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MATH 210, Finite and Discrete Mathematics

Bilkent University, Spring 2015, Laurence Barker

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MATH 210, Finite and Discrete Mathematics, Spring 2015

Course specification

Laurence Barker, Mathematics Department, Bilkent University,
version: 10 April 2015.

Course Aims: To introduce some concepts and techniques of finite, discrete and combinatorial mathematics. To reinforce skill at mathematical proof in areas of mathematics that are low in structure and routine methods, consequently high in the demand for creative thinking and clear reasoning.

Course Description: Discrete mathematics is an umbrella name for all areas of applicable mathematics where there is not much structure to work with. Although it is very diverse, certain kinds of technique tend to crop up frequently. We shall be studying three areas, superficially quite separate, but similar in style and technique: the first third will focus largely on graph theory; the middle third on enumerative study of relations; the last third on coding theory.

Course Requirements: There are no technical prerequisites, but there is an attitude prerequisite. Substantial mathematics cannot be learned just by listening. To take in the concepts and techniques, you must study the sources and do plenty of exercises.

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Assistant: Gökalp Alpan
e-mail: gokalp at fen dot bilkent dot edu dot tr.

Course Text: R. P. Grimaldi, "Discrete and Combinatorial Mathematics", 5th Ed. (Pearson, 2004).

Classes: Wednesdays 09:40 - 10:30 SA Z04, Fridays, 10:40 - 12:30 SA Z04.

Office Hours: Wednesdays, 08:40 - 09:30, SA-129.

Please note, Office Hours is not just for the strong students. I will have no sympathy for drowning students who do not come to me for help. Students who are having serious difficulties must come to Office Hours to discuss the mathematics. In fact, I need those students to come, so as to ensure that the classroom material does not lose touch with parts of the audience.

Syllabus:

Week number: Monday date: Subtopics. Section numbers

1: 2 Feb: Illustrations of problems in discrete mathematics. Mathematical induction, 4.1. Second order recurrence relations, 10.2

2: 9 Feb: Graphs 11.1. Trees, 11.2. Criteria for the existence of Euler paths or Euler circuits, 11.3.

3: 16 Feb: Euler's characteristic formula for planar graphs. Degrees of vertices of planar graphs, 11.4

4: 23 Feb: The non-planarity of the graphs K_5 and $K_{3,3}$, 11.4

5: 2 Mar: Review for Midterm 1.

6: 9 Mar: Binomial coefficients, 1.2, 1.3, 1.4. Midterm 1 on Friday 13 March.

7: 16 Mar: Sets and correspondences. Injective, surjective and bijective functions, 5.1, 5.2, 5.3, 5.6. Relations 7.1.

8: 23 Mar: Incidence matrices, enumeration of relations, 7.2. Partial orderings, 7.3. Equivalence relations, 7.4.

9: 30 Mar: Stirling numbers of the second kind, 5.3, 7.4.

10: 6 Apr: Isomorphism of binary relations, graphs, posets, 7.3, 11.2

11: 13 Apr: Review for Midterm 2. Midterm 2 on Friday 17 April.

12: 20 Apr: Coding theory, Hamming metric, parity-check and generator matrics, 16.5, 16.6, 16.7.

13: 27 Apr: Encoding and decoding linear codes using coset leaders, 16.8.

14: 4 May: Logic gates. Half-adder and full-adder, 15.2.

15: 11 May: Review for Final.

Assessment:

- Quizzes, Homework and Participation 15%.
- Midterm 1, 25%, Friday 13 March.
- Midterm 2, 25%, Friday 17 April.
- Final, 35%.

75% attendance is compulsory.

Class Announcements: All students, including any absentees from a class, will be deemed responsible for awareness of class announcements.

MATH 210, Finite and Discrete Mathematics

Homeworks and Quizzes, Spring 2015

Laurence Barker, Mathematics Department, Bilkent University,
version: 8 May 2015.

Office Hours: Wednesdays, 08:40 - 09:30, SAZ 129.

Office Hours would be a good time to ask me for help with the homeworks.

The course textbook is R. P. Grimaldi, "Discrete and Combinatorial Mathematics" 5th edition (Pearson, 2004). Many of the questions are taken from there. The solutions are to be discussed in class.

Quiz 1: *Friday, 13th February.* A graph has 32 vertices and every vertex has degree 4. How many edges are there?

Solution: Let e be the number of edges. Then $2e$ is the sum of the degrees of the vertices, which is 32.4. So $e = 32.4/2 = 64$.

