Archive for MATH 224, Linear Algebra 2, Spring 2022

Bilkent University, Laurence Barker, 10 June 2022.

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Course specification

MATH 224, *Linear Algebra 2*, Spring 2022

Laurence Barker, Bilkent University. Version: 14 April 2022.

Course Aims: To aquire theoretical and practical skill at some techniques of linear algebra that advance beyond a first course such as MATH 223 Linear Algebra 1.

Course Description: This is a second course in linear algebra, more advanced and more theoretically inclined than MATH 223. To obtain a satsfactory grade, it will be necessary to solve most of the routine problems and to answer brief theoretical questions. To obtain a high grade, it will also be necessary to solve some of the difficult problems and to give clear explanations.

Course Requirements: Knowledge of material in a first course in linear algebra, notably, the technique of diagonalization and the material behind it, such as the theory of coordinate transformations, the theory of eigenvalues and eigenvectors.

Instructor: Laurence Barker, Office SAZ 129, barker at fen dot bilkent dot edu dot tr.

Assistant: Serkan Sonel, serkan dot sonel at bilkent dot edu dot tr.

Recommanded textbooks:

- Stephen H. Friedberg, Arnold J. Insel, Lawrence E. Spence, "Linear Algebra", 5th edition, (Pearson, 2018).
- Bill Jacob, "Linear Algebra", (W. H. Freeman, New York, 1990).
- Tom M. Apostol, "Linear Algebra", (Wiley, New York, 1997).

Supplementary material on my homepage: [To be supplied later.]

- Homeworks, with solutions: homework224spr22.pdf .
- Quizzes, with solutions: quiz224spr22.pdf .
- Course notes on selected topics: PDF files with names of the form: LinAlgTheoNotes... .

Classes: Tuesdays 10:30 - 12:20, Thursdays 15:30 - 16:20, room SAZ-04.

Office Hours: Thursdays 16:30 - 17:20, in my office, room SA-129.

Office hours is for *all* the students on the course, not just the proficient. If you are having difficulty with the course, then it is best to come to see me for advice.

Weekly Syllabus

The format below is, *Week number; Monday date; Subtopics.*

1: Mon 30 Jan - Fri 4 Feb: Review of diagonalization over the real and complex numbers. Rings, fields, finite fields.

2: Mon 7 Feb - Fri 11 Feb: Vector spaces over arbitrary fields. Steinitz Exchange Lemma and applications. Error-correcting codes.

3: Mon 14 Feb - Fri 18 Feb: Error-correcting codes. Unique factorization of polynomial rings. Annihilating polynomials of a matrix and of a linear map.

4: Mon 21 Feb - Fri 25 Feb: Minimal polynomial of a linear map. Generalized eigenvectors.

5: Mon 28 Feb - Fri 4 Mar: Generalized eigenvectors. Jordan normal form.

6: Mon 7 Mar - Wed 9 Mar: Cayley–Hamilton Theorem and applications.

7: Mon 14 Mar - Fri 18 Mar: Inner product spaces. Cauchy–Schwartz inequality. Triangle inequality. Orthonormal bases. Pearson correlation coefficient.

8: Mon 21 Mar - Fri 25 Mar: Orthonormal bases. Gram–Schmidt orthogonalization

9: Mon 28 Mar - Fri 1 Apr: Quadratic forms. Bilinear forms. Sylvester's law of inertia.

10: Mon 4 Apr - Fri 8 Apr: Quadratic forms. Bilinear forms. Sylvester's law of inertia.

11: Mon 11 Apr - Fri 15 Apr: Symmetric, Hermitian, orthogonal and unitary operators, their diagonmalizability and their spectral theorems.

12: Mon 18 Apr - Fri 22 Apr: Symmetric, Hermitian, orthogonal and unitary operators, their diagonmalizability and their spectral theorems.

13: Mon 25 Apr - Fri 29 Apr: Skew-symmetric bilinear forms. Groups preserving bilinear forms.

14: Thur 5 May - Fri 6 May: Rational canonical form.

15: Mon 9 May - Fri 13 May: Review for Final.

Assessment: The method of assessment is by curve. Grades F, FX, FZ are to be for candidates judged to be thoroughly incompetent in the routine course material.
• Quizzes 10% (marks for worst 2 quizzes to be dropped).
• Midterm I, 30%, face-to-face exam, Thursday 17 March, 15:30 - 17:20, rooms SA-Z04, B-108.
• Midterm II, 30%, face-to-face exam, Thursday 21 April, 15:30 - 17:20, rooms SA-Z04, B-108.
• Final, 30%, face-to-face exam, Friday, 20 May.

FZ criteria: less than 30% of the sum of the Midterm 1 and Midterm 2 marks.

Masks: Masks must be worn at all times in class.

Speaking: All speaking in class must address everyone in the room. Questions and comments are welcome: they make learning easier, and they make teaching easier too.

Class Announcements: All students, including any absentees from a class, will be deemed responsible for awareness of class announcements.

Homework

MATH 224, *Linear Algebra 2*, Spring 2022, Laurence Barker

version: 15 April 2022

Homework 1

Solutions to be discussed in class on or after Tuesday 8 March.

Question 1.1: The **modulus** of a quaternion $q = t + ix + jy + kz$, with $t, x, y, z \in \mathbb{R}$, is defined by the Pythagorean formula

$$|q| = \sqrt{t^2 + x^2 + y^2 + z^2}\ .$$

As elements of $\mathrm{Mat}_2(\mathbb{C})$, write

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mathcal{I} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}, \qquad \mathcal{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad \mathcal{K} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

(a) Calculate the determinant of the matrix $tI + x\mathcal{I} + y\mathcal{J} + z\mathcal{K}$.
(b) Using part (a), show that, given $q, q' \in \mathbb{H}$, then $|qq'| = |q|.|q'|$.
(c) Let q be a non-zero quaternion. Again writing $q = t + ix + jy + kz$, give a formula for q^{-1}.

Question 1.2: A field F is said to be **finite** provided the underlying set F is finite, in other words, provided F has only finitely many elements. Show that, if F is finite, then $|F|$ is a power of a prime, in other words, $|F| = p^m$ for some prime p and some positive integer m. (Hint: First show that F contains a copy of the ring $\mathbb{Z}/n$ for some positive integer n. Then make use of the fact that every finite-dimensional vector space has a basis.)

Question 1.3: Suppose F is finite. Consider a system of F-linear equations $Ax = y$. Thus, x and y are column vectors over F and A is a matrix over F. Show that, for each y, the number $|\{x : Ax = y\}|$ is zero or a power of a prime.

Question 1.4: Prove the following two propositions concerning the polynomial ring $F[t]$, where F is a field:

(a) Given nonzero $a(t), b(t) \in F[t]$, then there exist unique $q(t), r(t) \in F[t]$ such that $a(t) = q(t)b(t) + r(t)$ and $\deg(r(t)) < \deg(b(t))$.

(b) Let $a(t), b(t) \in F[t]$. Let $c(t)$ be a polynomial of maximal degree such that $c(t)$ divides $a(t)$ and $b(t)$. Then there exist $u(t), v(t) \in F[t]$ such that $c(t) = u(t)a(t) + v(t)b(t)$. Furthermore, there exists a unique such monic $c(t)$.

Question 1.5: Consider the linear coding scheme with Hamming matrix

$$H = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

(a) Write down the generating matrix G for the coding scheme.

(b) Write down a decoding table, including the column of syndromes.

(c) Encode the message words 100, 110, 111.

(d) For the received words 00011, 00111, 01111, 11111, write down the syndromes, then write down the decoded words.

(e) What is the rate of the code?

(f) If a single codeword is transmitted, what is the maximum number of errors of transmission (the maximum number of inversions of binary digits) such that any error can be detected? What is the maximum number of errors of transmission (the maximum number of inversions of binary digits) such that any error can be corrected?

Solutions 1

1.1: Part (a). We have

$$\det(tI + x\mathcal{I} + y\mathcal{J} + z\mathcal{K}) = \begin{vmatrix} t - ix & -y + iz \\ y + iz & t + ix \end{vmatrix} = |t + ix|^2 + |y + iz|^2 = |q|^2 \ .$$

Part (b). The required equality follows from part (a) because $\det(AB) = \det(A)\det(B)$ for 2×2 matrices A and B over $\mathbb{C}$.

