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MATH 215 and MATH 500, Mathematical Analysis

Spring 2012, LJB

version: 31 October 2012

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MATH 215 Mathematical Analysis, Spring 2012

Handout 1: Course specification

Laurence Barker, Mathematics Department, Bilkent University,
version: 28 February 2012.

Course Aims: To provide an introduction to analysis for economists, engineers and natural scientists who may make substantial future use of analysis and advanced calculus in their work.

Course Description: The course supplies a foundation for sophisticated applications of analysis and topology in contexts where "advanced calculus" is inadequate. The material in the course has few direct applications in itself. Rather, it is a prerequisite for further study. Roughly the material has two kinds of uses:

• Rigorous treatment of theory and applications. This aspect of analysis arose during the late 19th century in connection with Fourier analysis, complex analysis and theory of partial differential equations.

• Conceptual background for areas of analysis, geometry and topology that would simply not be possible without a systematic approach. This aspect began to emerge during the 1920s, in connection with differential geometry (initially stimulated mainly by general relativity) and operator theory (initially stimulated mainly by quantum mechanics).

Course Instructor: Laurence Barker, Office SAZ 129.

Course Texts:

Primary: W. Rudin, "Principles of Mathematical Analysis", 3rd Ed. (McGraw-Hill, 1976).

Some other sources may be supplied for some small components of the syllabus material.

Classes: Tuesdays 13:40 - 14:30 SAZ 02, Thuesdays 15:40 - 17:30, SAZ 02.

Office Hours: Tuesdays, 14:40 - 15:30, SAZ 129.

Syllabus: Week number: Monday date, subtopics.

1: 6 Feb, properties of the real number system.

2: 13 Feb, countability and cardinality.

3: 20 Feb, real inner product spaces, metric spaces, continuity and completeness.

4: 27 Feb, construction of the real numbers and their completeness.

5: 5 Mar, convergent sequences, Cauchy sequences.

6: 12 Mar, connectedness. Midterm 1 on 15th March, Thursday 15:40.

7: 19 Mar, compactness, Bolzano–Weierstrass Theorem, Borel–Heine Theorem.

8: 26 Mar, convergent series, absolute convergence, the comparison test.

9: 2 Apr, further tests for convergence.

10: 9 Apr, continuity and uniform continuity.

11: 16 Apr, preservation of compactness and connectedness under continuity. Midterm 2 on 19th April, Thursday 15:40.

12: 23 Apr, sequences and series of functions.

13: 30 Apr, Presentations.

14: May, Presentations.

15: May, Review.

Assessment:

- Homeworks and Quizzes and Presentations 15%.
- Midterm I, 25%, 15 March.
- Midterm II, 25%, 19 April.
- Final, 35%.

(75% attendance is obligatory.)

Class Announcements: All students, including any absentees from a class, will be deemed responsible for awareness of class announcements.

MATH 215 Mathematical Analysis, Spring 2012

Handout 2: Further Notes on Countability

Laurence Barker, Mathematics Department, Bilkent University,
version: 31 October 2012.

These notes are merely a supplement to the course text:

Walter Rudin, *Principles of Mathematical Analysis*, 3rd Edition, (McGraw-Hill, 1976).

Summary of definitions and preliminary results

Given sets X and Y, we write $|X| \leq |Y|$ provided there exists an injection $X \to Y$. When $|X| \leq |Y|$ and $|Y| \leq |X|$, we write $|X| = |Y|$.

We call $|X|$ the **cardinality** of X. It is to be understood that $|X|$ is a mathematical object called a **cardinal number**. We make the identification $n = |\{1, 2, ..., n\}|$. A set X is said to be **finite** provided it can be written in the form $X = \{x_1, ..., x_n\}$ where the elements x_1, ..., x_n are mutually distinct. For such X, the cardinality of X, also called the **size** of X, is understood to be $|X| = n$. It is also to be understood that the empty set $\emptyset$ is a finite set with size $|\emptyset| = 0$. Thus, the cardinalities of the finite sets are the natural numbers.

We say that X is **countable** provided $|X| \leq |\mathbb{N}|$, in other words, provided there exists an injection $f : X \to \mathbb{N}$. In that case, we can enumerate the elements of X as x_0, x_1, ... in such a way that $f(x_0) < f(x_1) < ....$ Plainly, every finite set is countable. As a piece of jargon, when X is countable and infinite, we say that X is **countably infinite**. It is not hard to see that X is countably infinite if and only if $|X| = |\mathbb{N}|$.

Remark 1: *Let X, Y, Z be sets and let $f : X \to Y$ and $g : Y \to Z$ be functions. Consider the composite function $g \circ f : X \to Z$.*
(1) *If f and g are injective, then $g \circ f$ is injective.*
(2) *If f and g are surjective, then $g \circ f$ are surjective.*
(3) *If f and g are bijective, then $g \circ f$ is bijective.*

Proof: Suppose that f and g are injective. Given $x, x' \in X$ with $x \neq x'$ then $f(x) \neq f(x')$ by the injectivity of f, whence $g(f(x)) \neq g(f(x'))$ by the injectivity of g. We have shown that $g \circ f$ is injective. Part (1) is established.

Now suppose that f and g are surjective. Given $z \in Z$ then, since g is surjective, we have $z = g(y)$ for some $y \in Y$. Since f is surjective, $y = f(x)$ for some $x \in X$. We have $z = g(f(x))$, so $g \circ f$ is surjective. Part (2) is established.

Part (3) is immediate from parts (1) and (2). $\square$

The remark implies that, given sets X, Y, Z with $|X| \leq |Y|$ and $|Y| \leq |Z|$, then $|X| \leq |Z|$.

Proposition 2: *Given sets X and Y with $X \neq \emptyset$, then $|X| \leq |Y|$ if and only if there exists a surjection $Y \to X$.*

Proof: Suppose that $|X| \leq |Y|$, in other words, there exists an injection $f : X \to Y$. Since $X \neq \emptyset$, we can choose an element $x_0 \in X$. Let $g : Y \to X$ be the function such that $g(y) = x$ when $y = f(x)$ while $g(y) = x_0$ when y is not in the image of f. Plainly, y is surjective.

Conversely, if there exists a surjection $g : Y \to X$ then, for each $x \in X$, we can choose an element $f(x) \in Y$ such that $g(f(x)) = x$. The function $f : X \to Y$ is plainly injective. $\square$

For a set X, the **power set** $\mathcal{P}(X)$ is defined to be the set of all subsets of X. It is easy to see that, if X is finite, then $|\mathcal{P}(X)| = 2^{|X|}$ and, perforce, $|X| < |\mathcal{P}(X)|$. The next result says that the conclusion still holds for arbitrary sets.

