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MATH 220, Linear Algebra, Fall 2023

Bilkent University, Laurence Barker, 14 January 2024

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MATH 220, Section 1
Linear Algebra, Fall 2023
Course specification

Laurence Barker, Bilkent University. Version: 15 September 2023.

Classes: Wednesdayss 09:30 - 10:20, Fridays 13:30 - 15:20, room SA-Z01.

Office Hours: Wednesdays 08:30 - 09:20, my office, SA-129.
For all students, those doing well and aiming for an A, those doing badly and aiming for a C, Office Hours is an opportunity to come and ask questions.

Instructor: Laurence Barker
e-mail: barker at fen nokta bilkent nokta edu nokta tr.

Course Texts: The primary course text is:

> Bernard Kolman, David R. Hill, "Elementary Linear Algebra with Applications", 9th Edition, New International Edition, (Pearson 2014).

Since there is no formally assessed homework, and since suggested exercises will be separate from the textbook, any other edition of the above will do, or indeed any similar kind of textbook on the subject.

In fact, the internet has a vast supply of text and videos on the material convered. Now, if this course is only peripheral to your main interests, I can understand why you might wish to focus mainly on the textbook. Nevertheless, at a proficient level, full academic study does involve consultation of multiple sources, along with all the trouble which that entails: dealing with different notations, different terminology, different versions of the propositions and so on.

Supplementary material: Further texts supporting the course will be, from time to time, uploaded and updated on my university homepage (which is avaliable from the department webpages or, more quickly, by googling my name).

Syllabus: *Below is a tentative course schedule.* The format of the following details is *Week number: Monday date: Subtopics (Section numbers).*

1: 11 Sept: (Friday only.) Systems of linear equations, matrices 1.1 - 1.5.

2: 18 Sept: Echelon form of a matrix, nonsingular matrices 2.1 - 2.3.

3: 25 Sept: Elementary matrices 2.3 - 2.4.

4: 2 Oct: Determinants 3.1 - 3.5.

5: 9 Oct: Applications, catch-up.

6: 16 Oct: Vector spaces, subspaces 4.1 - 4.4.

7: 23 Oct: Linear independence, basis, dimension 4.5 - 4.6.

8: 30 Oct: Coordinates, homogenous systems 4.7 - 4.8.

9: 6 Nov: Rank of a matrix 4.9. Standard inner product 5.1 - 5.2.

10: 13 Nov: Inner product spaces, Gram–Schmidt process 5.3 - 5.4.

11: 20 Nov: Orthogonal complement 5.5, Linear transfomrations 6.1.

12: 27 Nov: Kernal and range of a matrix 6.2 - 6.3. Similarity 6.5.

13: 4 Dec: Diagonalization, eigenvalues and eigenvectors 7.1 - 7.3.

14: 11 Dec: Applications, catch-up.

15: 18 Dec: (Wednesday only.) Review.

Assessment:

- Midterm, 40%, Thursday 26 October, 20:00 - 22:00, rooms B-Z01, B-Z02, B-Z04, B-Z05.
- Final, 50%. Details to be announced.
- 8 Quizzes, 10%.

A Midterm score of least 20% (of the available Midterm marks) is needed to qualify to take the Final Exam, otherwise an FZ grade will be awarded.

75% attendance is compulsory.

Asking questions in class is very helpful. It makes the classes come alive, and it often improves my sense of how to pitch the material. The rule for talking in class is: if you speak, then you must speak to everyone in the room.

Quizzes, with solutions

MATH 220, *Linear Algebra, Section 1*, Fall 2023, Laurence Barker

version: 15 December 2023

Up to date versions of this file can be found on my homepage. Versions that will not always be up to date can also be found in the main course page for MATH 220 on Moodle.

Quiz 1: By Gaussian elimination, using our standard matrix notation, solve

$$x + y + z = 7\ , \qquad y + z = 5\ , \qquad y + 2z = 3\ ,$$

starting off by coding the problem as $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & 1 & 1 & 5 \\ 0 & 1 & 2 & 3 \end{array}\right]$.