Part (c). By the formula for the inverse of a 2×2 matrix with non-zero determinant, if $q \neq 0$, then

$$\begin{bmatrix} t - ix & -y + iz \\ y + iz & t + ix \end{bmatrix}^{-1} = (t^2 + x^2 + y^2 + z^2)^{-1} \begin{bmatrix} t + ix & y + iz \\ -y + iz & t - ix \end{bmatrix}$$

and hence $q^{-1} = (t - ix - jy - kz)/(t^2 + x^2 + y^2 = z^2)$.

1.2: Let K be the subring of F generated by the unity element 1_F. Let p be the smallest positive integer such that $p1_F = 0$. Then K is a copy of the ring $\mathbb{Z}/p$ of modulo p integers. For $x, y \in K$, if $xy = 0$ then $x = 0$ or $y = 0$. Therefore, p is prime and K is a field.

We can regard F as a vector space over K with the same addition operation and with scalar multiplication operation $K \times F \to F$ coming from the multiplication operation of F. Letting m be the dimension of F as a K-vector space, then $|F| = p^m$.

1.3: Let $S = \{x : Ax = y\}$. We may assume that $S \neq \emptyset$. Let $x_0 \in S$. For a column vector x with the appropriate number of coordinates, we have $x \in S$ if and only if $x - x_0$ belongs to the null space of A. So $|S| = |\text{Null}(A)| = |F|^n$ where $n = \dim(\text{Null}(A))$. In Question 2, we showed that $|F|$ is a power of a prime. Therefore, $|S|$ is a power of a prime.

1.4: Part (a). Let $m = \deg(a)$ and $n = \deg(b)$. We may assume that $m \geq n$. Write $a(t) = a_n t^n + ... + a_1 t + a_0$ and $b(t) = b_m t^m + ... + b_1 t + b_0$. Consder a polynomial $q(t) = q_{m-n} t^{m-n} + q_1 t + q_0$. The coefficient of t^m in $a - qb$ is $a_m - q_{m-n} b_n$. Putting $q_{m-n} = a_m / b_n$, we ensure that $a - qb$ has degree less than m. Inductively, for $m - n \geq s \geq 1$, having solved for $q_{m-n}, ..., q_s$ so as to ensure that $a - qb$ has degree less than $n + s$, we can consider coefficients of t^{n+s-1} to solve for q_{s-1} so as to ensure that $a - qb$ has degree less than $n + s - 1$. The case $s = 0$ is the required conclusion.

Part (b). We apply the Euclidian algorithm. Again, we may assume that $\deg(a) \geq \deg(b)$. Put $r_0 = a$ and $r_1 = b$. Inductively, having defined r_0, ..., r_i, we terminate the process when $r_i = 0$, otherwise, we define polynomials q_i and r_{i-1} such that

$$r_{i+1} = q_i r_i + r_{i-1}$$

and $\deg(r_i) > \deg(r_{i-1})$. Since the degrees $\deg(r_i)$ are strictly decreasing, the process must terminate. Let k be such that $r_{k+1} = 0$. Let $c = r_k$. If c divides r_{i-1} and r_i, then c divides r_{i+1} and r_i. So, by an inductive argument, c divides a and b. On the other hand, another inductive argument shows that each r_i has the form $u_i a + v_i b$ for polynomials u_i and v_i. In particular, $c = ua + vb$ for some u and v.

Finally, if d is a polynomial dividing a and b, then d must divide c because $c = ua + vb$. In particular, $\deg(d) \leq \deg(c)$ and if $\deg(d) = \deg(c)$ then d must be an F-multiple of c. The required conclusion follows.

1.5: Part (a). We have $G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

Part (b). The decoding table is as follows.

000	001	010	011	100	101	110	111	syndrome
00000	00101	01001	01100	10010	10111	11011	11110	00
00001	00100	01000	01101	10011	10110	11010	11111	01
00010	00111	01011	01110	10000	10101	11001	11100	10
00011	00110	01010	01111	10001	10100	11000	11101	11

Part (c). The encodings of 100, 110, 111 are 10010, 11011, 11110, respectively.

Part (d). The received words 00011, 00111, 01111, 11111 have syndromes 11, 10, 11, 01 and decodings 000, 001, 011, 111, respectively

Part (e). The rate is $3/5$.

Part (f). The minimum weight of a nonzero codeword is 2. So up to 1 error of transmission can always be detected, and up to 0 errors of transmission can always be corrected.

Homework 2

Solutions to be discussed in class on or after Tuesday 15 March.

Question 2.1: Consider the matrix $A = \begin{bmatrix} -1 & 2 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

Express A in the form $A = PEP^{-1}$ where E has Jordan normal form.

Question 2.2: Repeat Question 1 with $A = \begin{bmatrix} -2 & 1 & 0 \\ -1 & 0 & 0 \\ -1 & -1 & 1 \end{bmatrix}$.

Question 2.3: Let A be an $n \times n$ matrix over an algebraically closed field F.

(a) Show that, when $n = 3$, the Jordan normal form of A is determined by the characteristic polynomial and the minimal polynomial of A.

(b) In the case $n = 4$, give a counter-example to the conclusion of part (a).

Question 2.4: We write $\mathbb{F}_2$ for the field with order 2, and we write $\mathrm{Mat}_n(\mathbb{F}_2)$ to denote the ring of $n \times n$ matrices over $\mathbb{F}_2$. Let $\sim$ denote the equivalence relation on $\mathrm{Mat}_n(\mathbb{F}_2)$ such that, given $A, B \in \mathrm{Mat}_n(\mathbb{F}_2)$, then $A \sim B$ when A is similar to B, in other words, $A = PBP^{-1}$ for some invertible $P \in \mathrm{Mat}_n(\mathbb{F}_2)$.

(a) When $n = 2$, how many equivalence classes of $\sim$ include a matrix that has Jordan normal form?

(b) When $n = 3$, how many equivalence classes of $\sim$ include a matrix that has Jordan normal form?

(c) When $n = 4$, how many equivalence classes of $\sim$ include a matrix that has Jordan normal form?

Solutions 2

2.1: The eigenvalues are -1 and 2 with multiplicities 2 and 1, respectively.

Let f_1 be an eigenvector with eigenvalue -1. Write $f_1 = (x, y, z)$. Then

$$-x + 2y - z = -x \ , \qquad -y + z = -y \ , \qquad 2z = -z \ .$$

We have $y = z = 0$. So we can put $x = 1$ and $f_1 = (1, 0, 0)$.

Let f_2 be a generalized eigenvector such that $Af_2 = -f_2 + f_1$. Write $f_2 = (x, y, z)$. Then

$$-x + 2y - z = -x + 1 \ , \qquad -y + z = -y \ , \qquad 2z = -z \ .$$

Hence, $z = 0$ and $y = 1/2$. We can put $x = 0$. Then $f_2 = (0, 1/2, 0)$.

Let f_3 be an eigenvector with eigenvalue 2. Write $f_2 = (x, y, z)$. Then

$$-x + 2y - z = 2x \ , \qquad -y + z = 2y \ , \qquad 2z = 2z \ .$$

We can put $z = 1$. Then $y = 1/3$ and, since $3x = 2y - z = -1/3$, we have $x = -1/9$. In conclusion, we can put

$$P = \begin{bmatrix} 1 & 0 & -1/9 \\ 0 & 1/2 & 1/3 \\ 0 & 0 & 1 \end{bmatrix} \ , \qquad E = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \ .$$

Comment: As an aid to checking the answer, we mention that $P^{-1} = \begin{bmatrix} 1 & 0 & 1/9 \\ 0 & 2 & -2/3 \\ 0 & 0 & 1 \end{bmatrix}$.

2.2: The characteristic polynomial of A is

$$\begin{vmatrix} \lambda + 2 & -1 & 0 \\ 1 & \lambda & 0 \\ 1 & 1 & \lambda - 1 \end{vmatrix} = (\lambda + 2)\lambda(\lambda - 1) + \lambda - 1 = (\lambda^2 + 2\lambda + 1)(\lambda - 1) = (\lambda + 1)^2(\lambda - 1) \ .$$

So the eigenvalues are -1 and 1 with multiplicities 2 and 1, respectively.

Let f_1 be an eigenvector with eigenvalue -1. Write $f_1 = (x, y, z)$. Then

$$-2x + y = -x \ , \qquad -x = -y \ , \qquad -x - y + z = z \ .$$

We can put $x = y = z = 1$. Thus, $f_1 = (1, 1, 1)$.

