s,\ -2s-\tfrac{3}{2}u,\ t,\ u) = s(2,1,-2,0,0)+t(0,0,0,1,0)+u\left(2,0,-\frac{3}{2},0,1\right),\ \ s,t,u \in \mathbb{R}$$

Therefore, a basis for $V^\perp$ is the following

$$\left\{(2,1,-2,0,0),\ (0,0,0,1,0),\ \left(2,0,-\frac{3}{2},0,1\right)\right\}.$$

(b) Let $w_1 = (2,1,-2,0,0)$, $w_2 = (0,0,0,1,0)$, $w_3 = \left(2,0,-\frac{3}{2},0,1\right)$. We apply the Gram-Schmidt process to the basis $\{w_1, w_2, w_3\}$. Set

$$u_1 = w_1 = (2,1,-2,0,0).$$

Next,

$$u_2 = w_2 - \frac{\langle w_2, u_1 \rangle}{\|u_1\|^2} u_1 = w_2 = (0,0,0,1,0)$$

Since $\langle w_2, u_1 \rangle = 0$. Similarly, we compute u_3 as

$$u_3 = w_3 - \frac{\langle w_3, u_1 \rangle}{\|u_1\|^2} u_1 - \frac{\langle w_3, u_2 \rangle}{\|u_2\|^2} u_2.$$

We calculate each term as follows

$$\begin{aligned}
\langle w_3, u_1 \rangle &= \left(2,0,-\frac{3}{2},0,1\right) \cdot (2,1,-2,0,0) = 4+0+3+0+0 = 7. \\
\|u_1\|^2 &= 2^2 + 1^2 + (-2)^2 = 4+1+4 = 9. \\
\langle w_3, u_2 \rangle &= \left(2,0,-\frac{3}{2},0,1\right) \cdot (0,0,0,1,0) = 0, \\
\|u_2\|^2 &= 1.
\end{aligned}$$

Hence

$$u_3 = w_3 - \frac{7}{9}u_1 = \left(2,0,-\frac{3}{2},0,1\right) - \frac{7}{9}(2,1,-2,0,0) = \left(\frac{4}{9},-\frac{7}{9},\frac{1}{18},0,1\right).$$

Thus an orthogonal basis for $V^\perp$ is

$$\left\{(2,1,-2,0,0),\ (0,0,0,1,0),\ \left(\frac{4}{9},-\frac{7}{9},\frac{1}{18},0,1\right)\right\}.$$

(c) Let $z = (1, 1, 0, 0, 1)$. A vector belongs to $V^\perp$ if and only if it is orthogonal to both v_1 and v_2. Compute

$$\langle z, v_1 \rangle = (1, 1, 0, 0, 1) \cdot (-1, 2, 0, 0, 2) = -1 + 2 + 0 + 0 + 2 = 3.$$

Since $\langle z, v_1 \rangle \neq 0$, it follows that $z \notin V^\perp$.

Now we check whether $z \in V$. Assuming $z \in V$, there must exist scalars $c_1, c_2 \in \mathbb{R}$ such that

$$c_1 v_1 + c_2 v_2 = z.$$

That is,

$$c_1(-1, 2, 0, 0, 2) + c_2(2, 0, 2, 0, -1) = (1, 1, 0, 0, 1).$$

Comparing coordinates gives

$$-c_1 + 2c_2 = 1, \quad 2c_1 = 1, \quad 2c_2 = 0.$$

This is an inconsistent system for c_1, c_2. Therefore, $z \notin V$. Hence $(1, 1, 0, 0, 1)$ belongs to neither V nor $V^\perp$.

Final, with solutions

18:00 - 20:00, 16 May 2026.

The exam paper had a different format, with space for writing solutions. Because of that difference, the rules itemized below have been subsequently edited and are just for general illustration.

• This exam consists of 2 pages, with 4 questions that carry 25 points each. Some of the questions may be divided in subquestions, for which the corresponding number of points awarded is specified. All the questions are mandatory.

• Read each question carefully. Write your answer neatly.

• Show all your working. Correct answers without sufficient explanation will not get full credit.

• Electronic devices and cheat-sheets are prohibited.

Question 1: (5+15+5 points.) **(a)** For which real numbers c and d do the vectors $e_1 = (c, d, d)$ and $e_2 = (d, c, d)$ and $e_3 = (d, d, c)$ comprise an orthogonal basis for $\mathbb{R}^3$?

(b) Let $a, b \in \mathbb{R}$. Let $A = \frac{1}{9}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}$ and $B = \frac{1}{9}\begin{bmatrix} 8 & 2 & 2 \\ 2 & 5 & -4 \\ 2 & -4 & 5 \end{bmatrix}$.

Let $C = aA + bB$. By considering Ae_i and Be_i for $i \in \{1, 2, 3\}$, diagonalize C.