Solution: Starting from $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & 1 & 1 & 5 \\ 0 & 1 & 2 & 3 \end{array}\right]$,

we subtract row 2 from row 3, yielding $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & -2 \end{array}\right]$. The equations are now

$$x + y + z = 7\ , \qquad y + z = 5\ , \qquad z = -2\ .$$

So $y = 5 - z = 7$ and $x = 7 - y - z = 2$. In conclusion, $(x, y, z) = (2, 7, -2)$.

Quiz 2: Let n be a positive integers and A an $n{\times}n$ matrix whose (i, j) entry is $a_{i,j}$. Suppose $a_{i,j} = 0$ when $i + j \leq n$. (Thus, the entries above the bottom-left to top-right diagonal are 0.) Evaluate $\det(A)$.

Solution: We have

$$\det(A) = (-1)^k a_{1,n} a_{2,n-1} ... a_{n,1}\ .$$

where $n = 2k$ or $n = 2k + 1$. Indeed, by performing k row operations, each operation interchanging two rows, we replace A with an upper triangular matrix B whose diagonal entries are $a_{1,n}$, ..., $a_{n,1}$. We have $\det(A) = (-1)^k \det(B)$.

Comment: For another version of the same formula, note that $(-1)^k = (-1)^{n(n-1)/2}$.

Quiz 3: Let $U = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$. Is U a vector space? In other words, is U a subspace of $\mathbb{R}^3$?

Solution: Let $u, u' \in U$. Write $u = (x, y, z)$ and $u' = (x', y', z')$. Then

$$u + u' = (x + x', y + y', z + z')\ ,$$

which is plainly in U. So U is closed under addition. Given $\lambda \in \mathbb{R}$, then

$$\lambda u = (\lambda x, \lambda y, \lambda z)\ ,$$

plainly in U. So U is closed under scalar multiplication. Therefore, U is a subspace of $\mathbb{R}^3$.

Quiz 4: Does the set $\left\{ \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 16 \\ -5 \end{bmatrix}, \right\}$ span $\mathbb{R}^3$?

Solution: The question is as to whether, for all $\underline{y} \in \mathbb{R}^3$, there exists $\underline{x} \in \mathbb{R}^3$ satisfying $A\underline{x} = \underline{y}$, where $A = \begin{bmatrix} 1 & 2 & 4 \\ 5 & 6 & 16 \\ -3 & 1 & -5 \end{bmatrix}$. Subtracting multiples of the first row of A from the other rows, we obtain the matrix $\begin{bmatrix} 1 & 2 & 4 \\ 0 & -4 & -4 \\ 0 & 7 & 7 \end{bmatrix}$. Further row operations yield the matrix $B = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. An equivalent form of the question is as to whether, for all $\underline{v} \in \mathbb{R}^3$, there exists $\underline{u} \in \mathbb{R}^3$ such that $B\underline{u} = \underline{v}$. Putting $\underline{v} = (0, 0, 1)$, we see that the answer is: no.

Comment: The above solution follows the method described in the textbook section on spanning sets. When we have completed the theory of bases of vector spaces and when we have established the rank-nullity formula for matrices, it will be clear that the question is simply as to whether the matrix A is invertible. To show that the negative answer is correct, it is enough to show that $\det(A) = 0$.

Quiz 5: Find the rank and nullity of the matrix $\begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \\ 1 & 3 & 9 & 13 \\ 1 & 4 & 16 & 21 \end{bmatrix}$.

Solution: Subtracting multiples of column 1 from the other columns yields $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 3 & 4 \\ 1 & 2 & 8 & 10 \\ 1 & 3 & 15 & 18 \end{bmatrix}$.

Subtracting multiples of column 2 yields $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 2 \\ 1 & 3 & 6 & 6 \end{bmatrix}$.

Subtracting column 3 from column 4, then scaling column 2 yields $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 0 \end{bmatrix}$.

It is now clear that the rank is 3 and the nullity is 1.

Comment 1: As a check, the rank plus the nullity is $3 + 1 = 4$.

Comment 2: The question can equally well be done by using row operations to reduce to row echelon form.