Theorem 3: (Cantor) *Given a set X, then $|X| < |\mathcal{P}(X)|$.*

Proof: We have $|X| \leq |\mathcal{P}(X)|$ because the function sending each element $x \in X$ to the element $\{x\} \in \mathcal{P}(X)$ is injective. For a contradiction, suppose that $|X| = |\mathcal{P}(X)|$. Since $\mathcal{P}(X) \neq \emptyset$, Proposition 2 implies that there exists a surjection $f : X \to \mathcal{P}(X)$. Let

$$A = \{x \in X : x \notin f(x)\} .$$

Choose an element $a \in X$ such that $f(a) = A$. The conditions $a \in A$ and $a \notin A$ imply each other, which is absurd. $\square$

It follows, in particular, that the set $\mathcal{P}(\mathbb{N})$ is uncountable.

Some comments on the notion of proof

A proof is a very clear deductive explanation.

In particular, a proof is something communicated from one person to another person. So the validity of a proof depends on the audience. In this course, the audience are the students. So an argument is a proof if it would persuade those students who have mastered the appropriate prerequisites. (Incidentally, that is my criterion when marking exams.) By that criterion, some of my own arguments, presented during the course, will not be proofs. No doubt, some of the arguments produced by students, too, will not be proofs. That will not matter. Mathematics is discipline where important mistakes tend to get detected and then corrected.

Perhaps, the above justification of Remark 1 is not a proof. If Remark 1 was obvious already, then no persuasion was involved, and it would follow that the argument I presented could not have been a proof. Probably, Proposition 2 really did need some justification. Still, the argument I gave was not very interesting. Unquestionably, Theorem 3 has some genuine content. The notion of proof is important because of results like this. So, if the argument I gave for Theorem 3 is not a proof, then there are some important corrections that we need to make.

Some standard results

The following results are standard, which is to say, they are known by all mathematically competent people who have studied the topic. For that reason, you may assume them in an exam question, except when the question makes it clear that their proofs are required. It is important to learn proofs of these results. Proofs of theorems tend to be more general that statements of theorems. In applications, it often happens that one cannot directly apply known theorems but, instead, one can adapt the ideas behind the proofs of the theorems.

For all except one of the next five results, proofs were given in class.

Remark 4: *Let X be a subset of a set Y. Then $|X| \leq |Y|$. In particular, if Y is countable then X is countable.*

Proposition 5: *A countable union of countable sets is countable. That is to say, given a countable set I, and countable sets A_i for each $i \in I$, then the union $\bigcup_{i \in I} A_i$ is countable.*

Proof: We may assume that $I = \mathbb{N}$. Let p_0, p_1, ... be an enumeration of the primes. For each $i \in \mathbb{N}$, let f_i be an injection $A_i \to \mathbb{N}$. Write $A = \bigcup_i A_i$. Given $a \in A$, let i be the smallest natural number such that $a \in A_i$, and let $f(a) = p_i^{f_i(a)}$. By unique prime factorization, f is an injection $A \to \mathbb{N}$. $\square$

Proposition 6: *A finite direct product of countable sets is countable. That is, given countable sets A_1, ..., A_n, then the direct product $A_1 \times ... \times A_n$ is countable.*

Theorem 7: *The set of rational numbers $\mathbb{Q}$ is countable.*

Theorem 8: *The set of real numbers $\mathbb{R}$ is uncountable.*

Some material not on the examinable syllabus

The two theorems in this section are worth noting because, together, they clarify the notion of a cardinal number. However, we shall not be making use of them in the course, so they are excluded from the examinable syllabus.

Proof of the next result is elementary but somewhat complicated.

Bernstein–Cantor–Schröder Theorem: *Given sets X and Y such that $|X| = |Y|$, then there exists a bijection $X \to Y$.*

Note that this is a subtle theorem, not at all obvious. The assumption $|X| = |Y|$ is that there exists an injection $X \to Y$ and an injection $Y \to X$. Excluding the trivial case where $Y = \emptyset$, this is equivalent to the condition that there exists an injection $X \to Y$ and a surjection $X \to Y$. It is not at all straightforward to get from there to the conclusion that there exists a bijection between X and Y.

Proof of the next theorem requires some deeper set theory. In fact, it requires the Axiom of Choice. Some tricky philosophy is required and, really, the only way to establish this result properly is to define the notion of a set axiomatically.

Trichotomy Theorem: (Cantor) *Given sets X and Y, then exactly one of the following three conditions holds: $|X| < |Y|$; or $|X| = |Y|$; or $|X| > |Y|$.*

Another motive for the axiomatic definition of a set is that, all systematic approaches that are powerful enough for normal mathematics lead to counter-intuitive conclusions. And cavalier approaches lead to paradoxes.

Russell's Paradox: *Let S be the set whose elements are those sets T such that $T \notin T$. Then the statements $S \in S$ and $S \notin S$ are the negations of each other, yet they imply each other.*

Let us mention that Cantor introduced two different systems of numbers that could be applied to infinite sets: the *ordinal numbers* and the *cardinal numbers*. The ordinal numbers are no longer considered to be of importance, because their main purpose (a generalization of the Principle of Mathematical Induction) has been superceded by superior techniques (Zorn's Lemma). The notion of the cardinal numbers, though, is fundamental to much of mathematics. Anyway, that historical background indicates one reason why, for infinite sets, we tend to use the term *cardinality* rather than the more down-to-earth term *size*.

MATH 215 Mathematical Analysis, Spring 2012

Handout 3: Construction of the Real Numbers

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version: 31 October 2012.

These notes summarize some material that was presented with more background narrative in lectures. All of the material below is of fundamental importance to the so-called "conceptual" approach to mathematics. The parts on the syllabus for this Mathematical Analysis course are:

Section 2: Equivalence Relations.

Section 5: Construction of the real numbers.

(Section 6: Completion of a metric space. Omitted from this version of the notes, but discussed in class, and outlined in a step-by-step way in Exercise 3.24 of the textbook).

The parts not on the syllabus are:

Section 1: Axiomatic definition of the real numbers (because memorizing the axioms would be a waste of time).

Section 3: Construction of congruence classes of integers (but it is a nice model for a technique that is used frequently throughout mathematics, and I advise reading it in preparation for Sections 5 and 6).

Section 4: Construction of the natural numbers, the integers and the rational numbers (perhaps not "analysis", but still a necessary part of the story).

Note: in this draft, I find that I have included quite a lot of material, but it was written in haste, and there are sure to be many minor mistakes.

1: Axiomatic definition of the real numbers

The following two very general definitions will be needed. A **binary operation** $*$ on a set S, often simply called an **operation** on S, is defined to be a function $* : S \times S \to S$. Often, we write the value at (s, t) not as $*(s, t)$ but as $s * t$. A **relation** $\sim$ on S is formally defined to be a subset of $S \times T$. Usually, we write $s \sim t$ instead of $(s, t) \in \sim$. Thus, $\sim$ associates each element $(s, t) \in S \times T$ with a statement $s \sim t$.