Comment: Drawing an example and counting the edges does not suffice. You need to prove the conclusion for *all* graphs satisfying the assumptions. One graph satisfying the conditions is the 4×4 toroidal lattice, but it is not enough just to count the edges in that case. Another graph satisfying the conditions is the 4-dimensional cube.

Quiz 2: *Wednesday 25th February.* Let T be a tree with at least 3 vertices with degree 1. Prove that T has a vertex with degree at least 3.

Solution: Let n and e be the number of vertices and edges, respectively. We can number the vertices x_1, ..., x_n such that $d(x_{n-2}) = d(x_{n-1}) = d(x_n) = 1$. Since $e = n - 1$, we have

$$2n - 2 = 2e = d(x_1) + ... + d(x_n) = d(x_1) + ... + d(n-3) + 3 \, .$$

Hence $2(n-3) + 1 = d(x_1) + ... + d(x_{n-3})$. For $1 \leq i \leq n-3$, the average of the degrees $d(x_i)$ is greater than 2. Therefore, at least one of those integers $d(x_i)$ is at least 3. □

Homework 1 Set Friday 13th February, due Friday 27th February.

Question 1.1: (11.3.2 in Grimaldi.) Let G be a connected graph with 17 edges and $d(x) \geq 3$ for all vertices x, what is the maximum possible number of vertices?

Question 1.2: (11.3.18 in Grimaldi.) Let k be a positive integer and let G be a graph such that every vertex of G has degree at least k. Show that G has a path of length k.

Question 1.3: (Corrected version of 11.3.28 in Grimaldi.) Let G be a connected directed multigraph. For a vertex x, we write $\mathrm{id}(x)$ to denote the number of edges into x (the "in-degree"), and we write $\mathrm{od}(x)$ to denote the number of vertices out of x (the "out-degree".) Show that G has an Euler path if and only if one of the following two conditions holds:

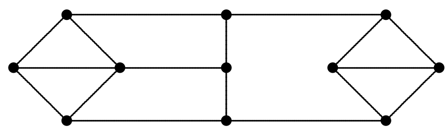
(a) $\mathrm{id}(z) = \mathrm{od}(z)$ for all vertices z,
(b) $\mathrm{id}(z) = \mathrm{od}(z)$ for all vertices z except for two vertices x and y such that $\mathrm{id}(x)+1 = \mathrm{od}(x)$ and $\mathrm{id}(y) = \mathrm{od}(y)+1$.

Question 1.4: (11.3.30 in Grimaldi.) Ali and Ece attended a party with three other married couples. During the party, no one shook hands with their spouse nor with themselves. At the end of the party, Ece asked each of the 7 others how many different people they shook hands with. She received 7 different answers. With how many people did Ece shake hands? With how many did Ali shake hands?

Solution 1.1: Let n be the number of vertices. Numbering the vertices x_1, ..., x_n, then

$$34 = 2.17 = 2e = d(x_1) + ...d(x_n) \ .$$

Each $d(x_i) \geq 3$, so $34 \geq 3n$. Therefore $n \leq 11$. The maximum possible value of n is 11 because the following graph satisfies the conditions and has 11 vertices.



Comment: To see why the example is necessary, consider the following question: For a connected graph with no Euler path, what is the minumim number of vertices of odd degree? What is wrong with the following argument? *Let r be the number of vertices with odd degree. By the Euler Path Theorem, $r \neq 0$ and $r \neq 2$. Since the sum of the degrees of the vertices is twice the number of edges, there does not exit a graph with $r = 1$. We have shown that $r \geq 3$. Therefore, the minimum possible value of r is* 3. How would you prove that, actually, the minimum possible value of r is 4? (Hopefully, you would dispose of the cases $r = 0$ and $r = 2$ using the Euler Path Theorem, you would dispose of the case where r is odd by considering the sum of the degrees, and you would deduce that $r \geq 4$. You would then show that the lower bound $r = 4$ is attained by giving an example, for instance, the graph K_4.)

Solution 1.2: Choose a vertex x_0. Then choose a vertex x_1 such that $x_0 x_1$ is an edge. Generally, having chosen vertices x_0, ..., x_{i-1} for $i \leq k$, the condition $d(x_{i-1}) \geq k$ implies that we can choose a vertex x_i distinct from x_0, ..., x_{i-1} and such that $x_{i-1} x_i$ is an edge. Thus, we construct a path x_0, ..., x_k with length k.

Solution 1.3: If G has an Euler circuit, then the number of times the circuit enters a given vertex z equals the number of times the circuit exits z. Hence condition (a) holds. A similar argument shows that, if G has an Euler path starting at a vertex x and finishing at a distinct vertex y, then condition (b) holds.