Comment 3: As a quick solution: The nullity must be at least 1 because the last column is the sum of the other columns. The rank must be at least 3 because the first three columns are linearly independent, indeed, the top-left 3×3 submatrix is a Vandemonde matrix.

Quiz 6: Recall, P_1 is the real vector space consisting of the functions $f : R \to \mathbb{R}$ that have the form $f(t) = a + bt$ with $a, b \in \mathbb{R}$. Regard P_1 as an inner product space with inner product given by

$$(f, g) = \int_0^1 f(t)g(t)\, dt \, .$$

Let $u_0(t) = 1$ and $u_1(t) = t$. Apply the Gram–Schmidt process to the basis $\{u_0, u_1\}$ to obtain an orthonormal basis for P_1.

Solution: We first obtain an orthogonal basis $\{v_0, v_1\}$. We put $v_0 = u_0$ and

$$v_1 = u_1 - \frac{(v_0, u_1)}{\|v_0\|^2} v_0 \ .$$

Thus $v_0(t) = 1$. We have

$$(v_0, u_1) = \int_0^1 t \, dt = \quad t^2/2 \Big|_{t=0}^{1} = 1/2 \ , \qquad \| v_0 \|^2 = \int_0^1 dt = 1 \ ,$$

hence $v_1(t) = u_1 - v_0/2 = t - 1/2$. To obtain an orthonormal basis $\{w_0, w_1\}$, we put $w_0 = v_0/\|v_0\|$ and $w_1 = v_1/\|v_1\|$. We have already seen that $\|v_0\| = \|u_0\| = 1$. So $w_0(t) = 1$. Now

$$\|v_1\| = \int_0^1 (t - 1/2)^2 \, dt = \int_{-1/2}^{1/2} s^2 \, ds = \quad s^3/3 \Big|_{s=-1/2}^{1/2} = 1/12 \ .$$

So $w_1(t) = \sqrt{12}(t - 1/2) = \sqrt{3}(2t - 1)$. In conclusion, the obtained orthonormal basis is $\{1, \sqrt{3}(2t - 1)\}$.

Quiz 7: Find the eigenvalues of the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Solution: The characteristic equation is

$$0 = \begin{vmatrix} 1 - \lambda & 1 \\ 1 & 1 - \lambda \end{vmatrix} = (1 - \lambda)^2 - 1 = \lambda(\lambda - 2)$$

which has solutions $\lambda = 0$ and $\lambda = 2$.

Quiz 8: Find invertible P and diagonal D such that $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = PDP^{-1}$.

Solution: The characteristic equation is

$$0 = \begin{vmatrix} 1 - \lambda & -1 \\ -1 & 1 - \lambda \end{vmatrix} = (1 - \lambda)^2 - 1 = \lambda^2 - 2\lambda$$

which has solutions $\lambda_1 = 2$ and $\lambda_2 = 0$. Some corresponding eigenvectors are $(1, -1)$ and $(1, 1)$, respectively. So we can put

$$P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} , \qquad D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \ .$$

October 26, 2023, Thursday
20:00–22:00 (2 hours)

NAME:..

STUDENT NO:..

Math 220 Linear Algebra – Midterm Exam

[ ] Laurence Barker (Section 1) [ ] Alex Degtyarev (Section 2)

Rules for the exams

(1) This exam consists of 5 questions of equal weight.

(2) Each question is on a separate sheet. Please read the questions carefully and write your answers under the corresponding questions. Be neat.

(3) Show all your work. Correct answers without sufficient explanation might not get full credit.

(4) Calculators are not allowed.

Please do not write anything below this line.