We may understand the set of real numbers $\mathbb{R}$ to be equipped an operation called addition, written $(x, y) \mapsto x + y$, an operation called multiplication, written $(x, y) \mapsto xy$ and a relation $\leq$ called the ordering, such that the following axioms hold:

F1: Additive Associativity Axiom: $x + (y + z) = (x + y) + z$ for all $x, y, z \in \mathbb{R}$.

F2: Zero Axiom: There is an element $0 \in \mathbb{R}$ such that $0 + x = x$ for all $x \in \mathbb{R}$.

F3: Negation Axiom: For all $x \in \mathbb{R}$, there is an element $-x \in \mathbb{R}$ such that $x + (-x) = 0$.

F4: Additive Commutativity Axiom: $x + y = y + x$ for all $x, y \in \mathbb{R}$.

F5: Distributivity Axiom: $x(y + z) = xy + xz$ for all $x, y, z \in \mathbb{R}$.

F6: Multiplicative Associativity Axiom: $x(yz) = (xy)z$ for all $x, y, z \in \mathbb{R}$.

F7: Unity Axiom: There is an element $1 \in \mathbb{R}$ such that $1x = x$ for all $x \in \mathbb{R}$.

F8: Inversion Axiom: For all non-zero $x \in \mathbb{R}$, there is an element $x^{-1} \in \mathbb{R}$ such that $x^{-1}x = 1$.

F9: Multiplicative Commutativity Axiom: $xy = yx$ for all $x, y \in \mathbb{R}$.

T1: Reflexivity Axiom: $x \leq x$ for all $x \in \mathbb{R}$

T2: Anti-symmetry Axiom: For all $x, y \in \mathbb{R}$ satisfying $x \leq y$ and $y \leq x$, we have $x = y$.

T3: Transitivity Axiom: For all $x, y, z \in \mathbb{R}$ satisfying $x \leq y$ and $y \leq z$, we have $x \leq z$.

T4: Commensurability Axiom: $x \leq y$ or $y \leq x$ for all $x, y \in \mathbb{R}$.

O1: Additive Ordering Axiom: For all $x, y, z \in \mathbb{R}$ satisfying $y \leq z$, we have $x + y \leq x + z$.

O2: Multiplicative Ordering Axiom: For all $x, y \in \mathbb{R}$ satisfying $0 \leq x$ and $0 \leq y$, we have $0 \leq xy$.

O3: Least Upper Bound Axiom: Any non-empty subset of $\mathbb{R}$ with an upper bound has a least upper bound.

All of these axioms, except for the last three, crop up in many different kinds of important mathematical structures. For instance, anything satisfying F1, F2, F3, F4 is called an **abelian group**. Anything satisfying F1 to F6 is called a **ring**. Anything satisfying F1 to F9 is called a **field**. Anything satisfying T1, T2, T3 is called a **partial ordering**. Those four concepts appear very frequently throughout most of mathematics. Anything satisfying T1, T2, T3, T4 is called a **total ordering**.

Of course, there are some other important features of $\mathbb{R}$, such as subtraction $x - y$, division x/y, absolute value $|x|$, taking limits $\lim_n x_n$. But these can be defined in terms of the equipment specified above. For instance, subtraction is defined to be the operation on $\mathbb{R}$ sending (x, y) to the real number $x - y = x + (-y)$.

It can be shown that, given two structures $\mathbb{R}_1$ and $\mathbb{R}_2$ satisfying all the above axioms, then there exists a unique bijection $\theta : \mathbb{R}_1 \to \mathbb{R}_2$ such that $\theta(x+y) = \theta(x)+\theta(y)$ and $\theta(xy) = \theta(x)\theta(y)$ for all $x, y \in \mathbb{R}_1$. It can also be shown that θ satisfies $x \leq y$ if and only if $\theta(x) \leq \theta(y)$. Thus, $\mathbb{R}_1$ and $\mathbb{R}_2$ are copies of each other. That allows us to avoid the metaphysical question as to what, exactly, the real numbers are. All copies of $\mathbb{R}$ are copies of each other, and no practical meaning can be assigned to any notion of, so to speak, the genuine original $\mathbb{R}$. Our concern, essentially, is with the common structure shared by all the copies of $\mathbb{R}$.

Below, we shall give a construction of $\mathbb{R}$, starting from the notion of a set. The motive for doing this is not just to check that such a structure $\mathbb{R}$ exists. The construction is useful in itself. It also illustrates various techniques that are often employed elsewhere in mathematics.

2: Equivalence Relations

A relation $\equiv$ on a set S is called an **equivalence relation** provided it satisfies the following four conditions for all $x, y, z \in S$.

Reflexivity Axiom: We have $x \equiv x$.

Symmetry Axiom: If $x \equiv y$ then $y \equiv x$.

Transitivity Axiom: If $x \equiv y$ and $y \equiv z$ then $x \equiv z$.

Plenty of examples will be given in later sections. Given an equivalence relation $\equiv$ on a set S and an element $x \in S$, the set

$$[x] = \{y \in S : y \equiv x\}$$

is called the **equivalence class** of x under $\equiv$.

Remark 2.1: *With the notation above, S is the disjoint union of the equivalence classes under $\equiv$. In other words, each element of S belongs to a unique equivalence class.*

Proof: Given $x \in S$ then, by reflexivity, $x \in [x]$. So S is the union of the equivalence classes. To show that the union is disjoint, it suffices to show that, given an equivalence class $[z]$ then, for all $y \in [z]$, we have $[y] = [z]$. For all $x \in [y]$, we have $x \equiv y$ and $y \equiv z$ hence, by transitivity, $x \equiv z$, in other words, $x \in [z]$. We have shown that $[y] \subseteq [z]$. By the symmetry condition, $z \in [y]$. Repeating the above argument with z and y interchanged, we deduce that $[z] \subseteq [y]$. Therefore $[y] = [z]$, as required. $\square$

3: Construction of congruence classes of integers

In this section, we presume familiarity with the set $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$ of integers, and we shall discuss the set $\mathbb{Z}/n$ of integers modulo n, where n is a positive integer. We shall equip this set with addition and multiplication operations.

The definition of $\mathbb{Z}/n$ is based on the relation $\equiv_n$ on $\mathbb{Z}$ such that, given $x, y \in \mathbb{Z}$ then $x \equiv_n y$ if and only if n divides $x - y$. We claim that $\equiv_n$ is an equivalence relation. Consider $x, y, z \in \mathbb{Z}$. If $x \equiv y$ and $y \equiv z$, in other words, if n divides $x - y$ and $y - z$, then n divides the integer $x - y + y - z = x - z$, in other words, $x \equiv z$. We have shown that $\equiv_n$ satisfies the Transitivity Axiom. Similar and easier arguments show that $\equiv_n$ satisfies the Reflaxivity and Symmetry Axioms. The claim is established.