(c) Let n be a positive integer. Evaluate C^n.

Question 2: (15+10 points.) **(a)** Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} + y = \dfrac{1}{e^{2x} - 3e^x + 2}$ for $x > \ln(2)$ with $y(1) = 0$.

(b) Find the implicit general solution to $(3x^2 + 4xy + 3y^2)\,\mathrm{d}x + (2x^2 + 6xy + 12y^2)\,\mathrm{d}y = 0$.

Question 3: (20+5 points.) **(a)** Find the general real solution to $\dfrac{\mathrm{d}\underline{x}}{\mathrm{d}t} = A\underline{x}$ where $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

(b) Using a Wronkskian, check that your solutions span a 4-dimensional space.

Question 4: (10+10+5 points.) Solve the following second-order nonhomogeneous linear differential equation with constant coefficients using the method of variation of parameters:

$$y'' - 4y' + 4y = e^{2x} \ln x, \qquad x > 0.$$

(a) Find the general solution of the associated homogeneous equation.

(b) Using the method of variation of parameters, find a particular solution of the nonhomogeneous equation.

(c) Hence write the general solution of the nonhomogeneous equation.

Hint: You may use the integration by parts formula: $\int u\,dv = uv - \int v\,du$.

Solutions to Final

Solution 1: Part (a). For $i \neq j$, we have $\langle e_i, e_j \rangle = 2cd + d^2$. So the specified basis condition holds if and only if $c \neq 0$ and $d \in \{0, -2c\}$.

Part (b). Put $d \neq 0$. We have $Ae_1 = e_1$ and $Be_1 = 0$. We have $Ae_2 = Ae_3 = 0$ and $Be_2 = e_2$ and $Be_3 = e_3$. So $C = PDP^{-1}$ where P has columns e_1, e_2, e_3 in that order and D is diagonal with diagonal entries a, b, b in that order.

Part (c). The diagonal entries of D^n are a^n, b^n, b^n. So C^n can be formed from C by replacing a and b with a^n and b^n, respectively. We deduce that $C^n = a^n A + b^n B$.

Alternative for part (c): We have $A^2 = A$ and $B^2 = B$. Also, $AB = BA = 0$. The above formula for C^n follows.

Solution 2: Part (a). Multiplying by the integrating factor $\exp(\int \mathrm{d}x) = e^x$, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}(e^x y) = \frac{e^x}{e^{2x} - 3e^x + 2} \ .$$

Integrating and substituting $u = e^x$, then $\mathrm{d}u = e^x \, \mathrm{d}x$, so

$$e^x y = \int \frac{\mathrm{d}u}{u^2 - 3u + 2} = \int \frac{\mathrm{d}u}{u-2} - \int \frac{\mathrm{d}u}{u-1} = \ln\left(\frac{u-2}{u-1}\right) - C$$

for some C. The initial condition yields $C = (e-2)/(e-1)$. So

$$y(x) = e^{-x} \ln\left(\frac{(e^x - 2)(e-1)}{(e-2)(e^x - 1)}\right) \ .$$

Part (b). Put $M = 3x^2 + 4xy + 3y^2$ and $N = 2x^2 + 6xy + 12y^2$. We have

$$\partial M/\partial y = 4x + 6y = \partial N/\partial x \ .$$

So we have exactness, and the implicit solution has the form $F(x, y) = C$ where C is a constant and F is such that $M = \partial F/\partial x$ and $N = \partial F/\partial y$. Write

$$F(x, y) = g(y) + \int M \, \mathrm{d}x = g(y) + x^3 + 2x^2 y + 3xy^2 \ .$$

We have $N = g'(y) + 2x^2 + 6xy$. So $g'(y) = 12y^2$ and, up to a constant summand, $g(y) = 4y^3$. In conclusion, the implicit solution is

$$x^3 + 2x^2 y + 3xy^2 + 4y^3 = C \ .$$

Alternative for part (b): One could guess that F has the form $F(x, y) = ax^3 + bx^2 y + cxy^2 + dy^3$, the coefficients to be evaluated by partially differentiating with respect to x and y.