1	2	3	4	5	TOTAL

NAME: STUDENT NO: DEPARTMENT:

Q-1) Find all real solutions to the linear system

$$\begin{array}{rcrcrcrcrcr} 2x_1 & - & x_2 & + & x_3 & + & 2x_4 & + & 3x_5 & = & 2 \\ 6x_1 & - & 3x_2 & + & 2x_3 & + & 4x_4 & + & 5x_5 & = & 3 \\ 6x_1 & - & 3x_2 & + & 4x_3 & + & 8x_4 & + & 13x_5 & = & 9 \\ 4x_1 & - & 2x_2 & + & x_3 & + & x_4 & + & 2x_5 & = & 1 \end{array}$$

Solution: Convert the augmented matrix to reduced row echelon form:

$$\left[\begin{array}{rrrrr|r} 2 & -1 & 1 & 2 & 3 & 2 \\ 6 & -3 & 2 & 4 & 5 & 3 \\ 6 & -3 & 4 & 8 & 13 & 9 \\ 4 & -2 & 1 & 1 & 2 & 1 \end{array}\right] \sim \left[\begin{array}{rrrrr|r} 2 & -1 & 1 & 2 & 3 & 2 \\ 0 & 0 & -1 & -2 & -4 & -3 \\ 0 & 0 & 1 & 2 & 4 & 3 \\ 0 & 0 & -1 & -3 & -4 & -3 \end{array}\right]$$

$$\sim \left[\begin{array}{rrrrr|r} 2 & -1 & 1 & 2 & 3 & 2 \\ 0 & 0 & 1 & 2 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{rrrrr|r} 2 & -1 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 0 & 4 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{rrrrr|r} 1 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & 4 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

We conclude that x_2 and x_5 are free variables, and the solutions are

$$\begin{aligned} x_1 &= \frac{1}{2}(-1 + r + s), \\ x_2 &= r \in \mathbb{R}, \\ x_3 &= 3 - 4s, \\ x_4 &= 0, \\ x_5 &= s \in \mathbb{R}. \end{aligned}$$

NAME: STUDENT NO: DEPARTMENT:

Q-2) If exists, find the inverse of the matrix

$$\begin{bmatrix} 1 & 0 & 3 & 4 \\ 2 & 1 & 6 & 8 \\ 3 & 4 & 10 & 12 \\ 2 & 5 & 9 & 9 \end{bmatrix}.$$

Solution: We set up the problem as $\left[\begin{array}{cccc|cccc} 1 & 0 & 3 & 4 & 1 & 0 & 0 & 0 \\ 2 & 1 & 6 & 8 & 0 & 1 & 0 & 0 \\ 3 & 4 & 10 & 12 & 0 & 0 & 1 & 0 \\ 2 & 5 & 9 & 9 & 0 & 0 & 0 & 1 \end{array}\right]$.

Subtracting multiples of row 1 from the other rows, $\left[\begin{array}{cccc|cccc} 1 & 0 & 3 & 4 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 & -3 & 0 & 1 & 0 \\ 0 & 5 & 3 & 1 & -2 & 0 & 0 & 1 \end{array}\right]$.

Subtracting multiples of row 2 from rows 3 and 4 gives $\left[\begin{array}{cccc|cccc} 1 & 0 & 3 & 4 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 5 & -4 & 1 & 0 \\ 0 & 0 & 3 & 1 & 8 & -5 & 0 & 1 \end{array}\right]$.

Subtracting 3 times row 3 from row 4 gives $\left[\begin{array}{cccc|cccc} 1 & 0 & 3 & 4 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 5 & -4 & 1 & 0 \\ 0 & 0 & 0 & 1 & -7 & 7 & -3 & 1 \end{array}\right]$.

Subtracting multiples of rows 3 and 4 from row 1 gives $\left[\begin{array}{cccc|cccc} 1 & 0 & 3 & 4 & 14 & -16 & 9 & -4 \\ 0 & 1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 5 & -4 & 1 & 0 \\ 0 & 0 & 0 & 1 & -7 & 7 & -3 & 1 \end{array}\right]$.

So the inverse matrix is $\begin{bmatrix} 14 & -16 & 9 & -4 \\ -2 & 1 & 0 & 0 \\ 5 & -4 & 1 & 0 \\ -7 & 7 & -3 & 1 \end{bmatrix}$.