We define $\mathbb{Z}/n$ to be the set of equivalence classes under $\equiv$. That is to say,

$$\mathbb{Z}/n = \{[x]_n : x \in \mathbb{Z}\}$$

where $[x]_n$ denotes the equivalence class of x under $\equiv_n$. We define an operation on $\mathbb{Z}/n$, called addition, given by

$$[x]_n + [y]_n = [x + y]_n$$

and another operation on $\mathbb{Z}/n$, called multiplication, given by

$$[x]_n \, [y]_n = [xy]_n \, .$$

We must show that these operations are well-defined, in other words, unambiguous. Before doing so, let us be clear about what the problem is. We defined addition to be such that, given elements X and Y of $\mathbb{Z}/n$, then $X + Y$ is to be the element of $\mathbb{Z}/n$ such that, choosing x and y satisfying $X = [x]$ and $Y = [y]$, then $X + Y = [x + y]$. What we must check is that, if we had chosen different integers x' and y' such that $X = [x']$ and $Y = [y']$, then we would have arrived at the same element $X + Y = [x' + y']$. Now, since $[x] = [x']$ and $[y] = [y']$, the integers $x - x'$ and $y - y'$ are divisible by n, hence $(x+y) - (x' + y')$ is divisible by n. Therefore $[x + y] = [x' + y']$, as required. The well-definedness of the multiplication is almost as easy. Again assuming that $[x] = [x']$ and $[y] = [y']$, then n divides the integer $(x - x')y = xy - x'y$, so

$[xy] = [x'y]$. Similarly, $[x'y] = [x'y']$. We conclude that $[xy] = [x'y']$ and that the multiplication operation is well-defined.

Let us point out that $\mathbb{Z}$ and $\mathbb{Z}/n$ are rings, we mean to say, they satisfy Axioms F1 to F6. In fact, they also satisfy Axioms F7 and F9. It is not hard to see that, furthermore, $\mathbb{Z}/n$ is a field if and only if n is prime.

Exercise 3.A: Let X be a set and let $d : X \times X \to \mathbb{R}$ be a function such that, for all $x, y, z \in X$, we have: $d(x, y) \geq 0$ and $d(x, x) = 0$; also $d(x, y) = d(y, x)$; also $d(x, z) \leq d(x, y) + d(y, z)$. Consider the relation $\equiv$ on X such that $x \equiv y$ if and only if $d(x, y) = 0$.
(1) Show that $\equiv$ is an equivalence relation.
(2) Letting $\mathcal{X}$ denote the set of equivalence classes under $\equiv$, show that there is a well-defined function $\delta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ such that $\delta([x], [y]) = d(x, y)$.
(3) Show that δ is a metric on $\mathcal{X}$.

One application of the latest exercise is in the notion of the Hilbert space $L^2(\mathbb{R})$. Often, $L^2(\mathbb{R})$ is casually described as the metric space consisting of the functions $f : \mathbb{R} \to \mathbb{C}$ such that $\int_{-\infty}^{\infty} |f(x)|^2 \, \mathrm{d}x$ is defined and finite, the metric being given by $d(f, g) = \int_{-\infty}^{\infty} \overline{g(x)}\, f(x) \, \mathrm{d}x$. But, letting c run over the real numbers, consider the functions f_c such that $f_c(x) = 1$ when $x = c$ and $f_c(x) = 0$ otherwise. These functions are mutually distinct but mutually equivalent in the sense of the above exercise, $d(f_c, f_{c'}) = 0$ for any $c, c' \in \mathbb{R}$. Strictly speaking, the metric space $L^2(\mathbb{R})$ is the set of equivalence classes under that equivalence relation.

4: Construction of the natural numbers, the integers and the rational numbers

Let us start from the notion of a set. A set S is said to be **finite** provided every injection $S \to S$ is a bijection. Two finite sets are said to be **equipotent** provided there is a bijection between them. Ignoring some deep roubles with set theory (the notion of the "set of finite sets" is logically tricky), equipotency of finite sets is an equivalence relation. Let $\mathbb{N}$ be the set of equivalence classes here. We write $|X|$ to denote the equivalence class of a finite set X. The elements of $\mathbb{N}$ are $0 = |\emptyset|$ and $1 = |\{0\}|$ and $2 = |\{0, 1\}|$ and so on. Addition on $\mathbb{N}$ is defined to be such that $|X| + |Y| = |X \cap Y|$ when X and Y are disjoint. Multiplication on $\mathbb{N}$ is defined to be such that $|X|\,|Y| = |X \times Y|$. The ordering on $\mathbb{N}$ is defined to be such that $|X| \leq |Y|$ provided there exists an injection $X \to Y$. It is laborious but easy to check that, equipped with that structure, $\mathbb{N}$ satisfies Axioms F1, F2 and F4 to F7, also F9 and T1 to T4 and O1 to O3.

We define $\mathbb{Z}$ to be the set of equivalence classes under the equivalence relation on $\mathbb{N} \times \mathbb{N}$ whereby $(a, b) \equiv (a', b')$ provided $a + b' = a' + b$. To check that $\equiv$ is an equivalence relation, note that reflexivity and symmetry are obvious, and transitivity holds because, if $a + b' = a' + b$ and $a' + b'' = a'' + b'$ then

$$a + b' + b'' = a' + b + b'' = a'' + b + b'$$

hence $a + b'' = a'' + b$. Write $[a, b]$ for the equivalence class of (a, b). We define addition and multiplication operations on $\mathbb{Z}$ such that

$$[a, b] + [c, d] = [a + b, c + d] \; , \qquad [a, b][c, d] = [ac + bd, ad + bd] \; .$$

Proof of the well-definedness of addition is very easy. In class, we took the trouble to show that the multiplication is well-defined.

We impose an ordering relation $\leq$ on $\mathbb{Z}$ such that $[a, b] \leq [c, d]$ provided $a + d \leq b + c$. Via some long but easy arguments, it can be shown that the relation $\leq$ on $\mathbb{Z}$ is well-defined and that $\mathbb{Z}$ satisfies many but not all of the axioms listed in Section 1.

We regard $\mathbb{N}$ as a subset of $\mathbb{Z}$ by identifying each element $a \in \mathbb{N}$ with the element $[a, 0] \in \mathbb{Z}$. Note that the addition and multiplication operations on $\mathbb{N}$ extend to the addition and multiplication operations on $\mathbb{Z}$. There is an evident way of defining a subtraction operation on $\mathbb{Z}$ and, under the identification we have made,

$$[a, b] = a - b \ .$$

It may be felt that the elements of $\mathbb{N}$, as defined above, are not the usual natural numbers that we have always known, and likewise for the elements of $\mathbb{Z}$. There are many different ways of defining copies of $\mathbb{N}$, but any copy will do just as well as any other copy, likewise for $\mathbb{Z}$.

