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MATH 210, Finite and Discrete Mathematics

Bilkent University, Spring 2016, Laurence Barker

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MATH 210, Finite and Discrete Mathematics, Spring 2016

Course specification

Laurence Barker, Bilkent University, version: 20 May 2016.

Course Aims: To supply an introduction to some concepts and techniques associated with discrete methods in pure and applied mathematics; to supply an introduction or reintroduction to the art of very clear deductive explanation.

Course Description: The terms *combinatorics* and *discrete mathematics* have similar meanings. The former refers to an area of pure mathematics concerned with mathematical objects that do not have very much topological, geometric or algebraic structure. The latter refers to an area of applicable mathematics that rose to prominence with the advent of electronic computers and information technology. Of course, the two cultures overlap considerably and cannot be clearly distinguished from each other.

In the 1950s and 60s, pioneers of computing and computer science found that the established styles of applied mathematics were unsuitable for the new kinds of problem that were appearing. Unlike the classical applied fields such as differential calculus, linear algebra and statistics, the new kind of mathematics was not conducive to formalism, that is to say, methods of calculation based on manipulation of written symbols. Applied mathematicians found that they needed to adopt a conceptual approach which had previously been mostly confined to pure mathematics and some areas of physics.

In discrete mathematics, as opposed to classical applied mathematics, solutions to problems tend to comparatively unsystematic, though certain fundamental ideas do tend to be used quite frequently. For that reason, the study of discrete mathematics depends heavily on the art of *very clear deductive explanation*, which will be emphasized throughout the course.

The course is intended for students who have little or no previous experience of this kind of mathematics. There are no course prerequisites, in fact, proficiency at formal methods of symbolic manipulation will confer no advantage.

We shall be studying three main areas, separate but with some interactions: (1) graph theory; (2) relations and enumerative combinatorics; (3) coding theory.

Instructor: Laurence Barker, Office SAZ 129, barker ta fen tod bilkent tod edu tod tr.

Assistant: Gökalp Alpan, Office SA-144, gokalp ta fen tod bilkent tod edu tod tr .

Text: R. P. Grimaldi, "Discrete and Combinatorial Mathematics", 5th Ed. (Pearson, 2004).
Some notes will be supplied, on my webpage, for some of the syllabus material.

Classroom location and schedule: All the classes are in room SAZ 04. The times are Wednesdays 09:40 - 10:30 and Fridays 10:40 - 12:30.

Office Hours: In room SA-129 (same building as the classroom), Wednesdays 08:40 - 09:30.

One purpose of Office Hours is to get help with the homework. Usually, homeworks are due in on a Friday. I suggest you do what you can of the homework and then come at ask me for help with the troublesome exercises on Wednesday morning. I will not solve the exercise for you, but I can give you some guidance on how to do it.

Office Hours is not just for the stronger students. If you cannot do the easy questions, and if you do not even understand very much of the course material then, (provided you have at least thought about it and have something to talk about), come and see me during office hours. If you think the best grade you can get is a C, then I will help you get that C. If you are hopelessly lost and heading directly for an F grade, I will not be annoyed about that, because I already know that there are always some students who are hopelessly lost; I cannot be of much help to them if they do not come to see me!

Class Announcements: All students, including any absentees from a class, will be deemed responsible for awareness of announcements made in class.

Revision Aid: Some past exams, with solutions, can be found in discretepastpapers.pdf, on my homepage.

Assessment

Homeworks: The only way to pick up skill at mathematics is through lots of practise.

You may not copy homeworks and you may not do paraphrase rewrites of homeworks by other people. If you break this rule, then you will not catch up in time for the Midterm and Final exams. Just as you cannot learn to swim by watching other people do it, you cannot aquire mathematical skill by just writing out arguments produced by others, not even if you feel that you are "understanding" it as you copy.

You should discuss the homework with each other. In this way, you will teach each other. If you cannot do a homework question, ask another student or ask me during office hours.

Participation: This will be a mark awarded to the whole section for collective academic behaviour and participation. Asking questions is usually very helpful. All communication should be addressed to the whole class. (Making a distracting noise by murmuring to your neighbour is not proper academic behaviour.)

Principle of marking: In mathematics, marks for written work are not awarded according to guesses about what the student might have had in mind. They are awarded according to *how helpful the written explanation would be to other students in the class.*

Grading percentages:

- Quizzes, participation and homework, 15%,
- Midterm 1, 25%, Friday, 11th March, 10:30 - 12:30.
- Midterm 2, 25%, Friday, 8th April, 10:30 - 12:30.
- Final, 35%.

Letter Grades: This is done by the "curve method". A grade C requires an understanding of the concepts and competence at the most routine exam questions. That fulfills the aim of the course: a competent grasp at an introductory level.

Some of the exercises and exam questions will be quite difficult. It has to be that way, not only for the benefit of the strongest students, but also because, without difficult questions, it would be hard to see the purpose of the art of *very clear deductive explanation*. However, students aiming for a grade C need not worry about being unable to do the more difficult questions.

FZ Grades: These will be awarded to students satisfying at least one of the following conditions: (a) Very poor attendance (less than 50% as measured by quizzes); or (b) very poor

Midterm marks (incompetence routine questions); or (c) poor attendance and poor Midterm marks.

Attendance: A minimum of 75% attendance and 50% participation in quizzes is compulsory.

Syllabus

The topics in square brackets, below, are not in the textbook and are not on the examinable syllabus. The format for each item is: week number; Monday date; topics; textbook section numbers.

1: 25 Jan: Outline of course. Examples of problems in discrete mathematics.

2: 1 Feb: Further exercises in discrete mathematics. Argument by contradiction.

3: 8 Feb: Argument by minimal counter-example. The Principle of Mathematical Induction. Argument by mathematical induction, 4.1.

4: 15 Feb: Recursive definitions and various illustrations of mathematical induction, 4.2. Second order recurrence relations as an application of induction, 10.2.

5: 22 Feb: Graphs, sum of degrees formula. Circuits and Trees. 11.1, 11.2, 12.1. Multigraphs and directed multigraphs. Criteria for existence of Euler paths or Euler circuits, proved by mathematical induction, 11.3.

6: 29 Feb: Euler's characteristic formula for planar graphs, proved by mathematical induction. Techniques for proving non-planarity and, in particular, the non-planarity of the graphs K_5 and $K_{3,3}$, 11.4. [Discussion of Four-Colour Map Theorem and proof of a version with 5 colours.]

7: 7 Mar: Revision for Midterm 1, then Midterm 1 on Friday 11 March.

8: 14 Mar: Permutations, combinations, the Binomial theorem, 1.2, 1.3, 1.4.

9: 21 Mar: Sets and correspondences. Functions. Injections, surjections and bijections, 5.1, 5.2, 5.3, 5.6.

10: 28 Mar: Relations. Incidence matrices. Reflexive, irreflexive, symmetric, antisymmetric and transitive relations. Enumeration of relations using incidence matrices, 7.1, 7.2. Partial ordering relations, Hasse diagrams, [Dilworth's Theorem], 7.3.

11: 4 Apr: Revision for Midterm 2. (Date of Midterm 2 not yet confirmed.)

12: 11 Apr: Equivalence relations, 7.4. Stirling numbers of the second kind and enumeration of equivalence relations, 5.3. Inclusion-Exclusion Principle 8.1. [Proof of formula for Stirling numbers using Inclusion-Exclusion Principle.]

13: 18 Apr: Coding theory, Hamming metric, 16.5, 16.6. Hamming bound and Gilbert bound, 16.8.