NAME: STUDENT NO: DEPARTMENT:

Q-3) Evaluate

$$\begin{vmatrix} 1 & 2 & 2 & 0 & 2 \\ 0 & 1 & 2 & 0 & -1 \\ 2 & 3 & 7 & 2 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 3 & 3 & 0 & 6 \end{vmatrix}.$$

Solution: We do the expansion in the columns indicated, upon zeroing out as many entries of these columns as possible (the entries to be killed by elementary row operations are underlined):

$$\begin{vmatrix} 1 & 2 & 2 & 0 & 2 \\ 0 & 1 & 2 & 0 & -1 \\ 2 & 3 & 7 & \underline{2} & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 3 & 3 & 0 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 2 & 0 & 2 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 1 & 3 & 0 & -1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 3 & 3 & 0 & 6 \end{vmatrix} \overset{4}{=} \begin{vmatrix} 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 3 & -1 \\ \underline{1} & 3 & 3 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 3 & -1 \\ 0 & 1 & 1 & 4 \end{vmatrix}$$

$$\overset{1}{=} \begin{vmatrix} 1 & 2 & -1 \\ \underline{1} & 3 & -1 \\ \underline{1} & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 5 \end{vmatrix} \overset{1}{=} \begin{vmatrix} 1 & 0 \\ -1 & 5 \end{vmatrix} = 1 \cdot 5 = \boxed{5}.$$

NAME: STUDENT NO: DEPARTMENT:

Q-4) Consider the following matrices:

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 4 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & -1 & 3 \\ 4 & 1 & 5 \\ 2 & 1 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Which of the following expressions make sense? Compute them.

(a) $DA + B$; (b) $CA - 2A^T$; (c) $\det A$; (d) B^{-1}; (e) D^3.

Solution: (a) DA is a (2×3)-matrix, whereas A^T is (3×2). Their sum makes no sense. (Note that there is no need to compute DA !)

(b) C is (3×3) and A is (2×3); their product makes no sense.

(c) A is not square; $\det A$ makes no sense.

(d) B is not square; B^{-1} makes no sense.

(e) This one is straightforward:

$$D^2 := DD = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}, \qquad D^3 := D^2 D = \begin{bmatrix} 13 & 8 \\ 8 & 5 \end{bmatrix}.$$

NAME: STUDENT NO: DEPARTMENT:

Q-5) Find all matrices X such that $X^2 = A$, where

$$\text{(a)}\ \ A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}; \qquad \text{(b)}\ \ A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Solution: We start with a common argument. Clearly, X is a (2×2)-matrix. Let

$$X = \begin{bmatrix} x & y \\ z & w \end{bmatrix}, \quad \text{so that} \quad X^2 = \begin{bmatrix} x^2 + yz & (w+x)y \\ (w+x)z & w^2 + yz \end{bmatrix}.$$

In both cases, equating the pairs of extra-diagonal entries, we have

$$a_{21}: \quad (w+x)z = 0, \tag{5.1}$$

$$a_{12}: \quad (w+x)y = 1 \neq 0; \tag{5.2}$$

hence, $w + x \neq 0$ from (5.2), and then $z = 0$ from (5.1), simplifying X^2 to

$$X^2 = \begin{bmatrix} x^2 & (w+x)y \\ 0 & w^2 \end{bmatrix}.$$

(a) Equating the two pairs of diagonal entries we get $x = \pm 1$ and $w = \pm 1$ but, since $x + w \neq 0$ by (5.2), this leaves but two solutions $x = w = \pm 1$, and then $y = \pm\frac{1}{2}$:

$$X = \pm \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}.$$

(b) Likewise, $x = w = 0$ from the diagonal entries, contradicting to (5.2). Hence, the equation has no solutions.

January 4, 2024, Thursday
09:30–11:30 (2 hours)

NAME:..

STUDENT NO:...

Math 220 Linear Algebra – Final Exam

[ ] Laurence Barker (Section 1)   [ ] Alex Degtyarev (Section 2)

Rules for the exams

(1) This exam consists of 5 questions of equal weight.

(2) Each question is on a separate sheet. Please read the questions carefully and write your answers under the corresponding questions. Be neat.

(3) Show all your work. Correct answers without sufficient explanation might not get full credit.

(4) Calculators are not allowed.

Please do not write anything below this line.