Variants of the above construction of $\mathbb{N}$ occasionally appear elsewhere in mathematics, and variants of the above construction of $\mathbb{Z}$ appear frequently. (As it happens, the day after presenting some of the above material in class, I was giving a seminar where both of those constructions were adapted as part of the definition of a mathematical object called the monomial Burnsider ring.) Variants of the next construction, too, are frequently applied in many different contexts.

We follow the same approach to constructing the set of rational numbers $\mathbb{Q}$. Let S be the set of pairs (α, β) where $\alpha, \beta \in \mathbb{Z}$ and $\beta > 0$. We define an equivalence relation on S such that (α, β) is equivalent to (α', β') provided $\alpha\beta' = \alpha'\beta$. Writing $[\alpha, \beta]$ for the equivalence class of (α, β), we define

$$[\alpha, \beta] + [\gamma, \delta] = [\alpha + \gamma, \beta\delta] \qquad\qquad [\alpha, \beta]\,[\gamma, \delta] = [\alpha\gamma, \beta\delta] \ .$$

We define $\leq$ on $\mathbb{Q}$ such that $[\alpha, \beta] \leq [\gamma, \delta]$ provided $\alpha\delta \leq \gamma\beta$. The rest of the discussion for $\mathbb{Q}$ proceeds much as for $\mathbb{Z}$. We omit the details. Eventually, $\mathbb{Q}$ is shown to satisfy all of the axioms in section 1 except for Axiom O3, and $\mathbb{Z}$ is realized as a subset of $\mathbb{Q}$ by means of the identification

$$[\alpha, \beta] = \alpha/\beta \ .$$

We point out that the addition, multiplication and ordering on $\mathbb{Q}$ extend the addition, multiplication and ordering on $\mathbb{Z}$.

There are, of course, other ways of constructing $\mathbb{Q}$. For instance, instead of constructing $\mathbb{Z}$ followed by $\mathbb{Q}$, we could have constructed the set of non-negative rational numbers $\mathbb{Q}^+$ from $\mathbb{N}$ using a variant of the latest construction, and then we could have constructed $\mathbb{Q}$ from $\mathbb{Q}^+$ using a variant of the penultimate construction. That would yield a different copy of $\mathbb{Q}$ and, strictly speaking, we perhaps ought to denote it by a different symbol, say, $\mathbb{Q}'$. However, a long but routine argument shows that there is a bijective correspondence $\mathbb{Q} \leftrightarrow \mathbb{Q}'$ preserving the addition operations, the multiplication operations and the ordering operations on the two sets. Thus, $\mathbb{Q}$ and $\mathbb{Q}'$ have the same structure, and they amount to the same thing.

5: Construction of the real numbers

We now apply the above techniques to construct a copy of the real numbers. A sequence $\underline{a} = (a_0, a_1, ...)$ of rational numbers is called a **Cauchy sequence** provided, for all positive rational numbers E, there exists a natural number N such that, for all natural numbers n and m with $n \geq N \leq m$, we have $|a_n - a_m| < E$. As another way of saying this: for all rational $E > 0$, we have $|a_n - a_m| < E$ for sufficiently large n and m.

Let $\mathcal{R}$ be the set of Cauchy sequences. Given $\underline{a}, \underline{b} \in \mathcal{R}$, we define

$$\underline{a} + \underline{b} = (a_0 + b_0, a_1, b_1, ...) \ , \qquad \underline{a} \, . \, \underline{b} = (a_0 b_0, a_1 b_1, ...) \ .$$

As we saw in class, it is easy to check that $\underline{a} + \underline{b}$ is a Cauchy sequence. A little more work is needed to arrive at the same conclusion for $\underline{a} \, . \, \underline{b}$. For that, we need the following lemma. Let us mention that variants of the following proof appear several times in the course material.

Lemma 5.1: *Let $\underline{a}$ be a Cauchy sequence of rational numbers. Then $\underline{a}$ has an upper bound. In other words, there exists a positive rational number A such that $a_n < A$ for all n.*

Proof: Choose $N \in \mathbb{N}$ such that $|a_n - a_m| < 1$ for all $n \geq N \leq m$. Then $|a_n - a_N| < 1$ for all $n \geq N$. Put $A = 1 + \max\{a_0, a_1, ..., a_N\}$. $\square$

For $\underline{a}$ and $\underline{b}$ be as above, choose upper bounds A and B, respectively. Given rational $E > 0$, we have $|a_n - a_m| < E/2B$ and $|b_n - b_m| < E/2A$ for sufficiently large n and m, whence

$$|a_n b_n - a_m b_m| = |a_n b_n - a_m b_n + a_m b_n - a_m b_m| \leq |a_n b_n - a_m b_n| + |a_m b_n - a_m b_m|$$

$$\leq |a_n - a_m| B + A |b_n - b_m| < \frac{E}{2B} B + A \frac{E}{2A} = E \ .$$

We have shown that $\underline{a} + \underline{b}$ is a Cauchy sequence.

Let $\equiv$ be the relation on $\mathcal{R}$ such that $\underline{a} \equiv \underline{a'}$ provided, for all rational $E > 0$, we have $|a_n - a'_n| < E$ for sufficiently large N. We claim that $\equiv$ is an equivalence relation. Suppose that $\underline{a} \equiv \underline{a'}$ and $\underline{a'} \equiv \underline{a''}$. Then, for all rational $E > 0$, we have $|a_n - a'_n| < E/2 > |a'_n - a''_n|$, whence $|a_n - a'_n| < E$. We have established transitivity. The proof of the claim is completed by observing that $\equiv$ is plainly reflexive and symmetric.

We write $[\underline{a}] = [a_0, a_1, ...]$ to denote the equivalence class of $\underline{a}$. We define $\mathbb{R}$ to be the set of equivalence classes,

$$\mathbb{R} = \{[\underline{a}] : \underline{a} \in \mathcal{R}\} \ .$$

We define addition and multiplication to be such that

$$[\underline{a}] + [\underline{b}] = [\underline{a} + \underline{b}] = [a_0 + b_0, a_1 + b_2, ...] \ , \qquad [\underline{a}] \, [\underline{b}] = [\underline{a} \, . \, \underline{b}] = [a_0 b_0, a_1 b_2, ...] \ .$$

Exercise 5.A: Show that the addition and multiplication operations on $\mathbb{R}$ are well-defined.

We define a relation $<$ on $\mathbb{R}$ such that $[\underline{a}] < [\underline{b}]$ provided $[\underline{a}] \neq [\underline{b}]$ and $a_n < b_n$ for sufficiently large n. We define $\leq$ on $\mathbb{R}$ such that $\underline{a} \leq \underline{b}$ provided $\underline{a} < \underline{b}$ or $\underline{a} = \underline{b}$.

Exercise 5.B: Show that the relation $<$ is well-defined.