Let f_2 be a generalized eigenvector such that $Af_2 = -f_2 + f_1$. Write $f_2 = (x, y, z)$. Then

$$-2x + y = -x + 1 \ , \qquad -x = -y + 1 \ , \qquad -x - y + z = z + 1 \ .$$

We can put $y = 1$, whereupon $x = 0$ and $z = 1$. Thus, $f_2 = (0, 1, 1)$.

Putting $f_3 = (0, 0, 1)$, then f_3 is plainly an eigenvector with eigenvalue 1. We have

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \ , \qquad E = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \ .$$

Comment: As an aid to checking the answer, we mention that $P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$.

2.3: Part (a). Write $c_A(t)$ and $m_A(t)$, respectively, for the characteristic polynomial and the minimal polynomial of A in $F[t]$. We run through all the possible cases.

If A has 3 mutually distinct eigenvalues, λ_1, λ_2, λ_3, then A is diagonalizable and

$$c_A(t) = m_A(t) = (t - \lambda_1)(t - \lambda_2)(t - \lambda_3) \ .$$

Now suppose A has distinct eigenvalues λ_1 and λ_2 with multiplicities 1 and 2, respectively. Then $c_A(t) = (t - \lambda_1)(t - \lambda_2)^2$. Up to renumbering of basis elements, the two possible Jordan normal forms for A are

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} , \qquad \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

whose minimal polynomials are $(t - \lambda_1)(t - \lambda_2)$ and $c_A(t)$, respectively. Finally, suppose A has a unique eigenvalue λ. Then $c_A(t) = (t - \lambda)^3$. Up to renumbering, the three possible Jordan normal forms for A are

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} , \qquad \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} , \qquad \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

whose minimal polynomials are $(t - \lambda)$, $(t - \lambda)^2$, $c_A(t)$, respectively. Evidently, no two of the six possible cases have the same polynomials $c_A(t)$ and $m_A(t)$.

Part (b). Let $\lambda \in F$. The two distinct matrices

$$\begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} , \qquad \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

are both in Jordan normal form, they both have characteristic polynomial $(t - \lambda)^4 \in F[t]$ and minimal polynomial $(t - \lambda)^2$.

2.4: In all three parts, we are to count the equivalence classes of $\sim$ that include a matrix having Jordan normal form.

Part (a). The answer is 5, since there are 5 possible cases: 2 distinct eigenvalues; unique eigenvalue 0 and diagonalizable; unique eigenvalue 0 and non-diagonalizable; unique eigenvalue 1 and diagonalizable; unique eigenvalue 2 and non-diagonalizable.

Part (b). We can classify the equivalence classes according to Jordan normal form, bearing in mind that renumberings of the Jordan basis yield the same Jordan normal form up to permutation of the Jordan block matrices.

There are no cases with 3 mutually distinct eigenvalues. In the the case of 2 distinct eigenvalues, there are 2 choices for which of the eigenvalues 0 or 1 has multiplicity 2 and then, as we saw in Question 2.3, there are 2 choices for the Jordan normal form. That yields 4 equivalence classes. Meanwhile, in the case of a unique eigenvalue, there are 2 choices for the eigenvalue, then 3 choices for the Jordan normal form. That yields 6 equivalence classes. So the total number of equivalence classes is $4 + 6 = 10$.

Part (c). Let $(m_1, m_2, ...)$ be the dimensions of the Jordan blocks with eigenvalue 0, arranged such that $m_1 \geq m_2 \geq ....$ Let $(n_1, n_2, ...)$ be defined similarly for eigenvalue 1. Let $m = m_1 + m_2 + ...$ and $n = n_1 + n_2 + ...$, which are the multiplicities of the eigenvalues 0 and 1. The number of equivalence classes is the number of tuples having the form $(m_1, m_2, ...; n_1, n_2, ...)$ such that $m + n = 4$.

When $m = 0$, the possible tuples are $(;1,1,1,1)$, $(;2,1,1)$, $(;2,2)$, $(;3,1)$, $(;4)$.

When $m = 1$, the possilities are $(1;1,1,1)$, $(1;2,1)$, $(1;3)$.

When $m = 2$, they are $(1,1;1,1)$, $(1,1;2)$, $(2;1,1)$, $(2;2)$.

When $m = 3$, the number of possibilities is the same as for $m = 1$.

When $m = 4$, the number of possibilities is the same as for $m = 0$.

So the number of equivalence classes is $5 + 3 + 4 + 3 + 5 = 20$.

Comment: Consider a field F. Recall that, when F is algebraically closed, every square matrix over F is similar to a matrix having Jordan normal form. For an arbitrary field F and a square matrix A over F, we may have to extend to a larger field in order to express A in the form $A = PEP^{-1}$ where E has Jordan normal form.

Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \in \mathrm{Mat}_3(\mathbb{F}_2) \ .$$

We have $A^3 = I$. But there does not exist a non-identity matrix $E \in \mathrm{Mat}_3(\mathbb{F}_2)$ such that $E^3 = I$. So A cannot be expressed in Jordan normal form over $\mathbb{F}_2$. However, now let us view A as a matrix over the field $\mathbb{F}_4$ with order 4. The field $\mathbb{F}_4 = \{0, 1, \omega, \omega^2\}$ has the following addition and multiplication tables.

		y			
	$x+y$	0	1	ω	ω^2
x	0	0	1	ω	ω^2
	1	1	ω	ω^2	0
	ω	ω	ω^2	0	1
	ω^2	ω^2	ω	1	0

		y			
	xy	0	1	ω	ω^2
x	0	0	0	0	0
	1	0	1	ω	ω^2
	ω	0	ω	ω^2	1
	ω^2	0	ω^2	1	ω

It is not hard to check that, as a product of three matrices over $\mathbb{F}_4$, we have

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}^{-1} .$$

Thus, in fact, A is diagonalizable over $\mathbb{F}_4$.

Homework 3

Solutions to be discussed in class on Thursday 14 April and Tuesday 19th April.

Question 3.1: Let π be the linear map $\mathbb{R}^2 \leftarrow \mathbb{R}^2$ such that $\pi(x,y) = (x+y, x+y)/2$. Find:

(a) The matrix representing π with respect to the standard basis $\{(1,0),(0,1)\}$ of $\mathbb{R}^2$.

(b) Evaluate the trace $\mathrm{tr}(\pi)$ and the determinant $\det(\pi)$.

Question 3.2: Let $e_1 = (1,1,1)$ and $e_2 = (1,2,2)$ and $e_3 = (1,2,3)$ in $\mathbb{R}^3$. Apply the Gram–Schmidt process to the basis $\{e_1, e_2, e_3\}$ to obtain an orthonormal basis for $\mathbb{R}^3$.

Question 3.3: Let

$$U = \{(x_1, x_2, x_3, x_4) \in \mathbb{C}^4 : x_1 + x_3 = x_2 + x_4 = 0\}$$

as a subspace of the inner product space $\mathbb{C}^4$.

(a) Find an orthonormal basis for U.

(b) Find an orthonormal basis for the orthogonal complement $U^\perp$ of U in $\mathbb{C}^4$.

(c) Find the matrix A representing the orthogonal projection $U \leftarrow \mathbb{C}^4$ with respect to the standard basis of $\mathbb{C}^4$.

(d) Find the matrix B representing the orthogonal projection $U^\perp \leftarrow \mathbb{C}^4$ with respect to the standard basis of $\mathbb{C}^4$.

Question 3.4: Let V and W be subspaces of a finite-dimensional inner product space U over $\mathbb{C}$. We write $V^\perp$ for the orthogonal complement of V in U. Show that:

(a) We have $(V^\perp)^\perp = V$.

(b) We have $(V + W)^\perp = V^\perp \cap W^\perp$.

(c) We have $(V \cap W)^\perp = V^\perp + W^\perp$.

(d) The inner product space $\ell^2(\mathbb{C})$ consists of the sequences $(x_1, x_2, ...)$ of complex numbers such that $\sum_n |x_n|^2 < \infty$. Let V be the subspace of $\ell^2(\mathbb{C})$ consisting of those sequences $(x_1, x_2, ...)$ such that only finitely many of the x_i are non-zero. Show that, if we drop the assumption that U is finite-dimensional and put $U = \ell^2(\mathbb{C})$, then V is a counter-example to part (a).