14: 25 Apr: Parity-check and generator matrics, decoding using syndromes and coset leaders, 16.7, 16.8.

15: 2 May: [Proof of optimality of the decoding method.] Exercises in coding theory.

16: 21 Dec: Exercises in all topics and revision for Final Exam.

Midterm 1 Syllabus

The numberings are chapter and section numbers in Grimaldi.

• Mathematical Induction, 4.1, 4.2.

Test: Do you know what the term *inductive assumption* means? When writing out induction arguments, can you state the inductive assumption? (If not, ask me. Many people find this difficult.)

• Introduction to graph theory, 11.1.

Major result: Letting e be the number of edges, then $2e$ is equal to the sum of the degrees.

• Trees, 12.1.

Definition: A tree is a connected graph with no cycles.
Major result: Given a tree with n vertices and e edges, then $e = n - 1$.

• Euler paths, 11.3.

Euler's Path Theorem: Let r be the number of odd-degree vertices of a connected graph G. Then G has an Euler path if and only if $r = 0$ or $r = 2$. Also, G has an Euler circuit if and only if $r = 0$.

Test: Do you know how to find Euler paths for given graphs?

• Planar graphs, 11.4.

Euler's Characteristic Theorem: Given a connected planar graph G with n vertices and e edges, supposing some planar diagram of G has f faces, then $n - e + f = 2$.

Corollary: Given a connected graph G that is not a tree, and an integer c with $c \geq 3$, supposing that every cycle in G has length at least c, then $e \leq c(n-2)/(c-2)$.

• Second order recurrence relations, 10.1, 10.2.

Assuming $a \neq 0$ and $c \neq 0$, then formula for the solutions to $ax_{n+2} + bx_{n+1} + c = 0$ depends on whether or not the quadratic equation $aX^2 + bX + c = 0$ has a repeated solution. If there are distinct solutions α and β, then there exist A and B such that, for all n, we have $x_n = A\alpha^n + B\beta^n$. If there exists a unique solution, then there exist C and D such that, for all n, we have $x_n = (C + Dn)\alpha^n$.

Midterm 2 Syllabus

• Binomial coefficients, 1.3 (see also 1.1, 1.2).

• Relations 5.1. Injective, surjective, bijective functions. Composition of functions. 5.2, 5.3, 5.6.

• Incidence matrices, especially when used for counting, 7.1, 7.2.

• Equivalence relations, 7.4.

• Graph isomorphism, 11.2.

(Partial ordering relations and Dilworth's Theorem are not on the Midterm 2 syllabus but will appear on the Final syllabus. Embedding of graphs on real 2-manifolds such as the torus, Klein bottle and real projective plane are not on the syllabi of any of the exams.)

Final Syllabus

In the final, all topics on the course are examinable, but the focus will be on the following topics because we studied them after the Midterm 2 exam. The section numbers are from the Grimaldi textbook.

• Stirling numbers 5.3. (Counting arrangements of coloured balls in plain boxes, counting surjections, counting equivalence relations.)

• Partial orderings, Hasse diagrams, 7.3. Dilworth's Theorem (not in textbook).

• Coding theory.
  • Binary codes, Hamming metric 16.5, 16.6. (Highlight: Hamming bound and Gilbert bound in Exercise 16.7.12.)
  • Linear codes, decoding table, efficient decoding using syndromes, 16.7, 16.8.

Special Office Hours for Final Preparation: Wednesday 4 May, 08:40 - 10:30, initially in my Office SA-129, migrating to classroom SA-Z04 when my office becomes too crowded.

Midterm 2 Makeup: Thursday, 28 April, 17:30 - 19:30, SA-Z18.

MATH 210, Finite and Discrete Mathematics
Homeworks and Quizzes, Spring 2016

Laurence Barker, Mathematics Department, Bilkent University,
version: 20 May 2016.

Office Hours: Wednesdays 08:40 - 09:30 room SA-129 (on the first floor of the same building as the classroom). *Office Hours is a good time to ask me for help with the homeworks!*

Homework 1 due Friday 12 February.

1.1: (a) Let x_0, x_1, ... be an infinite sequence of complex numbers such that

$$x_{n+2} - 7x_{n+1} + 12x_n = 0$$

and $x_0 = 2$ and $x_1 = 7$. Show that $x_n = 3^n + 4^n$ for all n.
(b) Let y_0, y_1, ... be an infinite sequence of complex numbers such that

$$y_{n+2} - 7y_{n+1} + 12y_n = 0 \ .$$

Show that there exist complex numbers A and B such that, for all n, we have $y_n = A3^n + B4^n$.

1.2: The following theorem is called Philip Hall's Marriage Theorem. We shall be able to state it in a less recreational way later in the course. Prove the theorem.

Theorem: Suppose that, at a party, there are finitely many boys and girls and that, for any set S of the boys, there are at least $|S|$ girls g such that at least one of the boys in S knows g. Show that it is possible to monogamously marry all the boys in such a way that every boy is married to a girl who he knows.

Quiz 1: *Friday 12 February.* Let n be a positive integer. Show that

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1} \ .$$

One solution: Let A_n and B_n be the left-hand expression and the right-hand expression, respectively. Plainly, $A_1 = B_1$. Now suppose, inductively, that $n \geq 2$ and that $A_{n-1} = B_{n-1}$. We have

$$B_n - B_{n-1} = \frac{n}{n+1} - \frac{n-1}{n} = \frac{n^2 - (n+1)(n-1)}{n(n+1)} = \frac{1}{n(n+1)} = A_n - A_{n-1} \ .$$

Cancelling, we deduce that $A_{n+1} = B_{n+1}$. $\square$

Another solution: We have

$$\frac{1}{1.2} + \frac{1}{2.3} + ... + \frac{1}{n(n+1)} = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + ... + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1} = \frac{n}{n+1} \ . \qquad \square$$

Homework 2, Recurrence relations, due Friday 4 March.

These questions are based on Questions 4, 6, 14, 24 in Section 10.2 of Grimaldi.

Warning: Homework 3 is also due on Friday 4th March because we shall be using the following week for Midterm revision.

2.1: Cars and motorbikes are to be parked in a row of n spaces. The cars are indistinguishable from each other, and each car takes up 2 spaces. The motorbikes are all indistinguishable from each other, and each motorbike takes up 1 space. All n spaces are to be used. How many distinguishable parking arrangements are there? (For example, if $n = 3$, there is one arrangement with 0 cars and 2 arrangements with 1 car, and no other arrangements, hence $1 + 2 = 3$ arrangements in total.)

2.2: Answer the previous question under the change of rules where empty spaces are now allowed.

2.3: Consider an alphabet consisting of 7 numeric characters and k runic characters. For $n \geq 0$, let a_n be the number of words of length n that contain no consecutive runic characters. (That is to say, any two runic characters in the word are separated by at least one numeric character.) Suppose that $a_{n+2} = 7a_{n+1} + 63a_n$ for all $n \geq 0$. What is the value of k?

2.4: Let a_n be the number of ways of covering a $2 \times n$ rectangle using 1×2 rectangles and 2×2 rectangles. Thus, for example, $a_2 = 3$. Find and solve a recurrence relation for a_n.

Homework 3, Graph Theory, due Friday 4 March.

Questions 3.1 and 3.2 are based on Grimaldi 11.3.2 and 11.3.30.

3.1: Consider the connected graphs with 17 edges such that every vertex has degree at least 3. What is the maximum possible number of vertices? (Hint: To complete your proof, you may have to exhibit a graph with the appropriate number of vertices.)

