Archive of past papers, solutions and homeworks for

MATH 224, Linear Algebra 2,

Spring 2013, Laurence Barker

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Homeworks

Homework 1: Let u_0, u_1, ... and v_0, v_1, ... be sequences of real numbers such that $u_{n+1} = 3u_n - v_n$ and $v_{n+1} = 2u_n$. Express u_n and v_n in terms of u_0 and v_0.

Homework 2: Exercises in Section 5.1 of book, page 256. Question 1 compulsory (all 11 parts). Question 3 optional.

Homework 3: Question 3.1: Let A be a 2×2 matrix $\mathbb{C}$ such that $A^n = I$ for some positive integer n. Show that A is diagonalizable. For each $n \geq 3$, give a counter-example to show that, if we replace $\mathbb{C}$ with $\mathbb{R}$, then the conclusion can fail.

Question 3.2: Let α be an operator on a vector space V (not necessarily over an algebraically closed field). Suppose that $\alpha^2 = \alpha$. Show that α is diagonalizable.

Homework 4: Complete the proof that $\{c_0, c_1, ..., c_t, s_1, ..., s_t\}$ is orthogonal. (In class, we defined $c_j(x) = \cos(jx)$ and $s_j(x) = \sin(jx)$, with $\langle f, g \rangle = \int_0^{2\pi} f(x)g(x)\,\mathrm{dx}$.

Homework 5: Question 6.2.1 (all 7 parts) and Question 6.2.2 parts (a) and (i).

Homework 6: Question 6.1: Let $A = \begin{bmatrix} c & s \\ s & c \end{bmatrix}$ where $c = \cosh(t)$ and $s = \sinh(t)$ with $t \in \mathbb{R}$. Find a unitary matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Question 6.2: Repeat the previous question but now taking A to be 3×3 matrix whose 9 entries are all 1.

Question 6.3: Give a concise proof of the following standard theorem, assuming results already established in class: The eigenvalues of a real symmetric matrix are real and, furthermore, there is an orthonormal basis of real eigenvectors. (You may extract arguments from the textbook or from other sources.)

Question 6.4: Let F be the matrix with (j, k) entry ν^{jk} where $j, k \in \{0, 1, 2, 3, 4\}$ and $\nu = e^{2\pi/5}$. (See discussion in class.) Calculate the matrix F^2, find the eigenvalues of F^2 and hence find the eigenvalues of F up to multiplicity.

Quizzes

Quiz 1: Let x_0, x_1, ... and y_0, y_1, ... be sequences of real numbers such that $x_{n+1} = x_n + y_n$ and $y_{n+1} = -2x_n + 4y_n$ and $x_0 = 2$ and $y_0 = 3$. Show that $x_n = 2^n + 3^n$ and $y_n = 2^n + 2\,.\,3^n$.

Quiz 2: Let α be an operator on a vector space. Let e_1 and e_2 be eigenvalues of α with corresponding eigenvalues λ_1 and λ_2, respectively. Show that $e_1 + e_2$ is an eigenvector if and only if $\lambda_1 = \lambda_2$.

Quiz 3: Give an example of a 2×2 matrix with eigenvalues 2 and 3.

Quiz 4: Let s_1, ..., s_n be the functions $[-1, 1] \to \mathbb{R}^2$ given by $s_j(x) = \sin(jx/\pi)$. Show that $\{s_1, .., s_n\}$ is an orthogonal set with respect to the inner product $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,\mathrm{dx}$.

Quiz 5: Find $a \in \mathbb{R}$ such that $\{(1, 1), (1, a)\}$ is an orthonormal set in $\mathbb{R}^2$.

Quiz 6: Give an example of a 2×2 unitary matrix.

Quiz 7: Write down two distinct orthonormal bases for the matrix $\mathrm{diag}(2, 2, 2)$.

MATH 224 Linear Algebra 2, Midterm 1

11 March 2013, LJB, Bilkent University.

Time allowed: 110 minutes.

Please make sure your name is on every sheet of your script. In Questions 1 and 3, each part of the question carries 10% of the total marks for the exam.

1: 30%: Let n be a positive integer, let V be an n-dimensional vector space over $\mathbb{R}$ and let α be a linear map $V \to V$.