1	2	3	4	5	TOTAL

Q-1) If possible, diagonalize the matrix and find a basis where it has a diagonal form:

$$A := \begin{bmatrix} 4 & 0 & 3 \\ -4 & 1 & -4 \\ -2 & 0 & -1 \end{bmatrix} .$$

Solution: The characteristic equation is

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 0 & 3 \\ -4 & 1-\lambda & -4 \\ -2 & 0 & -1-\lambda \end{vmatrix} = (4-\lambda)(1-\lambda)(-1-\lambda) + 6(1-\lambda)$$

$$= (1-\lambda)(2 - 3\lambda + \lambda^2) = (1-\lambda)^2(2-\lambda) .$$

So the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 1$ and $\lambda_3 = 2$. We shall find corresponding eigenvectors f_1 and f_2 and f_3, respectively. Any eigenvector (x, y, z) with eigenvalue 1 satisfies

$$\begin{bmatrix} 3 & 0 & 3 \\ -4 & 0 & -4 \\ -2 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \, ; .$$

So we can put $f_1 = (0, 1, 0)$ and $f_2 = (1, 0, -1)$. Meanwhile, any eigenvector (x, y, z) with eigenvalue 2 satisfies

$$\begin{bmatrix} 2 & 0 & 3 \\ -4 & -1 & -4 \\ -2 & 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \, ; .$$

So we can put $f_3 = (-3, 4, 2)$. In conclusion, a diagonalizing basis is $\{(0, 1, 0), (1, 0, -1), (-3, 4, 2)\}$, furthermore, $A = PDP^{-1}$, where

$$P = \begin{bmatrix} 0 & 1 & -3 \\ 1 & 0 & 4 \\ 0 & -1 & 2 \end{bmatrix} , \qquad D = \begin{bmatrix} 1 & 0 \; 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} .$$

Comment: As usual, many candidates were unable to deal with a repeated eigenvalue. For the eigenvalue 1, one has to choose 2 linearly independent eigenvectors. And the zero vector $(0, 0, 0)$ is not an eigenvector. If your purported invertible matrix P has a column whose entries are all 0, then your P cannot be invertible.

Q-2) Let M_n be the space of $(n \times n)$-matrices, $B \in M_n$ a fixed matrix, and $L\colon M_n \to M_n$ the operator given by $L(A) = BA$.

(c) Find the eigenvalues and eigenvectors of L in terms of those of B.

(d) Prove that L is diagonalizable if and only if so is B.

Solution: Part (a). Given a scalar λ and nonzero $A \in M_n$, then A is a λ-eigenvector of L if and only if every nonzero column of A is a λ-eigenvector of B. In particular, the eigenvalues of L coincide with the eigenvalues of B.

Part (b). From the above description ot the eigenvectors of L, it is clear that the eigenvectors of L span M_n if and only if the eigenvectors of B span the vector space of n-dimensional column vectors.

Comment: This was intentionaly a hard question, designed to distinguish the strongest candidates. It requires skill at describing mathematical objects.

Q-3) Find an orthogonal basis (do not normalize) for the space P_2 of polynomials of degree at most 2 with the inner product

$$(p, q) = \int_0^2 p(t) \cdot q(t)\, dt.$$

Solution: Putting $u_0(t) = 1$ and $u_1(t) = t - 1$ and $u_2(t) = (t-1)^2$, we shall apply the Gram–Schmidt process to the basis $\{u_0, u_1, u_2\}$ of P_2, obtaining an orthonormal basis $\{v_0, v_1, v_2\}$. The formula to be used is $v_0 = u_0$ and

$$v_i = u_i - \frac{(v_{i-1}, u_i)}{\|v_{i-1}\|^2}\, v_{i-1} - \dots - \frac{(v_0, u_i)}{\|v_0\|^2}\, v_0\ .$$