Exercise 5.C: Show that $\underline{a} \leq \underline{b}$ if and only if, for all rational $E > 0$, we have $b_n - a_n < E$ for all sufficiently large n.

There is a long list of further exercises in checking that $\mathbb{R}$, as defined in this section, satisfies all the axioms listed in Section 1. For all except one of those axioms, the argument is straightforward and makes use of the fact, noted above, that the axiom is satisfied by $\mathbb{Q}$. The exception is the Axiom O3, which we shall deal with below.

We regard $\mathbb{Q}$ as a subset of $\mathbb{R}$ by identifying each rational number a with the element $[a, a, a, ...]$ of $\mathbb{R}$. Plainly, the addition, multiplication and ordering on $\mathbb{R}$ are extensions of those operations and that relation on $\mathbb{Q}$.

Before demonstrating that our constructed $\mathbb{R}$ satisfies O3, the Least Upper Bound Axiom, we prove another fundamental property. Let us take it as granted, now, that $\mathbb{R}$, as constructed above, satisfies all of the other specified axioms. Let us also take it for granted that all necessary preliminary features, such as the subtraction operation $(x, y) \mapsto x - y$ and the absolute value $x \mapsto |x|$ have been defined and have had their basic properties established: for instance, the Triangle Inequality $|x - z| \leq |x - y| + |y - z|$.

We define a **Cauchy sequence** of real numbers to be a sequence $\underline{x} = (x_0, x_1, ...)$ of real numbers such that, for all $\epsilon > 0$, we have $|x_n - x_m| < \epsilon$ for sufficiently large n. Evidently, this extends the notion of a Cauchy sequence of rational numbers.

Theorem 5.2: (Completeness of the Real numbers; also called Cauchy's Criterion for Convergence.) *A sequence of real numbers is convergent if and only if it is a Cauchy sequence.*

Proof: Let $\underline{x}$ be a sequence of real numbers. An easy application of the Triangle Inequality shows that, if $\underline{x}$ is convergent, then $\underline{x}$ is Cauchy. The converse is similar to an argument in the next section. □

Homework 1

Questions 2, 3, 4 are from Rudin, page 45, at the end of Chapter 2.

1: Give a definition of a **countable set**. Directly from the definition you gave, show that a countable union of finite sets is countable. In other words, letting I be a countable set and letting A_i be a finite set for each $i \in I$, then the union $\bigcup_{i \in I} A_i$ is countable.

2: A complex number z is said to be **algebraic** provided there exist integers a_n, a_{n-1}, ..., a_1, a_0 such that $a_n \neq 0$ and $a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0 = 0$. Show that the set of all algebraic numbers is countable. (Hint: for every positive integer m, there are only finitely many integers n and a_n, a_{n-1}, ..., a_1, a_0 such that $n + |a_n| + |a_{n-1}| + ... + |a_1| + |a_0| = m$.)

3: Prove that there exist real numbers that are not algebraic.

4: Is the set of all irrational numbers countable?

Solutions 1

1: A set X is said to be **countable** provided X has the form $X = \{x_0, x_1, ...\}$ where x_0, x_1, ... is a finite or infinite sequence.

We are to show that, given sets A_0, A_1, ..., then the union $A_0 \cup A_1 \cup ...$ is countable. Inductively, suppose we have enumerated the elements of $A_0 \cup ... \cup A_k$ as a_0, ..., a_{m_k}. Then we can enumerate the elements of $A_{k+1} - (A_0 \cup ... \cup A_k)$ as a_{m_k+1}, ..., $a_{m_{k+1}}$.

Comment A: Alternatively, one could adapt the proof of Proposition 5.

Comment B: It is implicit from the phrasing of Question 1 that you are required to prove this standard result, and that you may not simply invoke Proposition 5.

2: For each positive integer m, let A_m be the set consisting of the complex numbers z such that $a_n z^n + ... + a_1 z + a_0 = 0$ for integers a_n, ..., a_0 satisfying $n + |a_n| + ... + |a_0| = m$. Given m, then there are only finitely many integers n and a_n, a_{n-1}, ..., a_1, a_0 satisfying $n + |a_n| + ... + |a_0| = m$. Furthermore, given n, a_n, ..., a_0, then there are at most n complex numbers satisfying $a_n z^n + ... + a_1 z + a_0 = 0$. Therefore A_m is finite. The set of algebraic integers is the union $\bigcup_{m=1}^{\infty} A_m$ and, by Question 1, this union is countable.

Comment C: In the latest argument, did we really need to explain why each A_m is finite, or was it obvious? Well, that is a moot question, and it is not very important anyway. If we were to omit just that part of the explanation, then all the key ideas in the proof would still be communicated to the reader.

3: Let A be the set of algebraic integers. In the previous question, we showed that A is countable. Perforce, $A \cap \mathbb{R}$ is countable. But $\mathbb{R}$ is uncountable. Therefore $A \cap \mathbb{R} \neq \mathbb{R}$.

4: Let I be the set of all irrational numbers. Then $\mathbb{R} = \mathbb{Q} \cap I$. But $\mathbb{Q}$ is countable and the union of two countable sets is countable. Yet $\mathbb{R}$ is uncountable. So I must be uncountable.

Other homeworks

Homework 2: Rudin 2.5, 2.6, 2.7, 2.8, 2.9 (page 43).

Homework 3: Rudin 2.10, 2.13, 2.15, 2.16, 2.17 (page 44).

Homework 4: Rudin 3.20, 3.21, 3.22 (page 82).

Homework 5: Rudin 3.23, 3.24, 3.25 (page 82).

Homework 6: Rudin 4.1, 4.2, 4.3, 4.4 (page 98).

MATH 215: Mathematical Analysis. Midterm 1. LJB, 15 March 2012, Bilkent University.

Time allowed: Two hours. Please put your name on EVERY sheet of your manuscript. The use of telephones and other electronic devices is prohibited. The use of very faint pencils is prohibited too.

1: 10% What is an *open set* in a metric space? (You may chose whichever of the standard definitions you prefer. Only one definition is needed.) Arguing from the definition you gave, explain why the set $(0, \infty) = \{x \in \mathbb{R} : 0 < x\}$ is an open set in the metric space $\mathbb{R}$.

2: 25% (a) What is a *countable set*? (In this course, we adopt the convention that every finite set is countable. Subject to that, you may choose whichever of the standard definitions you prefer. Only one definition is needed.)

(b) Directly from the definition you gave, show that, given countable sets A and B, then $A \cap B$ and $A \cup B$ and $A \times B$ are countable.

3: 25% Let X be a non-empty set and let d_1 and d_2 be two metrics on X. Thus, (X, d_1) and (X, d_2) are metric spaces.

(a) For $x, y \in X$, define $d(x, y) = \max\{d_1(x, y), d_2(x, y)\}$. Must (X, d) be a metric space? Justify your answer with a proof or a counter-example.