3.2: Ali and Eli attend a party with three other couples. At the party:

- No-one shook hands with his or her spouse.
- No-one shook hands with himself or herself.
- No two people shook hands with each other more than once.

After the party, Eli phoned each of the seven other people to ask how many times he or she had shaken hands. She received seven different answers. How many times did Eli shake hands at the party? How many times did Ali shake hands?

3.3: Let G be a directed multigraph.
(a) A **directed circuit** of G is a circuit of G that traverses every edge in the given direction. A **directed Euler circuit** is a directed circuit that uses every edge exactly once. Show that G has a directed Euler circuit if and only if G is connected and $\mathrm{id}(x) = \mathrm{od}(x)$ for all vertices x of G.
(b) Defining the notions of a **directed path** and a **directed Euler path** similarly, give a similar necessary and sufficient condition for G to have a directed Euler path but no directed Euler circuit.

3.4: For any positive integer n, the n**-cocube** C_n^* is the graph in real n-dimensional space such that: the vertices are the points having the form $z = (z_1, ..., z_n)$ where all of the coordinates z_i

are zero except one of them which is 1 or -1; any two vertices z and z' have an edge between them except when $z = z'$ or $z = -z'$. (Thus, $zandz'$ are adjacent if and only if the distance between them is $\sqrt{2}$. As examples, note that C_2^* is the graph of a square and C_3^* is the graph of an octahedron. For which positive integers n does C_n^* have an Euler circuit?

Quiz 2: *Friday 19 February.* Find a general formula for x_n where $x_0 = 2$ and $x_1 = 3i$ (as usual, $i = \sqrt{-1}$) and $2x_{n+2} - 4ix_{n+1} - 2x_n = 0$ for all natural numbers n.

Solution: The quadratic equation $2X^2 - 4iX - 2 = 0$ has repeated solution $X = i$. So $x_n = (C + nD)i^n$ for some C and D. Putting $n = 0$ yields $C = 2$. Putting $n = 1$ yields $C + D = 3$, hence $D = 1$. Therefore $x_n = (2 + n)i^n$ for all n.

Quiz 3: *Friday 26 February.* Does there exist a graph with 100 vertices such that, for all integers m in the range $0 \leq m \leq 99$, there is a vertex with degree m?

Solution: No, such a graph does not exist. Indeed, consider a graph with 100 vertices. Plainly, some vertex has degree 0, then every vertex has degree at most 98. Also, if some vertex has degree 99, then every vertex has degree at least 1.

Past paper revision questions suitable for Midterm 1

These questions can be found in the file discretepastpapers.pdf on my homepage. That file also has solutions and comments on common mistakes made by candidates who took the exam.

- Page 2 (MATH 132, Fall 2015, Midterm). Quesions 2, 3, 5.
- Page 6 (MATH 132, Fall 2015, Midterm Makeup). Quesions 1, 2, 4, 5.
- Page 2 (MATH 210, Spring 2015, Midterm 1). Quesions 1, 2, 3, 4, 5, 6.
- Page 2 (MATH 110, Fall 2014, Midterm). Quesions 1, 2, 3, 4.

Quiz 4: *Wednesday 23 March.* Complete the proof of the following proposition: *Let n be a natural number and m an integer. Given a finite set S of size $|S| = n$, then the number of subsets $T \subseteq S$ with size $|T| = m$ is* $\binom{n}{m}$. Our proof begins as follows: *Let t be the number of such subsets T. The number of ways of arranging m distinct elements of S in order is ...*

Solution: ... $n(n-1)...(n-m+1) = tm(m-1)...1$. $\square$

Homework 4, Enumerative Combinatorics, due Wednesday 6 April.

4.1: Let W, X, Y, Z be sets. Let $f : X \leftarrow Y$ be a function. Which two of the following three statements are equivalent to each other. (Give a proof to show that they are equivalent, and give a counter-example to show that the other statement is not equivalent.)
(1) For all functions $g, h : Y \leftarrow Z$ satisfying $f_\circ g = f_\circ h$, we have $g = h$.
(2) For all functions $d, e : W \leftarrow X$ satisfying $g_\circ f = h_\circ f$, we have $g = h$.
(3) The function f is injective.

4.2: How many solutions $(x_1, x_2, x_3, x_4, x_5)$ are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 12$ where each x_i is an integer with $x_i \geq 1$?

4.3: Given an integer $n \geq 2$, how many distinguishable ways are there of arranging n mutually distinct objects in a circle if

(a) Two arrangements are the same if one can be obtained from the other by rotation (in other words, the arrangement is determined if we know the left-hand neighbour of each object)?

(b) Two arrangements are the same if one can be obtained from the other by rotation or reflection (in other words, the arrangement is determined if we know the two neighbours of each object, possibly without knowing which neighbour is on the left and which on the right)?

4.4: Consider the graph C_{16} of the 16-dimensional cube. Recall, the vertices of C_{16} are the binary strings of length 16, and two vertices have an edge between then if and only if they differ at exactly one digit. Let x be a vertex of C_{16}. How many vertices y are there such that there is a path of length 6 from x to y?

Equivalent version of the problem: A string of 16 binary digits x is transmitted to us. Suppose we know the received binary string and we also know that, exactly 6 times during transmission, a binary digit was changed. How many possibilities are there for x?

Quiz 5: *Friday 1 April.* Consider the graph represented by a 5×5 square lattice. How many paths of length 8 are there from the bottom left corner to the top right corner?

Bonus part, 3 extra marks: Consider an 8×8 lattice with vertices (x, y) where x, y are integers in the range $1 \leq x \leq 8 \geq y \geq 1$. How many paths of length 8 from $(2, 2)$ to $(6, 6)$ are there if we add edges so that, identifying edges of a surronding square in the evident appropriate ways (depicted in class),

(a) The square becomes a torus? (Both pairs of opposite edges of the square are glued with the same alignment.)

(b) The square becomes a Klein bottle? (Only one of the two pairs of opposite edges are glued with the same alignment.)

(c) The square becomes a real projective plane? (Neither of the pairs of opposite edges are glued with the same alignment.)

Solution: Main part: Each path involves 4 eastward and 4 northward moves. The path, determined by the ordering of those moves, can be expressed as $X_1 ... X_8$ where $X_i = E$ or $X_i = N$ depending on whether the i-th move is due west or north, respectively. Choosing a path amounts to choosing the 4 indices i such that $X_i = E$. So the number of paths is

$$\begin{pmatrix} 8 \\ 4 \end{pmatrix} = \frac{8.7.6.5}{4.3.2.1} = 7.2.5 = 70 \; .$$

Part (a). Using the main part, we see that there are 70 such paths due north-east, 70 due south-east, 70 due north-east and 70 due north-west, thus 280 such paths altogether.

Part (b). There exactly 140 such paths, namely, the 70 paths due north-east and the 70 due south-east. Any other path would have to cross the boundary where the edges are glued in opposite directions. Crossing that boundary an odd number of times changes the parity of the sum of the coordinates, so the boundary must be crossed an even number of times. Plainly, any path from $(2, 2)$ to $(6, 6)$ crossing the boundary at least 2 times must have length greater than 8.

Part (c). The answer is 70. As before, there are 70 paths due north-east. We must show that there are no other paths satisfying the required conditions. Any other such path must cross a glued boundary at least once. Crossing either of the two boundaries changes the parity of the sum of the coordinates. So, again, there must be at least 2 crossings. And, again, that contradicts the requirement that the path has length 8.

Comment: As was made clear in class, the bonus part is ridiculously hard, and was set just for entertainment.