Thus, $v_0(t) = u_0(t) = 1$. We remark that $\int_{-1}^1 f(t)\,\mathrm{d}t = 0$ for any integrable $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(t) = -f(t)$. In particular, $(v_0, u_1) = \int_0^2 (t-1)\,\mathrm{d}t = \int_{-1}^1 s\,ds = 0$. So $v_1(t) = u_1(t) = t - 1$. Similarly, $(v_1, u_2) = 0$. Since $(v_0, u_2) = \int_0^2 (t-1)^2\,\mathrm{d}t = (t-1)^3/3\,|_0^2 = 2/3$ and $\|v_0\|^2 = \int_0^2 \mathrm{d}t = 2$, we have

$$v_2(t) = u_2(t) - \frac{(v_1, u_2)}{\|v_1\|^2}\, v_1 - \frac{(v_0, u_2)}{\|v_0\|^2}\, v_0 = (t-1)^3 - 1/3 = t^2 - 2t + 2/3\ .$$

So an orthogonal basis for P_2 is $\{1, t-1, t^2 - 2t + 2/3\}$ as a set of functions of t.

Comment: A more obvious choice of basis elements is with $u_0(t) = 1$ and $u_1(t) = t$ and $u_2(t) = t^2$. Then the calculations are a little more complicated but the obtained orthogonal basis is the same.

Q-4) Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ 4 \\ 0 \\ 2 \\ -2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -3 \\ 6 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 2 \\ -4 \\ -1 \\ -3 \\ 4 \end{bmatrix}$$

be vectors in $\mathbb{R}^5$ with the standard inner product and $W := \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. Find a basis for the space $W^\perp$. What are $\dim W$ and $\dim W^\perp$?

Solution: The space $W^\perp$ is the kernel of the matrix

$$A = \begin{bmatrix} 1 & -2 & 1 & 0 & -1 \\ -2 & 4 & 0 & 2 & -2 \\ -3 & 6 & -1 & 2 & -1 \\ 2 & -4 & -1 & -3 & 4 \end{bmatrix}.$$

Row operations do not change the kernel. Subtracting multiples of higher rows from lower rows, then dividing the second row by 2, we obtain

$$\begin{bmatrix} 1 & -2 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Substracting the second row from the first yields

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore $W^\perp = \ker(A) = \{(2b + d - e, b, -d + 2e, d, e) : b, d, e \in \mathbb{R}\}$, and a basis for the kernel is

$$\{(2, 1, 0, 0, 0), (1, 0, -1, 1, 0), (-1, 0, 2, 0, 1)\}.$$

We have $\dim(W) = \mathrm{rank}(A) = 2$ and $\dim(W^\perp) = \mathrm{null}(A) = 3$.

Q-5) Find the rank and a basis for the kernel (the null space) of the transformation $L\colon P_4 \to P_2$ given by

$$p(t) \mapsto p(1) + tp'(1) + t^2p''(1).$$

(Recall that P_d is the space of polynomials of degree at most d.)

Solution: Writing $p(t) = a + b(t-1) + c(t-1)^2 + d(t-1)^3 + e(t-1)^4$, then $p(1) = a$ and $p'(1) = b$ and $p''(1) = 2c$. So, with respect to the bases $\{1, t, t^2\}$ for P_2 and $\{1, t, t^2, t^3, t^4\}$ of P_4, the matrix representing L is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{bmatrix} .$$

Therefore $\ker(L)$ has basis

$$\{(t-1)^3, (t-1)^4\} .$$

Since L is surjective $\operatorname{rank}(L) = \dim(P_2) = 3$. (Alternatively, since $\ker(L)$ has dimension 2, the rank-nullity formula implies that $\operatorname{rank}(L) = \dim(P_4) - 2 = 3$.)

Comment: Many candidates instead considered the basis for P_4 consisting of the functions having the form 1, t, t^2, t^3, t^4. With respect to this basis for P_4 and the same basis as above for P_2 , the matrix representing L is

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & 6 & 12 \end{bmatrix} .$$

Using the standard Gauss–Jordan procedure, one finds that the elements of the kernel have the form $(-u-3v, 3u+8v, -3u-6v, u, v)$. Putting $(u, v) =\in \{(1,0), 0, 1)\}$ yields the basis

$$\{-3 + 8t - 6t^2 + t^4, (t-1)^3\} .$$