(e) Again dropping the assumption that U is finite-dimensional, give a counter-example to part (c). (Hint: Put U and V as in part (d) and find a suitable W.)

Question 3.5: Let n be a positive integer. The $n \times n$ **discrete Fourier transform matrix** F is the matrix with rows and columns indexed by the ring of modulo n integere $\mathbb{Z}/n$, with (s,t)-entry $\zeta^{st}/\sqrt{n}$, where $\zeta = e^{2\pi i/n}$.

(a) Why is F diagonalizable?

(b) Evaluate F^2. Hence show that every eigenvalue of F belongs to the set $\{1, i, -1, -i\}$.

(c) Suppose n is odd. Evaluate $\dim_{\mathbb{C}}(E_1 \oplus E_{-1})$ and $\dim_{\mathbb{C}}(E_i \oplus E_{-i})$, where E_λ denotes the eigenspace for eigenvalue λ.

Question 3.6: Let ρ be an orthogonal operator on a finite-dimensional real inner product space V. Show that, with respect to some basis of V, the matrix representing ρ has the form

$$\begin{bmatrix} A_1 & 0 & ... & 0 \\ 0 & A_2 & ... & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & ... & A_m \end{bmatrix}$$

where each A_s is either a 1×1 matrix $[1]$ or $[-1]$ or else a 2×2 matrix $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$. (Hint: Extend to the complex numbers and adapt the trick we used to show that real symmetric matrics are diagonalizable.)

Solutions 3

3.1: Part (a). The required matrix is $P = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Part (b). We have $\mathrm{tr}(\pi) = \mathrm{tr}(P) = 1$ and $\det(\pi) = \det(P) = 0$.

Comment: Many invariants of a linear map, such as trace, determinant, characteristic polynomial, minimal polynomial, are features of any matrix representing the linear map. Given a linear map, then the representing matrix depends on the choice of basis. But, when we change the basis, the invariants do not change. That is why they are called *invariants.* (I emphasize this fundamental point because grasping it is the beginning of a comprehension of the theory of linear algebra.)

3.2: We calculate the orthogonal basis $\{f_1', f_2', f_3'\}$ given by

$$f_1' = e_1 \, , \qquad f_2' = e_2 - \frac{\langle f_1' \,|\, e_2 \rangle}{\| f_1' \|^2} f_1' \, , \qquad f_3' = e_3 - \frac{\langle f_1' \,|\, e_3 \rangle}{\| f_1' \|^2} f_1' - \frac{\langle f_2' \,|\, e_3 \rangle}{\| f_2' \|^2} f_2' \, .$$

and then the orthonormal basis $\{f_1, f_2, f_3\}$ given by $f_i = f_i' / \| f_i' \|$. We have $f_1' = (1, 1, 1)$. Since

$$\frac{\langle f_1' \,|\, e_2 \rangle}{\| f_1' \|^2} = \frac{1+2+2}{1+1+1} = \frac{5}{3}$$

we have $f_2' = (1, 2, 2) - \frac{5}{3}(1, 1, 1) = \frac{1}{3}(-2, 1, 1)$. Since

$$\frac{\langle f_1' \,|\, e_3 \rangle}{\| f_1' \|^2} = \frac{1+2+3}{3} = 2 \, , \qquad \frac{\langle f_2' \,|\, e_3 \rangle}{\| f_2' \|^2} = \frac{(-2+2+3)/3}{(4+1+1)/9} = \frac{3}{2}$$

we have $f_3' = (1, 2, 3) - 2(1, 1, 1) - \frac{1}{2}(-2, 1, 1) = \frac{1}{2}(0, -1, 1)$. Therefore,

$$f_1 = (1, 1, 1)/\sqrt{3} \, , \qquad f_2 = (-2, 1, 1)/\sqrt{6} \, , \qquad f_3 = (0, -1, 1)/\sqrt{2} \, .$$

3.3: We define

$$f_1 = (1, 0, -1, 0)/\sqrt{2} \, , \qquad f_2 = (0, 1, 0, -1)/\sqrt{2} \, , \qquad f_3 = (1, 0, 1, 0)/\sqrt{2} \, , \qquad f_4 = (0, 1, 0, 1)/\sqrt{2} \, .$$

Note that $U = \{x \in \mathbb{C}^4 : \langle f_1 \,|\, x \rangle = \langle f_2 \,|\, x \rangle = 0\}$.

Part (a). An orthonormal basis for U is $\{f_1, f_2\}$.

Part (b). An orthonormal basis for $U^\perp$ is $\{f_3, f_4\}$.

Part (c). Given $x = (x_1, x_2, x_3, x_4) \in \mathbb{C}^4$, then the orthogonal projection of x to U is

$$\langle f_1 \,|\, x \rangle f_1 + \langle f_2 \,|\, x \rangle f_2 = (x_1 - x_3, x_2 - x_4, -x_1 + x_3, -x_2 + x_4)/2 \, .$$

Therefore, $A = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$.

Part (d). The orthogonal projection of x to $U^\perp$ is

$$\langle f_3 \,|\, x \rangle f_3 + \langle f_4 \,|\, x \rangle f_4 = (x_1 + x_3, x_2 + x_4, x_1 + x_3, x_2 + x_4)/2 \, .$$

Therefore, $B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

3.4: Part (a). As a preliminary, recall that $\dim(U) = \dim(V) + \dim(V^\perp)$. (This is proved by extending a basis for V to a basis for U, then applying the Gram–Schmidt process, which yields an orthonormal basis for V that extends to a orthonormal basis for U.) Hence $\dim(V) = \dim((V^\perp)^\perp)$. Plainly, $V \leq (V^\perp)^\perp$. The required conclusion follows.

Part (b). Since $V + W \geq V$, we have $(V + W)^\perp \leq V^\perp$. Similarly, $(V + W)^\perp \leq W^\perp$. Hence $(V + W)^\perp \leq V^\perp \cap W^\perp$. For the reverse inequality, let $z \in V^\perp \cap W^\perp$. Given $v \in V$ and $w \in W$, then $\langle v \perp z \rangle = \langle w \,|\, z \rangle$, hence $\langle v + w \,|\, z \rangle = 0$ and $z \in (V + W)^\perp$.

Part (c). Using parts (a) and (b), we have

$$(V \cap W)^\perp = (V^{\perp\perp} \cap W^{\perp\perp})^\perp = (V^\perp + W^\perp)^{\perp\perp} = V^\perp + W^\perp \ .$$

Part (d). We have $V^\perp = \{0\}$, hence $V^{\perp\perp} = U \neq V$.

Part (e). Let U and V be as suggested in the question. Let $x = (x_1, x_2, ...)$ be any element of $\ell^2(\mathbb{C})$ such that infinitely many of the x_i are non-zero. (For instance, we can put $x_i = z^i$ where z is any complex number satisfying $0 < |z| < 1$.) Let $W = \mathrm{span}_\mathbb{C}\{x\}$. Then $V \cap W = \{0\}$, hence $(V \cap W)^\perp = U$. Meanwhile, $w \notin W^\perp = V^\perp \cap W^\perp$. Therefore, $(V \cap W)^\perp > V^\perp \cap W^\perp$.

Comment: For each positive integer m, let e_m be the element of $\ell^2(\mathbb{C})$ such that the m-th term of e_m is 1 and all the other terms are 0. In functional analysis, we write

$$(x_1, x_2, ...) = \sum_{m=1}^{\infty} x_m e_m$$

and we call the infinite set $\{e_1, e_2, ...\}$ a *topological basis* for $\ell^2(\mathbb{C})$.

However, a vector space S over a field F comes equipped with an addition operation $S \times S \ni (s, t) \mapsto s + t \in S$. By the associativity of that operation, we can form finite sums $s_1 + ... + s_n$ for s_1, ..., s_n in X. By the commutativity of the addition operation, the sum $s_1 + ... + s_n$ does not depend on the ordering of the terms s_m. In linear algebra, we define a **basis** for S to be a subset $\{e_i : i \in I\} \subseteq S$ such that every element of X can be uniquely expressed as a sum $\sum_i x_i e_i$ such that $x_i \in F$ and only finitely many of the indices x_i are nonzero. Sometimes, a basis in that sense is called an **algebraic basis**. Using Zorn's Lemma, it can be proved that every vector space has an algebraic basis.