(a) What is an *eigenvector* of α? What is an *eigenvalue* of α?

(b) Show that, if α has n distinct real eigenvalues, then V has a basis whose elements are eigenvectors of α.

(c) Show that, if any of the roots to the characteristic polynomial of α are not real, then V does not have a basis whose elements are eigenvectors of α.

2: 30% Let $A = \begin{bmatrix} 3 & -3 & 2 \\ -3 & 3 & -2 \\ 1 & -1 & 2 \end{bmatrix}$. Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. (You do not need to find the inverse of P.)

3: 40% A Markov process has three states, labelled 1, 2, 3. After 1 unit of time, the probability of remaining in the same state is 2/3. After 1 unit of time, the system cannot move from state 1 to state 3, nor from state 2 to state 1, nor from state 3 to state 2. Thus, the transition matrix is $T = \frac{1}{3}\begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$.

(a) Let $\omega = (-1 + i\sqrt{3})/2 = e^{2\pi i/3}$. Show that $(1, 1, 1)$ and $(1, \omega, \omega^2)$ and $(1, \omega^2, \omega)$ are eigenvectors for T. (Hint: note that $\omega^2 = (-1 - i\sqrt{3})/2$ and $\omega^3 = 1$.)

(b) Find a matrix Q and a diagonal matrix E such that $T = QEQ^{-1}$.

(c) Find the inverse of Q. (Hint: to avoid long calculations, consider the complex conjugate of Q. Note that $1 + \omega + \omega^2 = 0$.)

(d) Show that, independently of the initial state, the probability of the system being in any given state after exactly n units of time tends to $1/3$ as n tends to infinity.

MATH 224 Linear Algebra 2, Midterm 2

15 April 2013, LJB, Bilkent University.

Time allowed: 110 minutes.

Please make sure your name is on every sheet of your script.

1: 30%: Let a be a positive real number and let V be the real inner product space of continuous functions $[0, a] \to \mathbb{R}$ with inner product

$$\langle f, g \rangle = \int_0^a f(x)\, g(x)\, \mathrm{dx} \; .$$

Let w_0, w_1, w_2 be the elements of V given by $w_0(x) = 1$ and $w_1(x) = x$ and $w_2(x) = x^2$. Apply the Gram–Schmidt Process to the set $\{w_0, w_1, w_2\}$ to obtain an orthogonal basis for the span$\{w_0, w_1, w_2\}$.

2: 40% As vectors in $\mathbb{C}^4$, let

$$w_1 = (1, 1, 0, 0) \; , \quad w_2 = (1 + i, 1 - i, 0, 0) \; , \quad w_3 = (1, 1, 1, 1) \; , \quad w_4 = (1, 1, 1 + i, 1 - i) \; .$$

(a) Apply the Gram–Schmidt Process to the set $\{w_1, w_2, w_3, w_4\}$ to obtain an orthogonal basis $\{v_1, v_2, v_3, v_4\}$ for $\mathbb{C}^4$ and an orthonormal basis $\{u_1, u_2, u_3, u_4\}$ for $\mathbb{C}^4$.

(b) Let $x = (1, 0, 1, 0)$. Find the coefficients λ_j such that $x = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 + \lambda_4 u_4$.

3: 30% Let V be a finite-dimensional complex inner product space, and let U be a subspace of V. Let $\{w_1, ..., w_m\}$ be a basis for U, extended to a basis $\{w_1, ..., w_m, w_{m+1}, ..., w_n\}$ for V. Let $\{v_1, ..., v_n\}$ be the orthogonal basis for V obtained from that via the Gram–Schmidt Process and let $\{u_1, ..., u_n\}$ be the associated orthonormal basis.

(a) Briefly, explain why $\{u_1, ..., u_m\}$ is an orthonormal basis for U.