Solution 3: Part (a). The characteristic polynomial of A is $\lambda^4 - 1$. So the eigenvalues are i^k which have corresponding eigenvectors $(1, i^k, i^{2k}, i^{3k})$ where k is an integer. So the general complex solution is

$$\underline{x}(t) = b_0 e^t \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + b_1 e^{it} \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix} + b_2 e^{-t} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} + b_3 e^{-it} \begin{bmatrix} 1 \\ -i \\ -1 \\ i \end{bmatrix}$$

where b_0, b_1, b_2, b_3 are constant complex numbers. To obtain the real solutions, we note that

$$e^{it}\begin{bmatrix}1\\ i\\ -1\\ -i\end{bmatrix} = (c+is)\begin{bmatrix}1\\ i\\ -1\\ -i\end{bmatrix} = \begin{bmatrix}c\\ -s\\ -c\\ s\end{bmatrix} + i\begin{bmatrix}s\\ c\\ -s\\ -c\end{bmatrix}$$

where $c = \cos(t)$ and $s = \sin(t)$. So the general solution is

$$\underline{x}(t) = a_0 e^t\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + a_1\begin{bmatrix}c\\ -s\\ -c\\ s\end{bmatrix} + a_2 e^{-t}\begin{bmatrix}1\\ -1\\ 1\\ -1\end{bmatrix} + a_3\begin{bmatrix}s\\ c\\ -s\\ -c\end{bmatrix}$$

where a_0, a_1, a_2, a_3 are constant real numbers.

Part (b). The above general real solution is expressed as a linear combination of 4 solutions that have Wronkskian

$$W(t) = \begin{vmatrix} e^t & c & e^{-t} & s \\ e^t & -s & -e^{-t} & c \\ e^t & -c & e^{-t} & -s \\ e^t & s & -e^{-t} & -c \end{vmatrix}.$$

This is the determinant of a matrix whose columns are mutually orthogonal. So $W(t) \neq 0$, as required.

Alternative for part (a): The first coordinate $x_1(t)$ of $\underline{x}$ satisfies $x_1'''' - x_1 = 0$, which has auxilliary polynomial X^4-1 with roots 1, i, -1, $-i$. So $x_1(t) = a_0e^t+a_1\cos(t)+a_2e^{-t}+a_3\sin(t)$. Formulas for the other three coordinates of $\underline{x}$ can now be obtained by differentiation.

Solution 4: Part (a). The associated homogeneous equation is

$$y'' - 4y' + 4y = 0.$$

Its characteristic equation is

$$r^2 - 4r + 4 = 0.$$

Hence

$$(r-2)^2 = 0.$$

Therefore, $r = 2$ is a repeated root. For the repeated root $r = 2$, two linearly independent solutions are

$$y_1 = e^{2x}, \qquad y_2 = xe^{2x}.$$

Thus the general solution of the homogeneous equation is

$$y_h = C_1e^{2x} + C_2xe^{2x}, \quad C_1, C_2 \in \mathbb{R}$$

Part (b). We now find a particular solution using variation of parameters. Let

$$y_1 = e^{2x}, \qquad y_2 = xe^{2x}.$$

The Wronskian is

$$W = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix}.$$

Therefore,

$$W = e^{2x}\left(e^{2x} + 2xe^{2x}\right) - 2e^{2x}\left(xe^{2x}\right).$$

Hence

$$W = e^{4x}.$$

The differential equation is already in standard form:

$$y'' - 4y' + 4y = g(x),$$

where

$$g(x) = e^{2x}\ln x.$$

By the method of variation of parameters,

$$y_p = -y_1 \int \frac{y_2 g(x)}{W}\,dx + y_2 \int \frac{y_1 g(x)}{W}\,dx.$$

Substituting

$$y_1 = e^{2x}, \qquad y_2 = xe^{2x}, \qquad g(x) = e^{2x}\ln x, \qquad W = e^{4x},$$

gives

$$y_p = -e^{2x} \int \frac{xe^{2x}\cdot e^{2x}\ln x}{e^{4x}}\,dx + xe^{2x} \int \frac{e^{2x}\cdot e^{2x}\ln x}{e^{4x}}\,dx.$$

Thus,

$$y_p = -e^{2x} \int x\ln x\,dx + xe^{2x} \int \ln x\,dx.$$

Now,

$$\int x\ln x\,dx = \frac{x^2}{2}\ln x - \frac{x^2}{4},$$

and

$$\int \ln x\,dx = x\ln x - x.$$

Therefore,

$$y_p = -e^{2x}\left(\frac{x^2}{2}\ln x - \frac{x^2}{4}\right) + xe^{2x}\left(x\ln x - x\right).$$

Expanding and simplifying,

$$y_p = e^{2x}\left(-\frac{x^2}{2}\ln x + \frac{x^2}{4} + x^2\ln x - x^2\right).$$

Hence

$$y_p = e^{2x}\left(\frac{x^2}{2}\ln x - \frac{3x^2}{4}\right).$$

Therefore, a particular solution is

$$y_p = e^{2x}\left(\frac{x^2}{2}\ln x - \frac{3x^2}{4}\right).$$

Part (c). The general solution is

$$y = y_h + y_p.$$

Hence

$$\boxed{y(x) = C_1 e^{2x} + C_2 xe^{2x} + e^{2x}\left(\frac{x^2}{2}\ln x - \frac{3x^2}{4}\right)}$$

for $x > 0$ and $C_1, C_2 \in \mathbb{R}$.