In particular, $\ell^2(\mathbb{C})$ must have an algebraic basis. But it would be very difficult, perhaps impossible, to explicitly specify any algebraic basis for $\ell^2(\mathbb{C})$. The above infinite sum only makes sense because $\ell^2(\mathbb{C})$ has more structure than just that of a vector space. Since $\ell^2(\mathbb{C})$ is an inner product space, we can view $\ell^2(\mathbb{C})$ as a metric space where the distance between two vectors u and v is the norm $\| u - v \|$. Via the metric space structure, we can form limits. To give sense to the above infinite sum, we define it to be the limit

$$\sum_{m=1}^{\infty} x_m e_m = \lim_{n \to \infty} \sum_{m=1}^{n} x_m e_m \ .$$

But $\mathrm{span}\{e_1, e_2, ...\}$ is not the whole of $\ell^2(\mathbb{C})$. In fact, $\mathrm{span}\{e_1, e_2, ...\}$ is precisely the subspace V defined in the question. Thus, the topological basis $\{e_1, e_2, ...\}$ for $\ell^2(\mathbb{C})$ is an algebraic basis for V and it is not an algebraic basis for $\ell^2(\mathbb{C})$.

3.5: Part (a). The matrix F is unitary, hence diagonalizable.

Part (b). Given $r, t \in \mathbb{Z}/n$, then the (r,t) entry of F^2 is

$$\frac{1}{n}\sum_{s\in\mathbb{Z}/n}\zeta^{rs}\zeta^{st} = \frac{1}{n}\sum_{s\in\mathbb{Z}/n}\zeta^{r+t}s = \begin{cases} 1 & \text{if } r=-t, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that F^4 is the identity matrix. So the minimal polynomial of F must divide $X^4 \in \mathbb{C}[X]$. The required conclusion follows.

Part (c). Since F is diagonalizable, part (b) yields $\mathbb{C}^n = E_1 \oplus E_{-1} \oplus E_i \oplus E_{-i}$. So, letting $a = \dim(E_1 \oplus E_{-1})$ and $b = \dim(E_i \oplus E_{-i})$, we have $a + b = n$. On the other hand, $E_1 \oplus E_{-1}$ is the 1-eigenspace of F^2, while $E_i \oplus E_{-i}$ is the -1-eigenspace of F^2. The evaluation of F^2 in part (b) implies that $a - b = \mathrm{tr}(F^2) = 1$. Therefore,

$$\dim(E_1 \oplus E_{-1}) = (n+1)/2 \;, \qquad\qquad \dim(E_i \oplus E_{-i}) = (n-1)/2 \;.$$

3.6: We $\mathbb{C}$-linearly extend ρ to a unitary operator $\phi_{\mathbb{C}}$ on the $\mathbb{C}$-vector space $\mathbb{C}V = V \oplus iV$. Thus, any $\mathbb{R}$-basis $\mathcal{E}$ for V becomes a $\mathbb{C}$-basis for $\mathbb{C}V$, and the matrix representing ρ with respect to $\mathcal{E}$ is also the matrix representing $\rho_{\mathbb{C}}$ with respect to $\mathcal{E}$.

Let z be an eigenvector of $\rho_{\mathbb{C}}$. Since $\rho_{\mathbb{C}}$ is unitary, the corresponding eigenvalue has the form $e^{i\theta}$ with $\theta \in \mathbb{R}$. Write $z = x + iy$ where $x, y \in V$. Suppose $\dim(\mathrm{span}\{x,y\}) = 1$. Then we can put $z = x$ or $z = y$ whence, perforce, $z \in V$ and $e^{i\theta} = \pm 1$. We have $V = \mathrm{span}\{z\} \oplus \mathrm{span}(\{z\}^\perp)$ and ρ restricts to an orthogonal operator on $\mathrm{span}\{z\}^\perp$. The required conclusion follows, in this case, by an inductive argument on $\dim(V)$.

So we may assume that $\dim(\mathrm{span}\{x,y\}) = 2$. We have $\rho(x+iy) = e^{i\theta}(x+iy)$ and, applying complex conjugation, $\rho(x-iy) = e^{-i\theta}(x-iy)$. With respect to the basis $\{x,y\}$ of $\mathrm{span}\{x,y\}$, the matrix representing ρ is the 2×2 rotation matrix specified in the question. We have $V = \mathrm{span}\{x,y\} \oplus \mathrm{span}(\{x,y\}^\perp)$ and ρ restricts to an orthogonal operator on $\mathrm{span}(\{x,y\}^\perp)$. Again, the required conclusion follows by an inductive argument on $\dim(V)$.

Quizzes, with solutions

MATH 224, *Linear Algebra 2*, Spring 2022, Laurence Barker

version: 19 May 2022

Quiz 1: Diagonalize $\begin{bmatrix} 1 & 1/2 \\ -2 & 1 \end{bmatrix}$. That is, express the specified matrix in the form PDP^{-1} with D diagonal.

Solution: Write A for the specified matrix. The charactreristic polynomial of A is

$$\det(\lambda I - A) = \lambda^2 - 2\lambda + 2$$

which has roots $\dfrac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$. The eigenvalue $1+i$ has associated eigenvectors $\begin{bmatrix} x \\ y \end{bmatrix}$ where $x + y/2 = (1+i)x$ and $2x + y = (1+i)y$. We can put $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2i \end{bmatrix}$. Similarly (or by taking complex conjugates), we see that the eigenvalue $1-i$ has an associated eigenvector $\begin{bmatrix} 1 \\ -2i \end{bmatrix}$. So $A = PDP^{-1}$, where $P = \begin{bmatrix} 1 & 1 \\ 2i & -2i \end{bmatrix}$ and $D = \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix}$. □

Comment: The example illustrates fact that, to diagonalize a real square matrix, we may need to extend to the complex numbers.

Quiz 2: The linear code over $\mathbb{F}_2$ with Hamming matrix

$$H = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

has the following decoding table:

000	001	010	011	100	101	110	111	syndrome
00000	00101	01011	01110	10010	10111	11001	11100	00
00001	00100	01010	01111	10011	10110	11000	11101	01
00010	00111	01001	01100	10000	10101	11011	11110	10
01000	01101	00011	00110	11010	11111	10001	10100	11

(a) For the message words 011, 001, 000, write down the corresponding codewords.

(b) For the received words 00110, 01011, 11010, write down the corresponding syndromes and the corresponding decoded words.

Solution: Part (a). Respectively, those message words have codewords 01110, 00101, 00000.

Part (b). Respectively, those received words have syndromes 11, 00, 11 and decodings 011, 010, 100.

Quiz 3: Express the matrix $A = \begin{bmatrix} 1 & 1 \\ -4 & 5 \end{bmatrix}$ as $A = PEP^{-1}$ with E in Jordan normal form.

Solution: The characteristic polynomial of A is

$$\det(tI - A) = \begin{vmatrix} t-1 & -1 \\ 4 & t-5 \end{vmatrix} = (t-1)(t-5) + 4 = t^2 - 6t + 9 = (t-3)^2 \ .$$

Solving for an eigenvector $f_1 = (x, y)$ with eigenvalue 3, we have

$$\begin{bmatrix} 1 & 1 \\ -4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix} \ .$$

In other words, $x + y = 3x$ and $-4x + 5y = 3y$. Thus, $y = 2x$. We can put $f_1 = (1, 2)$. Solving for $f_2 = (x, y)$ such that $Af_2 = 3f_2 + f_1$, we have

$$\begin{bmatrix} 1 & 1 \\ -4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \ .$$

In other words, $x + y = 3x + 1$ and $-4x + 5y = 3y + 2$. Thus, $y = 2x + 1$. We can put $f_2 = (0, 1)$. The Jordan basis $\{f_1, f_2\}$ yields

$$P = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} , \qquad E = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \ .$$

Comment: By the uniqueness property of Jordan normal form, there is no other solution for E. Of course, there are infinitely many solutions for P.

Quiz 4: Let $e_1 = (1, 1)$ and $e_2 = (0, 1)$. Apply the Gram–Schmidt process to the basis $\{e_1, e_2\}$ of $\mathbb{R}^2$ to obtain an orthonormal basis $\{f_1, f_2\}$ for $\mathbb{R}^2$. (Recall, the orthonormality condition is $\langle f_1 \,|\, f_1 \rangle = \langle f_2 \,|\, f_2 \rangle = 1$ and $\langle f_1 \,|\, f_2 \rangle = 0$.)