(b) By considering the basis $\{u_1, ..., u_m\}$ for U and the basis $\{u_1, ..., u_n\}$ for V, show that, for each $v \in V$, there exists a vector $\pi(v) \in U$ such that, for all $u \in U$, we have

$$\| v - u \|^2 = \| v - \pi(v) \|^2 + \| \pi(v) - u \|^2 \; .$$

Solutions to Midterm 2

1: We shall show that

$$v_0(x) = 1 \; , \qquad v_1(x) = x - a/2 \; , \qquad v_2(x) = x^2 - ax + a^2/6 \; .$$

We shall calculate v_0, v_1, v_2 directly from the equality $v_k = w_k - \sum_{j=0}^{k-1} \frac{\langle v_j \,|\, w_k \rangle}{\| v_j \|^2} v_j$. The value for v_1 is clear already. Since

$$\| v_1 \|^2 = \int_0^a \mathrm{d}x = a \; , \qquad \langle v_0 \,|\, w_1 \rangle = \int_0^a x \, \mathrm{d}x = a^2/2$$

we have $v_1(x) = w_1(x) - \dfrac{\langle v_0 \,|\, w_1 \rangle}{\| v_0 \|^2}\, v_0(x) = x - \dfrac{a^2/2}{a} = x - a/2$. Since

$$\| v_1 \|^2 = \int_0^a (x - a/2)^2 \, \mathrm{d}x = \int_{-a/2}^{a/2} y^2 \, \mathrm{d}y = a^3/12 \; , \qquad \langle v_0 \,|\, w_2 \rangle = \int_0^a x^2 \, \mathrm{d}x = a^3/3 \; ,$$

$$\langle v_1 \,|\, w_2 \rangle = \int_0^a (x - a/2))x^2 \, \mathrm{d}x = x^4/4 + ax^3/6 \Big|_{x=0}^{x=a} = a^4(1/4 - 1/6) = a^4/12$$

we have $v_2(x) = x^2 - \dfrac{a^3/3}{a} - \dfrac{a^4/12}{a^4/12}\,(x - a/2) = x^2 - ax + a^2/6$.

2: Part (a). We shall show that

$$v_1 = (1, 1, 0, 0) \; , \qquad v_2 = (i, -i, 0, 0) \; , \qquad v_3 = (0, 0, 1, 1) \; , \qquad v_4 = (0, 0, i, -i) \; .$$

Since all four of those vectors have norm $\sqrt{2}$, it will follow that each $u_j = v_j/\sqrt{2}$. We calculate the vectors v_k using the same formula as before, but with the indexing starting at $j = 1$. The value for v_1 is clear. Noting that $\langle v_1 \,|\, w_2 \rangle = \| v_0 \|^2 = 2$, we have

$$v_2 = (1 + i, 1 - i, 0, 0) - \frac{2}{2}\,(1, 1, 0, 0) = (i, -i, 0, 0) \; .$$

The calculations to obtain v_3 and v_4 are similar.

Part (b). Using the equality $\lambda_j = \langle u_j \,|\, x \rangle = \langle v_j \,|\, x \rangle/\sqrt{2}$, we find that $\lambda_1 = \lambda_3 = 1/\sqrt{2}$ and $\lambda_2 = \lambda_4 = -i/\sqrt{2}$.

3: Part (a). Plainly, the set $\{u_1, ..., u_m\}$ is orthonormal. Directly from the formula associated with the Gram–Schmidt Process, we have

$$\mathrm{span}\{v_1, ..., v_m\} = \mathrm{span}\{w_1, ..., w_n\} = U \; .$$

Normalization of a set of vectors does not change the span, so

$$\mathrm{span}\{u_1, ..., u_m\} = \mathrm{span}\{v_1, ..., v_m\} \; .$$

Part (b). Writing $v = \nu_1 u_1 + ... + \nu_m u_m + \nu_{m+1} u_{m+1} + ... + \nu_n v_n$ with each $\nu_j \in \mathbb{C}$, we define $\pi(v) = \nu_1 u_1 + ... + \nu_m u_m$. Writing $u = \mu_1 u_1 + ... + \mu_m u_m$ with each $\mu_j \in \mathbb{C}$, we have

$$\begin{aligned} \| v - u \|^2 &= \| (\nu_1 - \mu_1)u_1 + ... + (\nu_m - \mu_m)u_m + \nu_{m+1}u_{m+1} + ... + \nu_n u_n \|^2 \\ &= |\nu_{m+1}|^2 + ... + |\nu_m|^2 + |\nu_1 - \mu_1|^2 + ... + |\nu_m - \mu_n|^2 \\ &= \| \nu_{m+1}u_{m+1} + ... + \nu_n u_n \|^2 \; + \; \| (\nu_1 - \mu_1)u_1 + ... + (\nu_m - \mu_m)u_m \|^2 \\ &= \| v - \pi(v) \|^2 \; + \; \| \pi(v) - u \|^2 \; . \end{aligned}$$