Quiz 6: *Friday 15 April.* On the set $\{000, 001, 010, 011, 100, 101, 110, 111\}$ of binary strings of length 3, we impose the partial ordering $\leq$ such that $x_1x_2x_3 \leq y_1y_2y_3$ provided, for each $i \in \{1, 2, 3\}$, if $x_i = 1$ then $y_i = 1$. Thus, for example, $010 \leq 011 \leq 111$. Draw a Hasse diagram of this poset.

Comment: The most popular answer was diagram was one which looks like a cube, suitably oriented. Another possibility is the Hasse diagram that appeared in lectures for the power set of $\{1, 2, 3\}$ and for the set of positive divisors of 30 partially ordered by divisibility. Indeed, the poset in the question is isomorphic to those two posets.

Quiz 7: *Friday 29 April.* Using the coset leaders and their corresponding syndromes as in the example discussed in the lecture, consider the received word $r = 010101$ and find the associate syndrome σ, coset leader s, codeword c and decoded message word w.

Solution: We have $\sigma = 010$ and $s = 000010$ and $c = r + s = 010111$ and $w = 010$.

Exercises suitable for Final Exam preparation

In the Final Exam, all topics in the course are examinable but the focus will be on the topics introduced after Midterm 2: partial orderings, Stirling numbers, coding theory.

The following six questions, and example solutions, can be found in the file discretepast-papers.pdf on my homepage:

• Questions 1, 2, 3 on page 8, • Questions 1, 2, 3 on page 25.

Practise Final

This will not be marked. We shall discuss the solutions in the Special Office Hours on Wednesday 4 May, 08:40 - 10:30.

Question 1: 40% Consider the coding scheme with parity-check matrix

$$H = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) Write down the generating matrix G and the codewords for each of the 8 message words.
(b) Without writing down the decoding table, explain why the set of coset leaders consists of: the 6-digit binary string with weight 0, all the 6-digit binary strings with weight 1, one 6-digit binary string with weight 2. How many possible choices of the weight 2 coset leader are there? (Recall, the *coset leaders* are the received words that appear in the first column of the decoding table.)
(c) Without writing down the decoding table, decode the words 000001, 000011, 000111, 001111.

(d) How many errors of transmission can be detected? How many errors of transmission can be corrected? Briefly justify your answers.

Question 2: 20% Suppose we have 16 balls such that each ball is labelled with a binary string of length 4 and no two balls are labelled with the same binary string. How many ways are there of putting the balls into 4 indistinguishable boxes such that:

- every box contains at least one ball labelled with a binary string whose first digit is 0.
- every box contains exactly two balls labelled with binary strings whose first digit is 1.

Question 3: 20% How many isomorphism classes of partial orderings are there on a set with size 4?

Question 4: 20% For sets X and Y, a **correspondence from** X **to** Y is defined to be a subset of $X \times Y$. Given a correspondence $\sim$ from X to Y and elements $x \in X$ and $y \in Y$, we write $x \sim y$ when $(x, y) \in \sim$.

Now suppose that X and Y are finite and that, given any subset A of X, there are at least $|A|$ elements b of Y for which there exists an element $a \in A$ satisfying $a \sim b$. Show that there exists an injection $f : X \to Y$ such that, for all $x \in X$, we have $x \sim f(x)$. (Hint: first deal with the case where, given any non-empty subset A of X, there are at least $|A| + 1$ elements b of Y for which there exists an $a \in A$ satisfying $a \sim b$.)

MATH 210: Finite and Discrete Mathematics. Midterm 1

Time allowed: 110 minutes. Please put your name on EVERY sheet of your manuscript. The use of telephones and other electronic devices is prohibited. The use of very faint pencils is prohibited too. You may take the question-sheet home.

Do not forget: Justify your answers. A proof is a very clear deductive explanation. Arguments will be marked according to how clearly and correctly they would communicate with other members of the class.

All graphs are understood to be finite ordinary graphs.

LJB, 11 March 2016, Bilkent University.

1: 10% Show that $1/1^2 + 1/2^2 + 1/3^2 + ... + 1/n^2 \leq 2 - 1/n$ for all positive integers n.

2: 20% Let r be a real number such that $r + 1/r$ is an integer. Show that, $r^n + 1/r^n$ is an integer for all positive integers n. (Hint: consider the expression $(r^n + 1/r^n)(r + 1/r)$.)

3: 20% Let c be a real number. Let x_0, x_1, ... be an infinite sequence of real numbers such that $x_{n+2} + 2x_{n+1} + cx_n = 0$ and $x_0 = -1$ and $x_1 = 10$.
(a) In this part of the question, suppose that $c = -8$. Give a general formula for x_n.
(b) In this part, suppose there exist real numbers C, D, γ such that $x_n = (C + nD)\gamma^n$ for all n. Find the possible values of c, C, D, γ.

4: 30% The graph of the cube C_3, sometimes called the 3-cube, can be described as the graph whose vertices are the binary strings 000, 001, 010, 011, 100, 101, 110, 111 where two vertices are adjacent if and only if they differ at exactly one digit. For instance 010 and 110 are adjacent because the differ only in the left-hand digit. The vertices 010 and 100 are not adjacent because they differ at both the left-hand digit and the middle digit.
(a) Does C_3 have an Euler circuit?
(b) Is C_3 a planar graph?
(c) Let C_4 be the graph described similarly but with binary strings of length 4. Thus, for instance, the vertices 0110 and 1110 are adjacent because they differ at only one digit, but 0110 and 1100 are not adjacent because they differ at 2 digits. Draw a diagram of G and find an Euler circuit. (To specify the Euler circuit, list the vertices in order. Numbering the edges is not recommended. If you draw a nasty little diagram with the edges numbered with teeny tiny numbers, the examiner will not strain his eyesight trying to read it.)
(d) Is C_4 a planar graph?
(e) Let C_8 be the graph described as above, but now with binary strings of length 8 (so that there are now 256 vertices and each vertex has 8 edges). Does C_8 have an Euler circuit?
(f) Is C_8 a planar graph?

5: 20% Let G be a graph with n vertices and $(n-1)(n-2)/2 + 1$ edges. Show that G is connected.

Solutions to Midterm 1, MATH 210, Spring 2016

Solution 1: Let $A_n = 1/1^2 + ... + 1/n^2$ and $B_n = 2 - 1/n$. We are to show that $A_n \leq B_n$ for all positive integers n. The inequality holds when $n = 1$. because $A_1 = 1 = B_n$. Now suppose that $n \geq 2$ and that $A_{n-1} \leq B_{n-1}$. We have

$$B_n - B_{n-1} = 1/(n-1) - 1/n = n/n(n-1) - (n-1)/n(n-1) = 1/n(n-1) > 1/n^2 = A_n - A_{n-1}$$

Hence, $B_n - A_n > B_{n-1} - A_{n-1} \geq 0$, as required. □

Comment: The inductive hypothesis, above, is that $n \geq 2$ and $A_{n-1} \leq B_{n-1}$.