Solution: We construct an orthogonal basis $\{f_1', f_2'\}$ where $f_1' = e_1$ and $f_2' = e_2 - \langle f_1 \,|\, e_2 \rangle f_1$. Then we construct an orthonormal basis $\{f_1, f_2\}$ where $f_i = f_i' / \parallel f_i' \parallel$. We have $\parallel f_1 \parallel^2 = 2$, so

$$f_1 = (1/\sqrt{2}, 1/\sqrt{2}) \ .$$

Hence $f_2' = (0, 1) - (1/\sqrt{2}, 1/\sqrt{2})/\sqrt{2} = (-1/2, 1/2)$. Since $\parallel f_2' \parallel^2 = 1/2$, we have

$$f_2 = (-1/\sqrt{2}, 1/\sqrt{2}) \ .$$

Quiz 5: Which of the following form a subspace of $\mathrm{Mat}_n(\mathbb{R})$ and, in those cases, what is the dimension?

(a) The set of symmetric matrices on $\mathbb{R}^n$?

(b) The set of orthogonal matrices on $\mathbb{R}^n$?

Solution: Part (a), obviously, yes. The dimension is $n(n+1)/2$. Letting $\epsilon_{i,j}$ be the matrix in $\mathrm{Mat}_n(\mathbb{R})$ with (i, j) entry 1 and all other entries 0, then the subspace of symmetric matrices has a basis consisting of the matrices $\epsilon_{i,j} + \epsilon_{j,i}$.

Part (b), obviously, no.

Quiz 6: What is the set of real numbers t such that t is the trace of an orthogonal $n \times n$ matrix over $\mathbb{R}$?

Solution: When $n = 1$, the set is $\{-1, 1\}$. Otherwise, the set is the closed interval $[-n, n]$. Indeed, this is because the real eigenvalues of an orthogonal matrix are 1 and -1, while the non-real eigenvalues can be arranged in pairs $\{z, z^{-1}\}$ where z and hence z^{-1} are complex numbers with modulus unity.

MATH 224: Linear Algebra 2. Spring 2022. Midterm 1

LJB, 17 March 2022, Bilkent University.

Time allowed: 110 minutes hours. Please put your name on EVERY sheet of your manuscript. The use of telephones, calculators or other electronic devices is prohibited. The use of red pens or very faint pencils is prohibited too. You may take the question sheet home.

1: 35 marks. Consider the linear coding scheme over the field $\{0, 1\}$ with Hamming matrix

$$H = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix} .$$

(a) What is the generating matrix G for the coding scheme?

(b) Encode the message words 000, 001, 011.

(c) Write down a decoding table, including the column of syndromes, ensuring that the received words 00001, 00010, 00100 all have decoding 000

(d) Using that decoding table, for the received words 10000, 11000, 11001, write down the syndromes and the decoded words.

(e) How many possible decoding tables are there, assuming that the received words with decoding 000 all have weight 0 and 1?

2: 35 marks. Let $A = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 2 & 1 \\ 0 & -2 & 4 \end{bmatrix}$ as a matrix over $\mathbb{C}$.

(a) Show that A has unique eigenvalue 2.

(b) Find P and E, with E in Jordan normal form, such that $A = PEP^{-1}$.

(c) What is the minumal polynomial of A?

3: 10 marks. Let F be a field, let m and n be positive integers with $m \leq n$, and let M and N be F-vector spaces with dimensions $m = \dim(M)$ and $n = \dim(N)$ such that M is a subspace of N. Let E be the F-vector space consisting of the linear maps $\alpha : N \to N$ such that $\alpha(M) \leq M$. Evaluate $\dim(E)$.

4: 20 marks. Let F be a field, n a positive integer, V an F-vector space with dimension $n = \dim(V)$. Let $\theta : V \to V$ be a linear map such that $\theta^2 = \mathrm{id}_V$.

(a) Suppose $1_F + 1_F \neq 0_F$. Let E_1 and E_{-1} denote the 1-eigenspace and the -1-eigenspace of θ, respectively. Show that $\dim(E_1) + \dim(E_{-1}) = n$.

(b) Now suppose that $1_F + 1_F = 0_F$. Let E_1 denote the 1-eigenspace of θ. Show that $\dim(E_1) \geq n/2$.

Solutions to Midterm 1

1: We have $G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

Part (b). Respectively, 000, 001, 011 have encodings 00000, 00111, 01110.

Part (c). The decoding table is as follows.

000	001	010	011	100	101	110	111	syndrome
00000	00111	01001	01110	10011	10100	11010	11101	00
00001	00110	01000	01111	10010	10101	11011	11100	01
00010	00101	01011	01100	10001	10110	11000	11111	10
00100	00011	01101	01010	10111	10000	11110	11001	11

Part (d). Respectively, 10000, 11000, 11001 have syndromes 11, 10, 11 and decodings 101, 110, 111.

Part (e). The received word with syndrome 00 and decoding 000 must be 00000. The received word with syndrome 01 and decoding 000 must be 00001 or 01000. The received word with syndrome 01 and decoding 000 must be 00010. The recevied word with syndrome 11 and decoding 000 must be 00100 or 10000. So the number of possible decoding tables is 1.2.1.2 = 4.

2: Part (a). The characteristic polynomial of A in $\mathbb{C}[t]$ is

$$\begin{vmatrix} t & -1 & 0 \\ 2 & t-2 & -1 \\ 0 & 2 & t-4 \end{vmatrix} = t((t-1)(t-4)+2) + 2(t-4)$$

$$= t(t^2 - 6t + 10) + 2t - 8 = t^3 - 6t^2 + 12t - 8 = (t-2)^3 \ .$$

So the unique eigenvalue is 2.

Part (b). Let f_1 be an eigenvector of A. Say $f_1 = (x, y, z)$. Then

$$y = 2x \ , \qquad -2x + 2y + z = 2y \ , \qquad -2y + 4z = 2z \ .$$

We have $z = y$. Putting $x = 1$, then $f_1 = (1, 2, 2)$. The argument shows that every eigenvector of A is a scalar multiple of f_1. So there is a Jordan basis $\{f_1, f_2, f_3\}$ of A such that $Af_2 = 2f_2 + f_1$ and $Af_3 = 2f_3 + f_2$.

Writing $f_2 = (x, y, z)$, then

$$y = 2x + 1 \ , \qquad -2x + 2y + z = 2y + 2 \ , \qquad -2y + 4z = 2z + 2 \ .$$

Hence $2x = y - 1$ and $z = y + 1$. Putting $y = 1$, then $f_2 = (0, 1, 2)$.

Now writing $f_3 = (x, y, z)$, then

$$y = 2x \ , \qquad -2x + 2y + z = 2y + 1 \ , \qquad -2y + 4z = 2z + 2 \ .$$

Again, $z = y + 1$. Putting $z = 1$, then $f_3 = (0, 0, 1)$. From the coordinates of the vectors f_i, we arrive at

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix} \ , \qquad E = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \ .$$

Part (c). In view of the Jordan normal form of A, the minimal polynomial of A is $(t-2)^3$.

3: Extending a basis $\{e_1, ..., e_m\}$ for M to a basis $\mathcal{E} = \{e_1, ..., e_n\}$ for N, then E is the vector space consisting of those linear maps $\alpha : N \to N$ such that, with respect to $\mathcal{E}$, the matrix representing α has (i,j)-entry 0 whenever $i > m \geq j$. The number of other entries is $\dim(E) = n^2 + m^2 - nm$.

4: Part (a). By the Cayley–Hamilton Theorem, the only possible eigenvalues of θ are 1 and -1. When we extend to a larger field containing F, the nullities of $\theta - \mathrm{id}_V$ and $\theta + \mathrm{id}_V$ do not change, in other words, $\dim(E_1)$ and $\dim(E_{-1})$ do not change. So we may assume that F is algebraically closed. Let T be the matrix representing θ with respect to a Jordan basis for θ. We are to show that T is diagonal. Supposing otherwise, we may assume that the top-left Jordan block of T is not a 1×1 matrix. Then the $(1,2)$ entry of T^2 must be 2 or -2, which is impossible because T^2 is the identity matrix.

Part (b). Arguing as before, we may assume that F is algebraically closed. Again let T be a matrix representing θ with respect to a Jordan basis for θ. It suffices to show that every Jordan block of T is a 1×1 matrix or a 2×2 matrix. Supposing otherwise, we may assume that the top-left Jordan block of T is a square matrix with at least 3 columns. Then the $(1,3)$-entry of T^2 is 1, which is again impossible because T^2 is the identity matrix.