Suppose that condition (a) holds. We shall prove, by induction on the number of edges, that G has an Euler circuit. When G has 0 edges, the connectedness of G implies that G has only one vertex. In this case, the existence of an Euler circuit (of length zero) is trivial. Now suppose that G has at least one edge. Then the out-degree of each vertex must be 1. So, choosing a vertex x_0 arbitrarily, we can choose vertices x_1, x_2, ... such that there is an edge ϵ_i from each x_{i-1} to x_i. By the finiteness of G, we have $x_i = x_j$ for some $i < j$. Then there is a circuit C with edges ϵ_{i+1}, ..., ϵ_j.

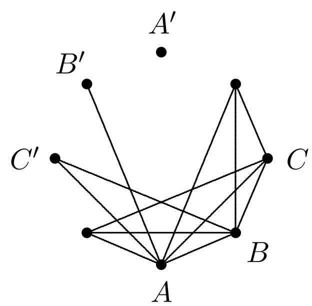
Let H be the graph obtained from G by deleting the edges of C. By the inductive assumption, every connected component of H has an Euler circuit. We can form an Euler circuit of G

by travelling around C and, every time we encounter a new component of H, we travel around an Euler circuit of that component before continuing with C. That completes the proof that G has an Euler circuit when condition (a) holds.

Finally, suppose that condition b holds. Introducing a new edge ϵ from y to x, we obtain a graph F satisfying condition (a). By what we have already proved, F has an Euler circuit B. Deleting ϵ from B, we obtain an Euler path from x to y in G.

Solution 1.4: Consider the graph whose vertices are the people and whose edges are the handshakes. Arrange the vertices in a circle, with each person opposite his or her spouse. When X denotes a person, X' denotes his or her spouse. The 7 vertices other than Ece must have degrees 0, 1, 2, 3, 4, 5, 6. The degree of Ece must be the degree of someone else.

Let A be a person with degree 6. Since A greets everyone except for A', the degree of A' must be zero. Let B be a person with degree 5. By considering the depicted graph, and noting that B must greet everyone except A' and B', we see that B' has degree 1. Similarly, letting C be a person with degree 4, then C' has degree 2. We obtain the depicted graph. Since the degrees of A, B, C, A', B', C' are determined and since the remaining two vertices are spouses, no more edges can be added. Since the degree of Ece is the degree of someone else, Ece and Ali each greet 3 other people.



Quiz 3: *Wednesday 4th March.* Let G be a connected planar graph with 12 vertices, every vertex having degree 5. How many faces does G have?

Solution: The number of vertices is $n = 12$. The number of edges e satisfies $2e = 12.5 = 60$, so $e = 30$. The number of faces f satisfies $n - e + f = 2$. We have $f = 2 - n + e = 2 - 12 + 30 = 20$.

Homework 2 Set Friday 27th February, due Friday 20th March.

Question 2.1: (11.4.2 in Grimaldi.) Show that, when any edge is removed from K_5, the resulting graph is planar. Is the same true for the graph $K_{3,3}$?

Question 2.2: (11.4.10 in Grimaldi.) A set S is said to be the **disjoint union** of two sets S_1 and S_2 provided $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$, in other words, every element of S_1 is in S, every element of S_2 is in S and every element of S is in exactly one of S_1 or S_2. A graph G is said to be **bipartite** provided the set of vertices V is the disjoint union of two sets V_1 and V_2 such that every edge has one end-point in V_1 and one end-point in V_2. Can a bipartite graph have a cycle of odd length?

Question 2.3: (11.4.18 in Grimaldi.) Let G be a connected graph which can be drawn on the plane with no intersecting edges and with 53 faces. Suppose that each face has at least 5 edges on its boundary. Show that the number of vertices of G is at least 82.

Question 2.4: (11.4.20 in Grimaldi.) Let G be a planar graph with n edges, e edges and k components. (a) State and prove an extension of Euler's Characteristic Theorem for such a graph. (b) Suppose that $e \geq 3$. Let f be the number of faces. Prove that $3f \leq 2e$ and $e \leq 3n - 6$.

Solution 2.1: Drawing a suitable planar representation of the graph reveals that it is planar. The same holds for $K_{3,3}$, for a similar reason. (The diagrams are omitted because they are difficult to draw electronically.)

Solution 2.2: No. Let G, V_1, V_2 be as specified in the question. Let v_0, ..., v_n be a path in G. For each $1 \leq i \leq n$, we have $v_{i-1} \in V_1$ if and only if $v_i \in V_2$. So, supposing n is odd, then $v_0 \in V_1$ if and only if $v_n \in V_2$. Perforce, $v_0 \neq v_n$.