Alternative Solution 1: When $n \geq 2$, we have $1/n^2 < 1/n(n-1) = 1/(n-1) - 1/n$, hence

$$1/1^2 + 1/2^2 + 1/3^2 + 1/4^2 + ... + 1/n^2 \leq$$

$$1 + (1/1 - 1/2) + (1/2 - 1/3) + (1/3 - 1/4) + ... + (1/(n-1) - 1/n) = 2 - 1/n\ . \qquad \square$$

Solution 2: We shall show, in fact, that $r^n + 1/r^n$ is an integer for all natural numbers n. The cases $n = 0$ and $n = 1$ are trivial. Now suppose, inductively, that $n \geq 1$ and that $r^n + 1/r^n$ and $r^{n-1} + 1/r^{n-1}$ are integers. The real number

$$(r^n + 1/r^n)(r + 1/r) = r^{n+1} + 1/r^{n+1} + r^{n-1} - 1/r^{n-1}$$

is an integers, hence $r^{n+1} + 1/r^{n+1}$ is an integer. □

Comment: The inductive hypothesis, here, is that $n \geq 1$ and $r^n + 1/r^n \in \mathbb{Z} \ni r^{n-1} + 1/r_{n-1}$.

Alternative Solution 2: The cases $n = 1$ and $n = 2$ are trivial. When $n \geq 2$, we have

$$(r + r^{-1})^n = r^n + r^{-n} + \left[\binom{n}{1} (r^{n-2} + r^{2-n}) + ... \right]$$

where the last term in the square brackets is $\binom{n}{m}(r + r^{-1})$ or $\binom{n}{m}$ when $n = 2m + 1$ or $n = 2m$, respectively. Assuming, inductively, that $r^k + r^{-k}$ is an integer for all $1 \leq k < n$, then the expression in square brackets is an integer, hence $r^n + r^{-n}$ is an integer.

Solution 3: Part (a). The recurrence relation $x_{n+2} + 2x_{n+1} - 8x_n$ has auxilliary quadratic equation $X^2 + 2X - 8 = 0$, which has distinct solutions $X = (-2 \pm \sqrt{36})/2$, in other words, $X = 2$ or $X = -4$. So $x_n = A2^n + B(-4)^n$ for some A and B. We have $-1 = x_0 = A + B$ and $10 = x_1 = 2A - 4B$, hence $A = 1$ and $B = -2$. Therefore, $x_n = 2^n - 2(-4)^n$.

Part (b). We shall show that $(80, -1, 0, -1)$ and (1, -1, -9, -1) are the only solutions for (c, C, D, γ). First suppose there exists a solution where $D = 0$. Putting $n = 0$, we deduce that $C = -1$. Putting $n = 1$, we deduce that $\gamma = -10$. It follows that -10 must be one of the solutions to $X^2 + 2X + c = 0$. By considering the coefficient of X, we deduce that the quadratic equation has one other solution, namely 8. The constant coefficient is the product of the two solutions, $c = 80$. Plainly, $(80, -1, 0, -1)$ is a solution.

Now suppose there exists a solution with $D \neq 0$. The discriminant of the quadratic equation $X^2 + 2X + c = 0$ is $\Delta = 4(1 - c)$. The form of the expression for x_n implies that $\Delta = 0$, in other words, $c = 1$. Then the unique solution to the quadratic equation is $X = -1$. We deduce that $\gamma = -1$. It follows that $x_n = (C + nD)(-1)^n$ for all $n \in \mathbb{N}$. Since $-1 = x_0 = C$ and $10 = x_1 = (C + D)(-1)$, we have $D = -9$. Conversely, it is clear that $(1, -1, -9, -1)$ is a solution.

Solution 4: Part (a). No, because C_3 has 8 vertices with odd degree.

Part (b). Yes. A planar diagram of C_3 was presented in class. (I omit it here because diagrams take a long time to draw in TeX.)

Part (c). This was done in class. (I omit it here because diagrams take a long time.)

Part (d). No, C_4 is non-planar. To see this, suppose otherwise. Since all cycles have length at least 4, we have $e \leq c(n-2)/(c-2)$ where e is the number of edges, n is the number of vertices and $c = 4$. But $n = 16$ and, since every vertex has degree 4, we have $2e = 4.16$ hence $e = 32$. We deduce that $32 \leq 4(16-2)/2 = 28$, which is a contradiction, as required.

Part (e). Yes, because every vertex of the connected graph C_8 has degree 8, which is even.

Part (f). No, by part (d) and the observation that C_8 has a copy of C_4 as a subgraph.

Solution 5: Let H be a disconnected graph with n vertices and a maximal number of edges e. Then H must be isomorphic to the disjoint union of the complete graphs K_a and K_b for some positive integers a and b with $a + b = n$. Then

$$2e = a(a-1) + b(b-1) = a^2 - a + (n-a)^2 - (n-a) = n(n-1) - 2a(n-a) = n(n-1) - 2ab\,.$$

Without loss of generality, $a \leq b$, hence

$$(a-1)(b+1) = ab + a - b - 1 \leq ab - 1 < ab\,.$$

Therefore, noting that a is a positive integer, the maximum value of e is achieved when $a = 1$. In that case, $e = (n-1)(n-2)/2$. We have shown that, among disconnected graphs with n vertices, the maximum possible number of edges is $(n-1)(n-2)/2$ (achieved only when the graph is the disjoint union of K_1 and K_{n-1}). The required conclusion follows. □

Alternative Solution 5: We may assume that $n \geq 3$ and that G has no vertex of degree $n-1$, because otherwise the required conclusion is trivial. Since $n(n-3)/2 < (n-1)(n-2)/2$, some vertex x of G must have degree $n-2$. Let y be the unique vertex of G not adjacent to x. All the vertices distinct from y lie in the same component as x. But those vertices have at most $(n-1)(n-2)/2$ edges between them. So $d(y) \geq 1$ and it follows that y lies in the same component as all the other vertices. □

MATH 210: Finite and Discrete Mathematics. Midterm 2

Time allowed: 110 minutes. Please put your name on EVERY sheet of your manuscript. The use of telephones and other electronic devices is prohibited. The use of very faint pencils is prohibited too. You may take the question-sheet home.

LJB, 8 April 2016, Bilkent University.

1: 24% How many integer solutions x_1, x_2, x_3, x_4 are there when:
(a) $x_1 + x_2 + x_3 + x_4 = 5$ and $-1 \leq x_i$ for each i?
(b) $x_1 + x_2 + x_3 + x_4 = 5$ and $-1 \leq x_i \leq 3$ for each i?
(c) $x_1 + x_2 + x_3 + x_4 = 6$ and $-1 \leq x_i \leq 3$ for each i?

2: 25% Which of the following statements are true for all functions $f : Y \leftarrow X$ and $g : X \leftarrow Y$? In each case, give a short proof or a counter-example. (Warning: In the cases where proofs are needed, there will be no marks for examples and no marks for diagrams. In the cases where counter-examples are needed, just give one simple counter-example, and do not try to give a general description of how the statement can fail.)
(a) If $g_\circ f$ is bijective, then g is surjective.
(b) If $g_\circ f$ is bijective, then g is injective.
(c) If $g_\circ f$ is bijective, then f is surjective.
(d) If $g_\circ f$ is bijective, then f is injective.
(e) If $g_\circ f_\circ g_\circ f$ is bijective, then $g_\circ f$ is bijective.

3: 16% Show that $\begin{pmatrix} n+1 \\ r+1 \end{pmatrix} = \sum_{m=r}^{n} \begin{pmatrix} m \\ r \end{pmatrix}$ for integers $0 \leq r \leq n$. (Hint: Interpret both sides of the equation conceptually.)

4: 10% How many isomorphism classes of graphs are there with:
(a) 4 vertices and 4 edges?
(b) 5 vertices and 5 edges?