(b) Define $e(x, y) = \min\{d_1(x, y), d_2(x, y)\}$. Must (X, e) be a metric space? Again, justify your answer with a proof or a counter-example.

4: 25% Let X be a metric space.

(a) Show that the following two conditions on X are equivalent to each other:

(i) there exist countably many open sets U_0, U_1, U_2, ... whose intersection $\bigcap_{i=0}^{\infty} U_i$ is not open,
(ii) there exist countably many closed sets V_0, V_1, V_2, ... whose union $\bigcup_{i=0}^{\infty} V_i$ is not closed.

(b) In the metric space $\mathbb{R}$, give an example of open sets U_0, U_1, U_2, ... whose intersection is not open. Give an example of closed sets V_0, V_1, V_2, ... whose union is not closed.

5*: 15% Let $\mathcal{S}$ be the set whose elements are those sequences $\underline{a} = (a_0, a_1, a_2, ...)$ such that each a_i is a rational number and $\sum_{n=0}^{\infty} |a_n| \leq 1$. Define

$$d(\underline{a}, \underline{b}) = \sum_{n=0}^{\infty} |a_n - b_n| \; .$$

Let $\mathcal{S}_0$ be the subset of $\mathcal{S}$ consisting of those sequences $\underline{a} = (a_0, a_1, a_2, ...)$ such that only finitely many of the a_i are non-zero.

(a) Show that $\mathcal{S}_0$ is countable and that $\mathcal{S}$ is uncountable.

(b) Show that $(\mathcal{S}, d)$ is a metric space. (You may make free use of ordinary rules for working with infinite sums. You do not need to justify any such rules in this question.)

(c) Is $\mathcal{S}_0$ a closed set in the metric space $(\mathcal{S}, d)$?

MATH 215: Mathematical Analysis. Midterm 2. LJB, 19 April 2012, Bilkent University.

Time allowed: 110 minutes. Please put your name on EVERY sheet of your manuscript. The use of telephones and other such devices is prohibited.

Announcement: Classroom times next week will be Tuesday 24th, 14:40 - 15:30 and Thursday 26th, 15:40 - 17:30 (because I must attend a seminar on Tuesday 13:40 - 14:30).

1: 25% (a) State the Bolzano–Weierstrass Theorem and the Heine–Borel Theorem.
(b) Let (x_0, y_0), (x_1, y_1), ... be elements of $\mathbb{R}^2$ such that $x_n^2 + y_n^2 \le 1$ for all n. Explain why there exists a strictly increasing sequence of natural numbers $n_0 < n_1 < \ldots$ such that the subsequence (x_{n_0}, y_{n_0}), (x_{n_1}, y_{n_1}), ... converges, in other words, we have a limit $(x, y) = \lim_{j \to \infty} (x_{n_j}, y_{n_j})$.
(c) Explain why $x^2 + y^2 \le 1$.
(d) If each $x_n^2 + y_n^2 < 1$, do we always have $x^2 + y^2 < 1$? (Give a proof or a counter-example.)

2: 25% Let V_1, ..., V_n be closed subsets of a Euclidian space $\mathbb{R}^n$, and let $A = V_1 \cup \ldots \cup V_n$ and $B = V_1 \cap \ldots \cap V_n$. Which of the following statements always hold? (In each case, give a proof or a counter-example. You may assume all standard results about compact sets.)
(a) If all the V_j are compact, then A is compact.
(b) If at least one of the V_j is compact, then A is compact.
(c) If all the V_j are compact, then B is compact.
(d) If at least one of the V_j is compact, then B is compact.

3: 25% Let $\mathbb{R}$ be the ordered field of real numbers as constructed by Cantor. Thus, each $x \in \mathbb{R}$ is the equivalence class $x = [\underline{x}]$ where $\underline{x} = (x_0, x_1, \ldots)$ is a Cauchy sequence with $x_j \in \mathbb{Q}$.
(a) Give definitions of the relations $<$ and $\le$ on $\mathbb{R}$. (Any correct definitions will do. You do not need to prove well-definedness.)
(b) Explain why, given $x, y, z \in \mathbb{R}$ such that $x \le y$ and $y \le z$, then $x \le z$.

4: 25% A subset P of a metric space X is said to be **perfect** provided $P = P'$. Let $q_0, q_1, \ldots$ be an enumeration of $\mathbb{Q}$. Let $P_0 = \mathbb{R}$. Choose irrational numbers a_0 and b_0 such that $a_0 < q_0 < b_0$. Let $P_1 = P_0 - (a_0, b_0)$ (the set obtained from P_0 by removing all those elements of P_0 that belong to (a_0, b_0)). Generally, if $q_n \in P_n$, we choose irrational numbers a_n and b_n such that $a_n < q_n < b_n$ and $b_n - a_n$ is less than $|q_n - a_m|$ and $|q_n - b_m|$ for all $m < n$. On the other hand, if $q_n \notin P_n$, then $q_n \in (a_m, b_m)$ for some $m < n$, and we put $a_n = a_m$ and $b_n = b_m$ where $m < n$ and $q_n \in (a_m, b_m)$. Let $P_{n+1} = P_n - (a_n, b_n)$ and let $P = \bigcap_{n=0}^{\infty} P_n$.
(a) Explain why each $a_n \in P$ and $b_n \in P$.
(b) Explain why P is closed.
(c) Explain why P is perfect and $P \cap \mathbb{Q} = \emptyset$. (Hint: given $x \in P$ and $\epsilon > 0$, then there exist rational numbers q_n and q_m such that $x - \epsilon < q_n < x < q_m < x + \epsilon$.)
(d) Explain why P is not compact.
(e) Briefly explain how to modify the construction so as to obtain a non-empty compact perfect subset Q of $\mathbb{R}$ such that $Q \cap \mathbb{Q} = \emptyset$.

Acknowledgement: Question 4 is based on a *Planet Math* entry "A non-empty perfect subset of $\mathbb{R}$ that contains no rational number" by Görkem Özkaya, (a PhD student at Princeton with research interests in signal processing and probability theory).

MATH 215: Mathematical Analysis. Makeup 1. LJB, 1 June 2012, Bilkent University.

Time allowed: 110 minutes. Please put your name on EVERY sheet of your manuscript. The use of telephones and other such devices is prohibited.

1: 20% Let X be a set. Show that the following two conditions are equivalent. What is the usual name for sets which satisfy these two equivalent conditions?

(a) $X = \emptyset$ or there exists a positive integer n and a bijection $X \to \{1, 2, ..., n\}$.

(b) Every injection $X \to X$ is a bijection.

2: 25% Which of the following exist? In each, give an example or a proof of the impossibility.

(a) A metric space with exactly 2 open sets.

(b) A metric space with exactly 3 open sets.

(c) A metric space with exactly 4 open sets.