Comment: In Question 3, π is the orthogonal projection from V to U. It is not hard to show that, given a vector $w \in U$, then $\| v - u \|^2 = \| v - w \|^2 + \| w - u \|^2$ if and only if $w = \pi(v)$. Thus, the equality determines the function π. Intuitively, one senses this already from Pythagoras' Theorem. However, the analytic argument is necessary in order to vindicate that intuition. Simply sketching a plane and a right-angled triangle with vertices labelled u and v and $\pi(v)$ does express some insight, but it amounts merely to an argument by analogy with the 3-dimensional special case, and it does not constitute a proof.

MATH 224 Linear Algebra 2, Final

25 May 2013, LJB, Bilkent University.

Time allowed: 2 hours. Please make sure your name is on every sheet of your script.

1: 10%: Let V be a finite-dimensional inner product space over $\mathbb{C}$. Which of the following statements hold? For each statement, give a proof or a counter-example.

(a) Given unitary operators α and β on V, then $\alpha + \beta$ is unitary.

(b) Given unitary operators α and β on V, then $\alpha\beta$ is unitary.

(c) Given Hermitian operators α and β on V, then $\alpha + \beta$ is Hermitian.

(d) Given Hermitian operators α and β on V, then $\alpha\beta$ is Hermitian.

2: 20%: Let α be an operator on a finite-dimensional inner product space V over $\mathbb{R}$. Suppose that α preserves right-angles, we mean, $\langle \alpha(x), \alpha(y) \rangle = 0$ for all $x, y \in V$ satisfying $\langle x, y \rangle = 0$.

(a) Let $\{e_1, ..., e_n\}$ be an orthonormal basis for V. Is $\{\alpha(e_1), ..., \alpha(e_n)\}$ a basis for V? Is $\{\alpha(e_1), ..., \alpha(e_n)\}$ an orthonormal basis for V?

(b) Show that there is a symmetric operator β and an orthogonal operator γ such that $\alpha = \beta\gamma$.

3: 40% Consider the matrix $H = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$.

(a) Using the observation that $(1, 0, -1, 0)$ and $(0, 1, 0, -1)$ and $(1, 1, 1, 1)$ are eigenvectors of H, find the eigenvalues of H and an orthonormal basis of eigenvectors of H. (Warning: the three specified eigenvectors are not normalized.)

(b) Find an orthogonal matrix U and a diagonal matrix D such that $H = UDU^{-1}$.

(c) Let E be the diagonal matrix such that $E^2 = D$ and all the eigenvalues of E are non-negative. Let $G = UEU^{-1}$. Show that $G^2 = H$.

(d) Evaluate the matrix G explicitly. (Hint: Recall that the inverse of an orthogonal matrix can be written down immediately without doing any work. If your calculations are correct, you will find that, although some of the entries of G are irrational, the entries do have an elegant symmetrical pattern.)

(Comment: The $n \times n$ version of H, for arbitrary n, is often used as a discrete analogue of the differential operator $\mathrm{d}^2/\mathrm{dx}^2$ for functions defined on a circle.)

4: 30% An $n \times n$ real symmetric matrix S is said to be **weakly positive-definite** provided $\langle x, Sx \rangle \geq 0$ for all nonzero vectors $x \in \mathbb{R}^n$. For such S, prove the following assertions. (You may assume standard general results about real symmetric matrices.)

(a) All the eigenvalues of S are non-negative,

(b) There exists a weakly positive-definite matrix R such that $R^2 = S$. (Hint: adapt the idea in Question 3. Be clear about what general results you are using.)

Comments on Final

In part (a) of Question 3, a pedestrian way of finding the fourth element of an eigenvector basis is to calculate the eigenvalues of the three given eigenvectors, find the fourth eigenvalue by considering the trace of H, then solve for the associated eigenvector. A quicker method is to note that the fourth eigenvector must be orthogonal to the three given eigenvectors. There was no need to find the characteristic polynomial of H.