5: 25% Let X be a set with size $|X| = 4$. Let S be the set of relations $\sim$ on X such that $\sim$ is symmetric and there are exactly 8 elements (x, x') of $X \times X$ satisfying $x \sim x'$. We define a relation $\cong$ on S such that, given elements $\sim_1$ and $\sim_2$ of S, then $\sim_1 \cong \sim_2$ provided there exists a bijection $f : X \to X$ with the property that, for all $x, x' \in X$ satisfying $x \sim_1 x'$, we have $f(x) \sim_2 f(x')$. When that condition on $\sim_1$ and $\sim_2$ holds, we say that $\sim_1$ is *isomorphic* to $\sim_2$. We call $\cong$ the *isomorphism* relation on S.
(a) Suppose that $\sim_1 \cong \sim_2$. Let $f : X \to X$ be a bijection with the property specified above. Show that, given $x, x' \in X$, then $x \sim x'$ if and only if $f(x) \sim f(x')$.
(b) Show that $\cong$ is an equivalence relation.
(c) Find the value of $|S|$.
(d) An equivalence class under $\cong$ is called an **isomorphism class**. How many isomorphism classes are there? (Hint: Invent a way of representing isomorphism classes using diagrams rather like diagrams of graphs. Note that this problem has a connection with part (a) of Question 4.)
(e) For each isomorphism class in S, find the number of elements of S belonging to that isomorphism class. Check that the sizes of the isomorphism classes add up to your answer to part (c).

<u>**Solutions to Midterm 2**, MATH 210, Spring 2016</u>

Solution 1: Part (a). Substituting $y_i = x_i + 1$, the answer is equal to the number of natural number solutions y_1, y_2, y_3, y_4 to the equation $y_1 + y_2 + y_3 + y_4 = 9$. This is the number of ways of putting 9 plain balls into 4 coloured boxes, $\binom{12}{9} = 12.11.10/3.2 = 220$.

Part (b). The condition $x_i \leq 3$ is equivalent to the condition $y_i \leq 4$. If that condition fails, then $y_i \geq 5$ for exactly one of the 4 possible values of i. For each i, the number of solutions in part (a) with $y_i \geq 5$ is equal to the number of natural number solutions to $z_1 + z_2 + z_3 + z_4 = 4$, which is $\binom{7}{4} = 7.6.5/3.2 = 35$. So the answer is $220 - 4.35 = 220 - 140 = 80$.

Part (c). Making the same substitution as before, we are to find the number of natural number solutions to $y_1 + ... + y_4 = 10$ satisfying $y_i \leq 4$. Arguing as in part (a), the number of solutions without the constraint $y_i \leq 4$ is $\binom{13}{10} = 13.12.11\ 3.2 = 13.22 = 286$. Meanwhile, arguing as in part (b), we see that, for each i, the number of solutions satisfying $y_i \geq 5$ is $\binom{8}{5} = 8.7.6/3.2 = 56$. But there are 6 solutions where $y_i = 5$ for two values of i and $y_i = 0$ for the other two values of i. Therefore, the answer is $286 - 4.56 + 6 = 292 - 224 = 68$.

Alternative Solution: The following approach can be applied to all three parts of the question, but let us just demonstrate it for part (c). Consider the natural number solutions $(y_1, ..., y_4)$ satisfying $y_1 + ... + y_4 = 10$ and $y_i \leq 4$ for all i. The solutions which also satisfy the ordering condition $y_1 \geq y_2 \geq y_3 \geq 4$ are:

$$(4,4,2,0),\ (4,4,1,1),\ (4,3,3,0)\ ,(4,3,2,1),\ (4,2,2,2),\ (3,3,3,1)\ ,(3,3,2,2)\ .$$

Now dropping the ordering condition, thos 6 solutions can be rearranged in 12, 6, 12, 24, 4, 4, 6 ways, respectively. So the answer is $12 + 6 + 12 + 24 + 4 + 4 + 6 = 68$.

Solution 2: Part (a), true. Indeed, given $x \in X$ then, since $g \circ f$ is bijective, we have $x = g(f(z))$ for some $z \in X$. Putting $y = f(x)$, we deduce that, for all x, we have $x = f(y)$ for some $y \in Y$.

Part (b), false. Any case where $|X| = 1$ and $|Y| = 2$ is a counter-example.

Part (c), false. Any case as in part (b) is a counter-example.

Part (d), true. Indeed, given distinct elements $x, x' \in X$, then $g(f(x)) \neq g(f(x'))$ because $g \circ f$ is injective. Therefore $f(x) \neq f(x')$.

Part (e), true. This follows by applying parts (a) and (d), with g and f both replaced by $g \circ f$.

Comment: "All ravens are black." If you think that this is *true*, then you must give a general *proof*: "Let R be any raven. ...". If you think it is *false*, then it is enough to give a single *counter-example*: "Let R be the following raven. ..."

In such "proof or counter-example" questions, do not write a long equivocal essay full of deep wisdom. Simply *say what the answer is* and then *justify the answer with a proof or a counter-example*.

The solutions to Questions 3, 4, 5 were discussed in class.

MATH 210: Finite and Discrete Mathematics. Makeup for Midterm 2

Time allowed: 110 minutes. Please put your name on EVERY sheet of your manuscript. The use of telephones and other electronic devices is prohibited. The use of very faint pencils is prohibited too. You may take the question-sheet home.

LJB, 28 April 2016, Bilkent University.

1: 10% How many integer solutions x_1, x_2, x_3, x_4 are there to the equation

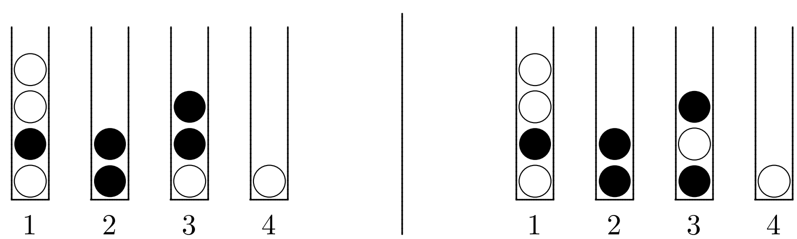
$$x_1 + x_2 + x_3 + x_4 = 18$$

where $x_i \geq i$ for all $i \in \{1, 2, 3, 4\}$?

2: 15% Show that, given integers m, n, k satisfying $1 \leq m \leq n \geq k \geq 1$, then

$$\sum_{j=0}^{k} \binom{m}{j} \binom{n-m}{k-j} = \binom{n}{k} .$$

3: 30% Suppose we have 5 white tennis-balls which are indistinguishable from each other, 5 black tennis-balls which are indistinguishable from each other, 4 tennis-ball tubes numbered from 1 to 4. Suppose also that each tube can hold a maximum of 8 balls. When we place balls in a tube, the balls have an ordering: the first ball put in is the ball at the bottom, the next ball put in is the ball immediately above that, and so on. How many distinguishable ways are there of placing all the balls in the tubes? (Note that we can distinguish between the two arrangements illustrated below; in the left-hand arrangement, the first ball placed in box 3 is white, whereas, in the right-hand arrangement, the first ball placed in box 3 is black.)



4: 20% How many isomorphism classes of graphs G are there:
(a) when G has 7 vertices, G is connected and every vertex of G has degree 2?
(b) when G has 7 vertices and every vertex of G has degree 0 or 1 or 2?