Alternative, faster version of the solution to 4: The following was used in the exam script of Buruk Yavus. Recall that, when λ_i runs over the eigenvalues of a linear map and the minimal polynomial is $\prod_i (t - \lambda_i)^{m_i}$, the largest Jordan block with eigenvalue λ_i is an $m_i \times m_i$ matrix. We have $\theta^2 - \mathrm{id} = 0$. So the minimal polynomial of θ must divide $t^2 - 1$. In part (a), we have $t^2 - 1 = (t-1)(t+1)$. So the minimal polynomial must be $t - 1$ or $t + 1$ or $(t-1)(t+1)$. It follows that all the Jordan blocks must be 1×1 matrices, in other words, θ is diagonalizable. In part (b), we have $t^2 - 1 = (t-1)^2$. So now the minimal polynomial must be $t-1$ or $(t-1)^2$. Hence all the Jordan blocks must be 1×1 or 2×2 matrices.

MATH 224: Linear Algebra 2. Spring 2022. Midterm 2

LJB, 21 April 2022, Bilkent University.

Time allowed: 110 minutes hours. Please put your name on EVERY sheet of your manuscript. The use of telephones, calculators or other electronic devices is prohibited. The use of red pens or very faint pencils is prohibited too. You may take the question sheet home.

1: (30 marks). Let $e_1 = (0, 1, 1)$ and $e_2 = (1, 0, 1)$ and $e_3 = (1, 1, 0)$. Let $\{f_1, f_2, f_3\}$ be the basis for $\mathbb{R}^3$ obtained from $\{e_1, e_2, e_3\}$ by the Gram–Schmidt process. Evaluate f_1 and f_2 and f_3.

2: (30 marks). Let $U = \text{span}_{\mathbb{R}}\{(1, 2, 3)\}$ as a subspace of $\mathbb{R}^3$. Let $\pi : U \leftarrow \mathbb{R}^3$ and $\pi' : U^{\perp} \leftarrow \mathbb{R}^3$ be the orthogonal projections. Express, with respect to the standard basis of $\mathbb{R}^3$, the matrices representing π and π'.

3: (12 marks). Let n be a positive integer. Determine the sets of:

(a) complex numbers z such that z is the determinant of a Hermitian operator on $\mathbb{C}^n$,

(b) complex numbers z such that z is the trace of a Hermitian operator on $\mathbb{C}^n$,

(c) complex numbers z such that z is the determinant of a unitary operator on $\mathbb{C}^n$,

(d) complex numbers z such that z is the trace of a unitary operator on $\mathbb{C}^n$.

4: Let n be a positive integer.

(a) (2 marks) Let $\{f_1, ..., f_n\}$ be an orthonormal basis for $\mathbb{R}^n$ and let $x \in \mathbb{R}^n$. Let x_i, for $1 \leq i \leq n$, be real numbers such that $x = \sum_i x_i f_i$. State a formula for x_i in terms of x and f_i.

(b) (26 marks) Let $f : \mathbb{R}^n \leftarrow \mathbb{R}^n$ be a distance-preserving function, in other words,

$$\| f(x) - f(y) \| = \| x - y \|$$

for all $x, y \in \mathbb{R}^n$. Show that there exist $a \in \mathbb{R}^n$ and an orthogonal $n \times n$ matrix A such that $f(x) = Ax + a$ for all $x \in \mathbb{R}^n$. (Warning: be careful not to make an invalid assumption of linearity.)

Solutions to Midterm 2

1: We put $f_1' = e_1 = (0,1,1)$ and

$$f_2' = e_2 - \frac{\langle f_1' \,|\, e_2 \rangle}{\| f_1' \|^2} f_1' \; .$$

Now $\langle f_1' \,|\, e_2 \rangle = 1$ and $\| f_1' \|^2 = 2$, so $f_2' = (1,0,1) - (0,1,1)/2 = (1,-1/2,1/2)$. We have

$$f_3' = e_3 - \frac{\langle f_1' \,|\, e_3 \rangle}{\| f_1' \|^2} f_1' - \frac{\langle f_2' \,|\, e_3 \rangle}{\| f_2' \|^2} f_2' \; .$$

Now $\langle f_1' \,|\, e_3 \rangle = 1$, also $\langle f_2' \,|\, e_3 \rangle = 1/2$ and $\| f_2' \|^2 = 3/2$. So

$$\begin{aligned} f_3' &= (1,0,0) - (0,1,1)/2 - (1,-1/2,1/2)/3 \\ &= \big((6,0,0) - (0,3,3) - (2,-1,1)\big) = (2,2,-2)/3 \; . \end{aligned}$$

Up to scaling, the obtained orthogonal basis is $\{(0,1,1),(2,-1,1),(1,1,-1)\}$. Normalizing, the associated orthonormal basis is $\{(0,1,1)/\sqrt{2},(2,-1,1)/\sqrt{6},(1,1,-1)/\sqrt{3}\}$.

Comment: In the phrasing of the question, I omitted to specify whether the basis should be orthonormal or just orthogonal. I deemed any basis consisting of multiples of $(0,1,1)$ and $(2,-1,-1)$ and $(1,1,-1)$ to be correct.

2: We have $\pi(x,y,z) = \dfrac{(1,2,3)(x,y,z)}{\| (1,2,3) \|^2}(1,2,3) = \dfrac{x+2y+3z}{1+4+9}(1,2,3)$

$$= (x+2y+3z, 2x+4y+6z, 3x+6y+9z)/14 \; .$$

So π and π are represented by $\dfrac{1}{14}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ and $\dfrac{1}{14}\begin{bmatrix} 13 & -2 & -3 \\ -2 & 10 & -6 \\ -3 & -6 & 5 \end{bmatrix}$, respectively.

3: The eigenvalues of a Hermitian operator on $\mathbb{R}^n$ are arbitrary real numbers, so the answer to (a) is $\mathbb{R}$ and the answer to (b) is $\mathbb{R}$. The eigenvalues of a unitary operator on $\mathbb{C}^n$ are arbitrary complex numbers with modulus unity. So the answer to (c) is $\{z \in \mathbb{C} : |z| = 1\}$ and the answer to (d) is $\{z \in \mathbb{C} : |z| \leq n\}$.

4: Part (a). Obviously, $x_i = \langle f_i \,|\, x \rangle$.

Part (b). Define $g : \mathbb{R}^n \leftarrow \mathbb{R}^n$ such that $g(x) = f(x) - f(0)$ for $x \in \mathbb{R}^n$. We have $\| g(x) - g(y) \| = \| f(x) - f(y) \| = \| x - y \|$. Replacing f with g, we may assume that $f(0) = 0$, and we are to show that $f(x) = Ax$ for some orthogonal matrix A.

We have $\| f(x) \| = \| f(x) - f(0) \| = \| x - 0 \| = \| x \|$. Thus, f preserves the norm. We have

$$\| f(x) \|^2 + \| f(y) \|^2 - 2\langle f(x) \,|\, f(y) \rangle = \| f(x) - f(y) \|^2 = \| x - y \|^2 = \| x \|^2 + \| y \|^2 - 2\langle x \,|\, y \rangle \; .$$

But $\| f(x) \| = \| x \|$ and similarly for y, so $\langle f(x) \,|\, f(y) \rangle = \langle x \,|\, y \rangle$. Thus, f preserves the inner product.

Let $\{e_1, ..., e_n\}$ be the standard basis for $\mathbb{R}^n$. Let $f_i = f(e_i)$. Since f preserves the inner product, $\{f_1, ..., f_n\}$ is an orthonormal basis for $\mathbb{R}^n$. Write $x = \sum_i e_i$. Again using the preservation of inner products, and also applying part (a), $x_i = \langle e_i \,|\, x \rangle = \langle f_i \,|\, f(x) \rangle$, hence $f(x) = \sum_i x_i f_i$. It is now clear that f is an operator and that, in fact, f is an orthogonal operator.

MATH 224: Linear Algebra 2. Spring 2022. Makeup 2

LJB, 12 May 2022, Bilkent University.

Time allowed: 110 minutes hours. Please put your name on EVERY sheet of your manuscript. The use of telephones, calculators or other electronic devices is prohibited. The use of red pens or very faint pencils is prohibited too. You may take the question sheet home.