(d) A metric space with countably many open sets and uncountably many closed sets.

3: 25% Let F be a finite set and let C be a countable set. (You may use standard results in this question, provided you are clear about which standard results you are using.)

(a) Show that the set of functions $F \to C$ is countable.

(b) Show that if $|F| \geq 2$ and C is infinite, then the set of functions $C \to F$ is uncountable.

4: 30% Let X and Y be metric spaces.

(a) Let d_1 and d_2 be the metrics on $X \times Y$ such that

$$d_1((x, y), (x', y')) = d(x, x') + d(y, y') , \qquad d_2((x, y), (x', y')) = \sqrt{d(x, x')^2 + d(y, y')^2} .$$

Show that d_1 and d_2 are metrics on $X \times Y$.

(b) Let $U \subseteq X \times Y$. Show that U is open with respect to the metric d_1 if and only if U is open with respect to the metric d_2.

MATH 215: Mathematical Analysis. Makeup 2. LJB, 1 June 2012, Bilkent University.

Time allowed: 110 minutes. Please put your name on EVERY sheet of your manuscript. The use of telephones and other such devices is prohibited.

1: 10% State the Heine–Borel Theorem. Give an example of a closed subset of $\mathbb{R}^n$ that is not compact. Give an example of a bounded set in $\mathbb{R}^n$ that is not compact.

2: 30% Let x_0, x_1, ... be a sequence in $\mathbb{R}$.

(a) Assume that, for all sufficiently large positive integers n, there exists an integer m such that $n < m$ and $x_n < x_m$. Show that the sequence x_0, x_1, ... has a strictly increasing subsequence $x_{n_1} < x_{n_2} < \dots$.

(b) Now suppose that the assumption in part (a) fails. Show that there is a decreasing subsequence $x_{n_1} \geq x_{n_2} \geq \dots$.

(c) Using parts (a) and (b), deduce the following special case of the Bolzano-Weierstrass Theorem: Every bounded sequence in $\mathbb{R}$ has a convergent subsequence. (Alternative proofs by different methods will not gain credit.)

3: 30% Let f, g, h be bounded functions $\mathbb{R}^2 \to \mathbb{R}$, and let

$$A = \{(x, y) \in \mathbb{R}^2 : f(x, y) + x = g(x, y) + y = h(x, y)\} .$$

(a) Give an example to show that A is not necessarily compact. (Hint: the functions f, g, h are not necessarily continuous.)

(b) Show that the closure of A is compact.

4: 30% Let X_1 and X_2 be compact metric spaces with metrics d_1, d_2, respectively. We make the set $X = X_1 \times X_2$ become a metric space with metric d such that

$$d((x_1, x_2), (y_1, y_2) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\} .$$

(a) Let $x \in X$, and let U be an open set in X such that $x \in U$. Show that there exists an open set U_1 in X_1 and an open set U_2 in X_2 such that $x \in U_1 \times U_2 \subseteq U$.

(b) Show that if X_1 and X_2 are compact, then $X_1 \times X_2$ is compact. (Hint: Let $X = \bigcup_\alpha U_\alpha$ be an open cover. For each $x \in X$, choose an α such that $x \in U_\alpha$ and choose open sets U_1^x in X_1 and U_2^x in X_2 such that $x \in U_1^x \times U_2^x \subseteq U_\alpha$. Consider the union $\bigcup_x (U_1^x \times U_2^x)$.)

MATH 215: Mathematical Analysis. Final. LJB, 30 May 2012, Bilkent University.

Time allowed: two hours. Please put your name on EVERY sheet of your manuscript. The use of telephones and other such devices is prohibited.

1: 20% What is a continuous function $\mathbb{R} \to \mathbb{R}$? Show that the set of continuous functions $\mathbb{R} \to \mathbb{R}$ is uncountable.

2: 25% Let a_0, a_1, ... and b_0, b_1, ... be sequences of complex numbers. Suppose that the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{m=0}^{\infty} b_m$ are absolutely convergent. Let $\gamma_k = |a_0 b_k| + |a_1 b_{k-1}| + ... + |a_k b_0|$ and $c_k = a_0 b_k + a_1 b_{k-1} + ... + a_k b_0$.

(a) Briefly explain why $\left(\sum_{n=0}^{K} |a_n|\right)\left(\sum_{m=0}^{K} |b_m|\right) \leq \sum_{k=0}^{2K} \gamma_k \leq \left(\sum_{n=0}^{2K} |a_n|\right)\left(\sum_{m=0}^{2K} |b_n|\right)$. Hence explain why the series $\sum_{k=0}^{\infty} \gamma_k$ is convergent.

(b) Let α_0, α_1, ... and β_0, β_1, ... be sequences of real numbers such that each $0 \leq \alpha_k \leq \beta_k$ and the series $\sum_{k=0}^{\infty} \beta_k$ is convergent. Explain why the series $\sum_{k=0}^{\infty} \alpha_k$ is convergent.

(c) Using parts (a) and (b), deduce that the series $\sum_{k=0}^{\infty} c_k$ is absolutely convergent.

3: 25% Let X be a compact metric space.

(a) Directly from the definition of compactness, show that, given a metric space Y and a continuous function $h : X \to Y$, then $h(X)$ is compact.

(b) Let $f : X \to \mathbb{R}$ be a continuous function. Using part (a), show that f has an upper bound, in other words, there exists a real number b such that $f(x) \leq b$ for all $x \in X$.

(c) Let b be a real number such that $f(x) < b$ for all $x \in X$. Let

$$U_x = \{x' \in X : 2\,|f(x') - f(x)| < b - f(x)\}$$

for each $x \in X$. By considering the open cover $X = \bigcup_{x \in X} U_x$, show that b is not a least upper bound for the image $f(X)$.

(d) Using parts (b) and (c), show that there exists an element $x_0 \in X$ such that $f(x) \leq f(x_0)$ for all $x \in X$.

4: 15% Let $f : \mathbb{R}^3 \to Y$ be a continuous function and let $g : \mathbb{R}^3 \to Y$ be the function such that $g(x, y, z) = f(\sin(x), \sin(y), \sin(z))$. Show that $g(\mathbb{R}^3)$ is compact.

5: 15% Give an example of a metric space Y and a continuous function $g : Y \to \mathbb{R}$ such that $g(Y)$ has an upper bound but there does not exist an element $y_0 \in Y$ with the property that $g(y) \leq g(y_0)$ for all $y \in Y$.

Presentations

Cansu Başçıl: *p-adic numbers and the p-adic topology*

Merve Merakli: *Probability spaces*

Efe Onarin: *Hilbert spaces*

Deniz Serdengeçti: *Fixed-point theorems in economics*

Doğuhan Şündal: *The Contraction Mapping Theorem*

Cüneyet Y{ilmaz: *Topological spaces*

Vedet Bayram (MATH 500): *Dedekind's construction of the reals*