5: 25% Let n be a positive integer and $X = \{1, 2, ..., n\}$. Let F be the set of functions $X \to X$. We define a relation $\equiv$ on F such that, given $f, g \in F$, then $f \equiv g$ if and only if there exists an integer c with the property that, for all $x \in X$, we have $f(x) - g(x) = c$. (In other words, $f \equiv g$ if and only if $f - g$ is a constant function.)
(a) Show that $\equiv$ is an equivalence relation.
(b) For $1 \leq m \leq n$, let $a_{n,m}$ be the number of functions $f \in F$ such that, as x runs over the elements of X, the minimum value of $f(x)$ is 1 and the maximum value of $f(x)$ is m. Find a recurrence relation expressing $a_{n,m}$ in terms of $a_{n,1}$, $a_{n,2}$, ..., $a_{n,m-1}$.
(c) Supposing that $n = 3$, how many equivalence classes does $\equiv$ have? (Hint: Use part (b).)
(d) Supposing that $n = 4$, how many equivalence classes does $\equiv$ have?

Solutions to Makeup for Midterm 2, MATH 210, Spring 2016

This exam was unusually difficult because the Midterm 2 exam was unusually difficult.

Solution 1: Substituting $y_i = x_i - i$, the answer is equal to the number of natural number solutions y_1, ..., y_4 to $y_1 + y_2 + y_3 + y_4 = 8$. Applying a standard formula, we obtain the answer

$$\binom{8+3}{3} = \frac{11.10.9}{3.2} = 11.5.3 = 165 \; .$$

Solution 2: Let N be a set of size $|N| = n$ and let $M \subseteq N$ with $|M| = m$. The right-hand side of the required equation is the number of subsets S of N such that $|S| = k$. To choose such S, we might first choose an integer j in the range $0 \leq j \leq k$. Then, when possible, we might choose subsets $S_1 \subseteq M$ and $S_2 \subseteq N - M$ with $|S_1| = j$ and $|S_2| = k - j$. Thereupon, we put $S = S_1 \cup S_2$. Interpreting degenerate binomial coefficients as 0, the numbers of choices for S_1 and S_2 are $\binom{m}{j}$ and $\binom{n-m}{k}$, respectively.

Solution 3: Let us define a **generalized arrangement** to be an arrangement of the balls in the tubes, placed in order, but without any constraint on the maximum number of balls in a tube. Let us define an **illegal arrangement** to be a generalized arrangement with 9 or 10 balls in some tube. The number of valid arrangements is the number a of generalized arrangements minus the number b of illegal arrangements.

Adapting a proof of the formula applying to balls of a single colour, we see that a is the number of strings of length 13 such that, respectively, 5, 5, 3 of the symbols are W, B, I (standing for white ball, black ball, divider between tubes). Thus, for example, the string associated with the arrangement depicted in the left-hand diagram of the question-sheet is $WBWWIBBIWBBIW$. We see that $a = 13!/5!\,5!\,3! = 13.11.9.8.7 = 8(10+3)(10-3)(10+1)(10-1) = 8(100-9)(100-1) = 8(10000-900-100+9) = 8.9009 = 72072$.

The number of illegal arrangements with 10 balls in tube 1 is

$$\binom{10}{5} = \frac{10.9.8.7.6}{5.4.3.2} = 9.7.4 = 252 \; .$$

So the number of illegal arrangements with 10 balls in some tube is 4.252. The number of illegal arrangements with 9 balls in tube 1 and 1 ball is tube 2 is, again, 252. To choose an illegal arrangement with 9 balls in some tube, there are 4 choices for the tube with 9 bals and 3 choices for the tube with 1 ball. So the number of illegal arrangements with 9 balls in some tube is $4.3.252 = 12.252$. Therefore $b = (4+12).252 = 16(252+4-4) = 2^{12} - 2^6 = 4096 - 64 = 4032$. Finally, the number of valid arrangements is $a - b = 68040$.

Solution 4: Part (a). There is exactly 1 isomorphism class, namely, that of the graphs represented by a 7-sided polygon.

Part (b). For a positive integer n, let $f(n)$ be the number of isomorphism classes of connected graphs that have n vertices and no vertices with degree greater than 2. We have $f(1) = f(2) = 1$. For $n \geq 3$, we have $f(n) = 2$, one of the isomorphism classes being represented by an n-sided polygon, the other being represented by a line with a vertex at each end-point.

Consider the graphs G satisfying the condition in the question. We associate each G with a partition of 7, we mean to say, a finite sequence $n_1, ..., n_r$ of positive integers such that $n_1 \geq ... \geq n_r$ with sum $n_1 + ... + n_r = 7$. We take r to be the number of components of G and take each n_i to be the size of the i-th largest component. Let us write $n_1 n_2 ... n_r$ in place of $n_1, ..., n_r$. Fixing a partition $n_1 ... n_r$ of 7, it is not hard to see that, if no positive integer greater than 2 appears twice in the partition, then the number of possible choices for the isomorphism class of G is $f(n_1)...f(n_2)$. But the number of isomorphism classes for the partition 331 is 3, the possible diagrams being: two triangles and a dot; a triangle, a line and a dot; two lines and a dot. The 15 partitions of 7 are

$$7\,,\ 61\,,\ 52\,,\ 511\,,\ 43\,;\quad 421\,,\ 4111\,,\ 331\,,\ 322\,,\ 3211\,;\quad 31111\,,\ 2221\,,\ 22111\,,\ 211111\,,\ 1111111\,.$$

The corresponding numbers of choices of G, up to isomorphism, are

$$2\,,\ 2\,,\ 2\,,\ 2\,,\ 4\,;\quad 2\,,\ 2\,,\ 3\,,\ 2\,,\ 2\,;\quad 2\,,\ 1\,,\ 1\,,\ 1\,,\ 1\,.$$

Summing those 15 numbers, we deduce that the number of possibilities for G is 29.

Solution 5: Part (a). Plainly, $\equiv$ is reflexive and symmetric. For transitivity, let $f, g, h \in F$ such that $f \equiv g$ and $g \equiv h$. Then there exist integers c and d such that, for all $x \in X$, we have $f(x) - g(x) = c$ and $g(x) - h(x) = d$. It follows that $f(x) - h(x) = f(x) - g(x) + g(x) - h(x) = c + d$. Therefore $f \equiv h$.

Part (b). The initial condition associated with the recurrence relation is $a_{n,1} = 1$. Let $n \geq m \geq 2$. Given $f \in F$, let $\max(f)$ and $\min(f)$ denote the maximum and minimum values of f, respectively. Then

$$|\{f \in F : \max(f) \leq m\}| = m^n\ .$$

On the other hand, $|\{f \in F : 1 = \min(f),\ \max(f) = k\}| = a_{n,k}$ for $1 \leq k \leq m - 1$, hence

$$|\{f \in F : \max(f) - \min(f) = k,\ \max(f) \leq m\}| = (m - k + 1)a_{n,k}\ .$$

Thus we obtain the recurrence relation

$$a_{n,m} = |\{f \in F : 1 = \min(f),\ \max(f) = m\}| = m^n - \big(m a_{n,1} + (m-1)a_{n,2} + ... + 2a_{n,m-1}\big)\ .$$

Part (c). For arbitrary $n \geq m \geq 1$, consider the functions $f \in F$ satisfying $\max(f) - \min(f) = m$. There are $(n - m + 1)a_{n,m}$ such functions, and the equivalence classes of such functions all have size $n - m + 1$. So there are $a_{n,m}$ equivalence classes of such functions. We deduce that the total number of equivalence classes of $\equiv$ is $a_{n,1} + a_{n,2} + ... + a_{n,n}$.

Putting $n = 3$, we have $a_{3,1} = 1$ and $a_{3,2} = 3^2 - 2a_{3,1} = 9 - 2 = 6$ and $a_{3,3} = 3^3 - (3a_{3,1} + 2a_{3,2}) = 27 - (3 + 12) = 12$, hence the number of equivalence classes is $1 + 6 + 12 = 19$.