Comically, a few candidates saw this part as a cue for an application of the Gram–Schmidt routine. Hilariously, two candidates felt that three eigenvectors was already good enough, took U to be a 4×3 matrix, and then supposedly inverted U by taking the conjugate-transpose. Another two candidates, wisely preferring U to be a 4×4 matrix, cunningly inserted a column of zeroes before supposedly inverting U.

For part (d) of that question, the unique correct answer is

$$G = \frac{1}{2} \begin{bmatrix} a & c & b & c \\ c & a & c & b \\ b & c & a & c \\ c & b & c & a \end{bmatrix}$$

where $a = 1 + \sqrt{2}$ and $b = 1 - \sqrt{2}$ and $c = -1$.

For Question 4, it is necessary to make use of the theorem asserting that, given a real symmetric matrix, then there is an orthonormal basis of real eigenvectors. In part (b), writing U for the orthogonal matrix whose columns comprise an eigenvector basis for S, then $S = UDU^{-1}$ where D is diagonal, say, $D = (\lambda_1, ..., \lambda_n)$. By part (a), each eigenvalue λ_j is non-negative Defining $E = \text{diag}(\sqrt{\lambda_1}, ..., \sqrt{\lambda_n})$ and $R = UEU^{-1}$, it is easy to confirm that R has the required properties. (Incidentally, it is not very hard to show that R is unique.)

MATH 224 Linear Algebra 2, Retake Final

4 June 2013, LJB, Bilkent University.

Time allowed: 2 hours. Please make sure your name is on every sheet of your script.

1: 20%: Let A be a 3×3 real orthogonal matrix such that $\det(A) = 1$. Show that there exists an orthogonal matrix P and a real number θ such that

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} .$$

(You may assume that every orthogonal operators on $\mathbb{R}^2$ is a rotation or a reflection.)

2: 40% Consider the matrix $H = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$.

(a) Show that, if two of x_1, x_2, x_3, x_4 are equal to 1 and if two of them are equal to -1, then (x_1, x_2, x_3, x_4) is an eigenvector for H.

(b) Hence find an orthonormal basis of eigenvectors for H. For each basis element, what is the corresponding eigenvalue? (Hint: there is no need to consider the characteristic polynomial.)

(c) Find an orthogonal matrix P and a diagonal matrix D such that $H = PDP^{-1}$.

(d) Let n be a positive integer. Using part (c), evaluate H^n. (Give a formula for the matrix entries in terms of n.)

3: 25% Let V be a finite-dimensional real inner product space. For non-zero vectors $x, y \in V$, the **correlation** of x and y is defined to be

$$\mathrm{corr}(x, y) = \frac{\langle x, y \rangle}{\| x \| \, . \, \| y \|} .$$

Let α be an invertible linear operator on V such that $\mathrm{corr}(\alpha(x), \alpha(y)) = \mathrm{corr}(x, y)$ for all $x, y \in V - \{0\}$. Show that there exists a non-zero real number λ and an orthogonal operator β such that $\alpha = \lambda\beta$.

4: 15% Let A be a Hermitian matrix. Consider the pairs (U, D) such that U is a unitary matrix, D is a diagonal matrix and $A = UDU^{-1}$. Show that there are infinitely many such pairs if and only if the characteristic polynomial of A has a repeated root.

Syllabus

Week 1: Eigenvalues and eigenvectors

Week 2: Diagonalizability. Jordan Normal Form (without proof).

Week 3: Quotient spaces and triangulation.

Week 4: Cayley–Hamilton Theorem.

Week 5: Applications of Markov chains.

Week 6: Real inner product spaces. Isometry. Midterm 1.

Week 7: Orthonormal bases. Gram–Schmidt Process. Orthogonal complements.

Week 8: Cauchy-Schwartz and Triangle Inequalities.

Week 9: Symmetric matrics and the Spectral Theorem

Week 10: Complex inner product spaces.

Week 11: Orthogonal and unitary operators. Midterm 2.

Week 12: Bilinear forms.

Week 13: Sylvester's Law of Inertia. Minkowsky metric on spacetime.

Week 14: Proof of Jordan Normal Form.