Solution 2.3: The number of vertices n, the number of edges e and the number of faces f satisfy $n - e + f = 2$. Since $f = 53$, we have $n = e - 51$. But $2e \geq 5f = 265$, so $e \geq 133$, hence $n \geq 51$.

Solution 2.4: Part (a). We have $n - e + f = 1 + k$. Indeed, numbering the components from 1 to k, letting n_i, e_i, f_i be the number of vertices, edges and internal faces of the i-th component, we have $n_i - e_i + f_i = 1$. The required equality follows because $n = n_1 + ... n_k$ and $e = e_1 + ... + e_k$ and $f = f_1 + ... + f_k + 1$ (remembering to count the external face).

Part (b). Consider the pairs (F, ϵ) where F is a face, ϵ is an edge with F on one side of ϵ and a different face on the other side of ϵ. Bearing in mind that $e \geq 3$, we see that number of such pairs is greater than or equal to $3f$ and is less than or equal to $2e$. So $3f \leq 2e$. Hence, using part (a) and the inequality $k \geq 1$, we obtain $n - e + 2e/3 \geq 2$, in other words, $e \leq 3n - 6$.

Quiz 4: *Friday 27 March.* How many ways are there of putting 5 indistinguishable trolls into 4 differently coloured bags?

Solution: By a standard formula, the answer is $\binom{5+4-1}{5} = 8.7.6/3.2.1 = 56$.

Homework 3 Set Friday 27th March, due Friday 10th April.

Question 3.1: (1.3.28 in Grimaldi.) For any positive integer n, evaluate:
(a) $\sum_{i=0}^{n} \frac{1}{i!(n-i)!}$, **(b)** $\sum_{i=0}^{n} \frac{(-1)^i}{i!(n-i)!}$.

Question 3.2: (4.1.18 in Grimaldi.) Consider the four equations

$$1 = 1 \qquad 2 + 3 + 4 = 1 + 8\ , \qquad 5 + 6 + 7 + 8 + 9 = 8 + 27$$

$$10 + 11 + 12 + 13 + 14 + 15 + 16 = 27 + 64\ .$$

Conjecture a general formula suggested by these four equations and then prove your conjecture.

Question 3.3: (5.2.4 in Grimaldi.) If there are 2187 functions $A \to B$ and $|B| = 3$, what is $|A|$?

Question 3.4: (10.2.4 in Grimaldi.) Find and solve a recurrence relation for the number of ways of parking indistinguishable motorbikes and indistinguishable cars in a row of n spaces if each motorbike requires one space, each car requires two spaces, and all n of the spaces are to be used.

Solution 3.1: Part (a). Multiplying by $n!$, we deduce that the sum is $2^n/n!$.
Part (b). Multiplying by $n!$, we deduce that the sum is 0.

Solution 3.2: We shall show that

$$\sum_{i=n^2+1}^{(n+1)^2} i = n^3 + (n+1)^3 \ .$$

The asserted equality holds because

$$\sum_{i=1}^{2n+1} (n^2+i) = (2n+1)n^2 + \sum_{i=1}^{n} i = 2n^3+n^2+(2n+1)(n+1) = 2n^3+3n^2+3n+1 = n^3+(n+1)^3 \ .$$

Solution 3.3: Since $3^7 = 2187$, we have $|A| = 7$.

Solution 3.4: Let x_n be the number of arrangements with n slots. Consider the case where there are $n+2$ slots. If the first slot is taken by a motorbike, then there are x_{n+1} ways of completing the arrangement. If the first and second slots are taken by a car, then there are x_n ways of completing the arrangement. Therefore $x_{n+1} = x_{n+1} + x_n$.

The initial conditions are $x_0 = x_1 = 1$. So x_0, x_1, ... is the Fibonacci sequence with the index shifted by 1, we mean to say, $x_n = F_{n+1}$. There is a well-known formula for the Fibonacci numbers but, instead of recalling it, let us recover a version of it by a direct argument. The quadratic equation $t^2 - t - 1 = 0$ has solutions $t = \phi$ and $t = \psi$ where $\phi = (1+\sqrt{5})/2$ and $\psi = (1-\sqrt{5})/2$. By the theory of second-degree recurrence relations,

$$x_n = A\phi^n + B\psi^n$$

for some A and B. Putting $n = 1$ and $n = 2$, we obtain

$$1 = A + B = A\phi + B\psi \ .$$

Since $\phi = A\phi + B\phi$, we have $-\psi = \phi - 1 = B(\phi - \psi) = B\sqrt{5}$. Hence $B = -\psi/\sqrt{5}$. Similarly, $A = \phi/\sqrt{5}$. Therefore

$$x_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right) \ .$$

Quiz 5: *Friday 3 April.* Solve the recurrence relation $x_{n+2} - 6x_{n+1} + 9x_n = 0$ with initial conditions $x_0 = x_1 = 1$.