Part (d). Now putting $n = 4$, we have $a_{4,1} = 1$ and $a_{4,2} = 4^2 - 2a_{4,1} = 16 - 2 = 14$ and $a_{4,3} = 4^3 - (3a_{4,1} + 2a_{4,2}) = 64 - (3 + 28) = 50$ and

$$a_{4,4} = 4^4 - (4a_{4,1} + 3a_{4,2} + 2a_{4,3}) = 256 - (4 + 42 + 100) = 110\ .$$

So the number of equivalence classes is $1 + 14 + 50 + 110 = 175$.

MATH 210: Finite and Discrete Mathematics. Final

Time allowed: two hours. Please put your name on EVERY sheet of your manuscript. The use of telephones and other electronic devices is prohibited. The use of very faint pencils is prohibited too. You may take the question-sheet home.

LJB, 14 May 2016, Bilkent University.

Notation and terminology: Recall that, for a positive integer n, we let $\mathbb{Z}_2^n$ denote the set of binary strings of length n. As usual, we define a **code** in $\mathbb{Z}_2^n$ to be a non-empty subset of $\mathbb{Z}_2^n$.

1: 10% Consider the set $\mathbb{Z}_2^2 = \{00, 01, 10, 11\}$. How many codes are there in $\mathbb{Z}_2^2$? How many linear codes are there in $\mathbb{Z}_2^2$?

2: 40% Consider 7 balls and 4 boxes. How many distinguishable arrangements of the balls in the boxes are there if:

(a) The balls are coloured (distinguishable) and the boxes are coloured?

(b) The balls are coloured, the boxes are coloured and no box is empty?

(c) The balls are plain (indistinguishable) and the boxes are coloured?

(d) The balls are plain, the boxes are coloured and no box is empty?

(e) The balls are coloured and the boxes are plain?

(f) The balls are coloured, the boxes are plain and no box is empty?

(g) The balls are plain and the boxes are plain?

(h) The balls are plain, the boxes are plain and no box is empty?

3: 15% Consider the posets P such that $|P| = 4$ and, given any three distinct elements of P, then at least one of them is greater than at least one of the others. (In other words, for distinct $x_1, x_2, x_3 \in P$, there exist distinct $i, j \in \{1, 2, 3\}$ such that $x_i > x_j$). Up to isomorphism, how many such posets P are there? Draw Hasse diagrams for each isomorphism class.

4: 35% Consider the linear encoding scheme $\mathbb{Z}_2^4 \to \mathbb{Z}_2^7$ with parity-check matrix

$$H = \left[\begin{array}{cccc|ccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] .$$

(a) Find the generating matrix. Encode the three message words 1010, 0101, 1111.

(b) Explain why the set of coset leaders is precisely the subset of $\mathbb{Z}_2^7$ consisting of the words of weight 0 or 1. Find the syndromes for each of those 8 coset leaders.

(c) Decode the three received words 1010101, 0101010, 1111000.

(d) What is the maximum number of single-digit errors of transmission such that a received word can always be correctly decoded?

(e) What is the rate of this code?

Solutions to Final, MATH 210, Spring 2016

1: The number of codes in $\mathbb{Z}_2^2$ is the number of non-empty subsets of a set of size 4. That number is $2^4 - 1 = 15$.

The linear codes in $\mathbb{Z}_2^n$ are the non-empty subsets that are closed under addition. Recall, any linear code must contain the zero vector. Observing that the sum of any two distinct non-zero elements of $\mathbb{Z}_2^n$ is the third non-zero element, we see that the linear codes are $\{00\}$ and $\{00, 01\}$ and $\{00, 10\}$ and $\{00, 11\}$ and $\mathbb{Z}_2^2$. In particular, the number of such codes is 5.

2: Part (a), $4^7 = 1024.16 = 16384$.

Part (b). The Stirling numbers $S(m, n)$, for $m \geq n$, are given by the recurrence relation

$$S(m+1, n) = S(m, n-1) + n\,S(m, n)\ .$$

when $m > n > 1$, together with the initial conditions $S(1, n) = S(n, n) = 1$. We now see that the answer is $4!\,S(7, 4) = 24.350 = 8400$.

$S(m,n)$		n = 1	2	3	4
m	1	1			
	2	1	1		
	3	1	3	1	
	4	1	7	6	1
	5	1	15	25	10
	6	1	31	90	65
	7	1	63	301	350

Part (c), $\begin{pmatrix} 7+3 \\ 3 \end{pmatrix} = \dfrac{10.9.8}{3.2} = 120.$

Part (d). After placing one ball in each box, the number of ways of arranging the remaining balls is $\begin{pmatrix} 3+3 \\ 3 \end{pmatrix} = \dfrac{6.5.4}{3.2} = 20.$

Part (e). Using the above table again, we see that the number of arrangements with exactly 4 and 3 and 2 and 1 box non-empty is, respectively, $S(7, 4) = 350$ and $S(7, 3) = 301$ and $S(7, 2) = 63$ and $S(7, 1) = 1$. The total number of arrangements is $350+301+63+1 = 715$.

Part (f), from the calculation in the previous part, we obtain the answer $S(7, 4) = 350$.

Part (g). Each arrangement can be represented by a string $n_1n_2n_3n_4$ where n_1, n_2, n_3, n_4 are the numbers of balls in the boxs and $n_1 \geq n_2 \geq n_3 \geq n_4$. The strings representing the possible arrangements are

$$7000\,,\ 6100\,,\ 5200\,,\ 5110\,,\ 4300\,,\ 4210\,,\ 4111\,,\ 3310\,,\ 3220\,,\ 3211\,,\ 2221\,.$$

In particular, the number of arrangements is 11.

Part (h). The answer is 3, since this is the number of strings in part (g) with $n_4 \neq 0$.

3: We are to count, up to isomorphism, the posets P of size 4 that have width 1 or 2. The unique P with width 1 is also the unique P with height 3. There are exactly 2 cases where P has a disconnected Hasse diagram. By considering ways of extending a chain of size 3, we see that there are exactly 5 cases where P has height 3 and the Hasse diagrams are connected. Finally, there are exactly 2 cases where P has height 2 and the Hasse diagram is connected. So the number of isomorphism classes is $1 + 2 + 5 + 2 = 10$. (We omit the Hasse diagrams because they are laborious to code.)

4: Part (a). The generating matrix is $G = \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$ and $\begin{array}{r} 1010.G = 1010010, \\ 0101.G = 0101101, \\ 1111.G = 1111111. \end{array}$

Part (b). The syndromes of the 8 received words

0000000 , 0000001 , 0000010 , 0000100 , 0001000 , 0010000 , 0100000 , 1000000

are, respectively, 000, 001, 010, 100, 110, 101, 011, 111. Since the syndromes are mutually distinct, each row of any decoding table must include exactly one of those 8 received words. Plainly, for each row, all 15 of the other received words are of greater weight.

Part (c). The following table shows the received words r and the corresponding syndrome σ, the coset leader s, the codeword $c = r + s$ and the decoded message word w.

r	σ	s	c	w
1010101	111	1000000	0010101	0010
0101010	111	1000000	1101010	1101
1111000	111	1000000	0111000	0111

Part (d). The smallest weight of the sum of 1 or 2 or 3 or 4 distinct rows of G is, respectively, 3 or 3 or 4 or 7. So the smallest weight of a non-zero codeword is 3. Since the code is linear, the minimal distance between distinct codewords is 3. For correction, therefore, the maximum number of transmission errors is 1.

Part (e). The rate is 4/7.