1: (30 marks). Let $e_1 = (0, 1, 2)$ and $e_2 = (1, 2, 1)$ and $e_3 = (2, 1, 0)$. Let $\{f_1, f_2, f_3\}$ be the basis for $\mathbb{R}^3$ obtained from $\{e_1, e_2, e_3\}$ by the Gram–Schmidt process. Evaluate f_1 and f_2 and f_3.

2: (30 marks). Let

$$U = \{(a, b, c, d) \in \mathbb{R}^4 : a + b + c + d = b + 2c + 3d = 0\} .$$

Express, with respect to the standard basis of $\mathbb{R}^3$, the matrix representing the orthogonal projection $U \leftarrow \mathbb{R}^4$.

3: (20 marks). Let m and n be positive integers with $m \leq n$. Let U be an m-dimensional subspace of an n-dimensional inner product space V over $\mathbb{C}$. Let E be the set of Hermitian operators $V \leftarrow V$ that restrict to operators $U \leftarrow U$. Regarding E as a vector space over $\mathbb{R}$ in the obvious way, evaluate the dimension of E.

4: (20 marks). Let θ be a unitary operator on $\mathbb{C}^n$.

(a) Under what conditions on θ do there exist only finitely many unitary operators ϕ on $\mathbb{C}^n$ such that $\theta = \phi^2$?

(b) Under those conditions, how many such ϕ exist?

MATH 224: Linear Algebra 2. Spring 2022. Final

LJB, 20 May 2022, Bilkent University.

Time allowed: 2 hours. Please put your name on EVERY sheet of your manuscript. The use of telephones, calculators or other electronic devices is prohibited. The use of red pens or very faint pencils is prohibited too. You may take the question sheet home.

1: (20 marks). Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

(a) Find the eigenvalues and eigenvectors of A.

(b) Find the rank and signature of the real quadratic form represented by A.

2: (40 marks). Let n be an integer. When $A^2 = B$ for $n \times n$ matrices A and B, we call A a **square root** of B.

(a) Find the multiplicity of 1 as an eigenvalue of $H = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$.

(b) Find all the other eigenvalues of H. (Hint: to avoid complicated calculations, consider the trace of H.)

(c) Show that H has a square root in $\mathrm{Mat}_4(\mathbb{R})$. (Hint: to avoid complicated calculations, use parts (a) and (b).)

(d) Show that every symmetric matrix in $\mathrm{Mat}_n(\mathbb{R})$ has a square root in $\mathrm{Mat}_n(\mathbb{C})$.

(e) Give an example of an integer n and a symmetric matrix B in $\mathrm{Mat}_n(\mathbb{R})$ such that B has no square root in $\mathrm{Mat}_n(\mathbb{R})$.

(f) Give an example of an integer n and a matrix C in $\mathrm{Mat}_n(\mathbb{R})$ such that C has no square root in $\mathrm{Mat}_n(\mathbb{C})$.

3: (20 marks). For a positive integer n,

(a) How many equivalence classes of symmetric bilinear forms $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ are there?

(b) For how many of those equivalence classes are the bilinear forms non-degenerate?

4: (20 marks). Let $\mathbb{F}$ denote the field with order $|\mathbb{F}| = 4$, let n be a positive integer, and let $\mathbb{F}^n$ denote the standard $\mathbb{F}$-vector space of dimension n.

(a) What is the number of n-tuples $(f_1, ..., f_n)$ such that the set $\{f_1, ..., f_n\}$ is a basis for $\mathbb{F}^n$?

(b) How many linear maps $\mathbb{F}^n \to \mathbb{F}^n$ are there?

(c) How many invertible linear maps $\mathbb{F}^n \to \mathbb{F}^n$ are there?

(d) What is the number of linear maps $\alpha : \mathbb{F}^n \to \mathbb{F}^n$ such that $\det(\alpha) = 1$?

Solutions to Final

1: Part (a). The characteristic polynomial of A is

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (\lambda-1)^2 - 4 = \lambda^2 - 2\lambda + 3 = (\lambda-3)(\lambda+1) \ .$$

Eigenvectors with eigenvalue 3 and -1 are $(1,1)$ and $(1,-1)$.

Part (b). Let Q be the quadratic form represented by A with respect to the standard basis of $\mathbb{R}^2$. With respect to the basis $\{(1,1),(1,-1)\}$, the matrix representing Q has diagonal entries 3 and -1. Normalizing the basis, then the matrix representing Q has diagonal entries 1 and -1. So the rank of Q is 2 and the signature of Q is 0.

2: Part (a). The matrix $H-1$ has rank 1 and nullity 3. So the multiplicity of the eigenvalue 1 is 3.

Part (b). By part (a), H has one other eigenvalue λ, with multiplicity 1. The trace of H is $8 = 3 + \lambda$, so $\lambda = 5$.

Part (c). Since H is symmetric, H is diagonalizable. By parts (a) and (b), there exists invertible $P \in \mathrm{Mat}_4(\mathbb{R})$ such that $H = P\,\mathrm{diag}(1,1,1,5)P^{-1}$, where $\mathrm{diag}(a_1, a_2, ...)$ denotes the diagonal matrix with (i,i) entry a_i. One square root of H is $P\,\mathrm{diag}(1,1,1,\sqrt{5})P^{-1}$.

Part (d). Let S be a symmetric matrix in $\mathrm{Mat}_n(\mathbb{R})$. Then S is diagonalizable over $\mathbb{R}$, say $S = Q\,\mathrm{diag}(d_1, ..., d_n)Q^{-1}$. Letting c_i be a square root of d_i in $\mathbb{C}$, then $Q\,\mathrm{diag}(c_1, ..., c_n)Q^{-1}$ is a square root of S in $\mathrm{Mat}_n(\mathbb{C})$.

Part (e). Put $n = 1$ and let B be the 1×1 matrix with entry -1.

Part (f). Let $C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. For a contradiction, suppose C has a square root A in $\mathrm{Mat}_2(\mathbb{C})$. Then $A^4 = 0$. By considering the minimal polynomial of A, we find that 0 is the unique eigenvalue of A. By considering the Jordan normal form of A, we deduce that $A^2 = 0 \neq C$, which is a contradiction, as required.

3: Part (a). Sylvester's Law of Inertia tells us that each equivalence class is represented by a unique diagonal matrix whose first a entries are 1, whose next b entries are -1 and whose next c entries are 0, where a, b, c are natural numbers such that $a+b+c=n$. Putting $m = a+b = n-c$, then $0 \leq m \leq n$ and, for each m, the number of possibilities for (a,b,c) is $m+1$. So the number of equivalence classes is

$$\sum_{m=0}^{n} (m+1) = 1 + 2 + ... + n + (n+1) = (n+1)(n+2)/2 \ .$$

Part (b). The non-degenerate cases are those where $c = 0$. The number of such cases is $n+1$.

4: Part (a). There are $4^n - 1$ choices for f_1, then $4^n - 4$ choices for f_2 and so on, finally $4^n - 4^{n-1}$ choices for f_n. So the number of such n-tuples is

$$(4^n - 1)(4^n - 4)...(4^n - 4^{n-1}) = 4^{n(n-1)/2}(4_n - 1)(4^{n-1} - 1)...(4-1) \ .$$

Part (b). Since $\mathrm{Mat}_n(\mathbb{F})$ has dimension n^2 as a vector space over $\mathbb{F}$, the specified number is $|\mathrm{Mat}_n(\mathbb{F})| = 4^{n^2}$.

Part (c). Fixing an ordered basis $\{e_1, ..., e_n\}$ for $\mathbb{F}^n$, each Each invertible map α on $\mathbb{F}^n$ is uniquely determined by the ordered basis $\{\alpha(e_1), ..., \alpha(e_n)\}$. So the number of such maps is $4^{n(n-1)/2}(4^n - 1)(4^{n-1} - 1)...(4 - 1)$.

Part (d). Given a nonzero element λ of $\mathbb{F}$ and $A \in \mathrm{Mat}_n(\mathbb{F})$, then multiplying the top row of A by λ produces a matrix with determinant $\lambda \det(A)$. So the number of invertible $n \times n$ matrices with determinant λ is independent of λ. There are exactly 3 non-zero elements of $\mathbb{F}$. So the number of α as specified is the answer to (c) divided by 3. In other words, the number of such α is $4^{n(n-1)/2}(4^n - 1)(4^{n-1} - 1)...(4^2 - 1)$.