Solution: The quadratic equation $t^2 - 3t + 9 = 0$ has unique solution $t = 3$. So there exist C and D such that $x_n = (C + nD)3^n$. Putting $n = 0$, we deduce $C = 1$. Putting $n = 1$, we have $1 = 3(C + D)$, hence $D = -2/3$. Therefore $x_n = (1 - 2n/3)3^n$.

Quiz 6: *Wednesday 22 April.* In the graph of the 5-cube (identifiable with binary strongs with length 5), how many vertices are there at distance 3 from the vertex 00000?

Solution: $\begin{pmatrix} 3 \\ 5 \end{pmatrix} = 10.$

Quiz 7: *Wednesday 6th May.* Solve the recurrence relation

$$x_{n+3} - 2x_{n+2} - 3x_{n+1} + 6x_n = 0$$

with initial conditions $x_0 = x_1 = x_2 = 0$.

Solution: Obviously, $x_n = 0$ for all natural numbers n.

MATH 210: Finite and Discrete Mathematics. Midterm 1

Time allowed: 110 minutes. Please put your name on EVERY sheet of your manuscript. The use of telephones and other electronic devices is prohibited. The use of very faint pencils is prohibited too. You may take the question-sheet home.

Do not forget to justify your answers in terms which could be understood by people who know the background theory but are unable to do the questions themselves.

All graphs are understood to be finite ordinary graphs.

LJB, 13 March 2015, Bilkent University.

1: 10% Let G be a graph such that every vertex has degree 4 and the number of edges is 12. How many vertices does G have?

2: 10% Let G be a connected graph with at least 2 vertices. Show that there exists a vertex x of G such that, when we delete x and all its edges, the resulting graph is connected.

3: 20% The **cone** of a graph G is defined to be the graph $\Delta(G)$ that is obtained from G by adding a new vertex v and a new edge vx for each vertex x of G. Recall that the 3-cube is the graph C_3 with vertices 000, 001, 010, 011, 100, 101, 110, 111 where two vertices are adjacent provided they differ by exactly one digit. Thus, C_3 has 12 edges and its cone $\Delta(C_3)$ has 9 vertices and 20 edges. Find an Euler circuit for $\Delta(C_3)$. (Specify the Euler circuit by listing the vertices in order.)

4: 20% State, without proof, a theorem saying when a graph has an Euler circuit. Is the following statement true? "*Given any graph G, then the cone $\Delta(G)$ has an Euler circuit if and only if every vertex of G has odd degree*". Give a proof of a counter-example.

5: 20% Let G be a connected planar graph with n vertices, e edges and f faces.

(a) State a formula relating n and e and f.

(b) One proof of the correct answer to part (a) begins as follows. "*Suppose, for a contradiction, that G is a counter-example with n as small as possible. Plainly, $n \geq 2$. By Question 2, we can choose a vertex x of G such that, letting G' be the graph obtained by deleting x and all its edges from G, then G' is connected. Let n', e', f', respectively, be the number of vertices, edges and faces of G'.*" Complete this proof. (No marks will be awarded for presenting a different proof.)

6: 20% Let G be a connected planar graph with 90 edges. Suppose that, for exactly 60 of the edges, the face on one side has 3 edges and the face on the other side has 10 edges. Also suppose that, for exactly 30 of the edges, the two faces on each side are distinct from each other and both of those faces have 10 edges. How many vertices does G have?

Solutions to Midterm 1

Solution 1: Letting n be the number of vertices, then $4n = 2.12 = 24$, hence $n = 6$.

Solution 2: Remove edges from circuits until a tree is obtained. Let x be a vertex which, in the tree, has degree 1.

Solution 3: An Euler circuit is

$$v, 110, 111, 101, 100, 110, 010, 011, 001, 000, 010, v, 111, 011, v, 101, 001, v, 100, 000, v\ .$$

Comment: Specifying an Euler circuit suffices, because it is easy for the reader to check. To find an Euler circuit, the most straightforward method is to adapt the proof of the existence of Euler circuits. To find the above Euler circuit, I began with the obvious circuit

$$v, 110, 010, v, 111, 011, v, 101, 001, v, 100, 000, v$$

and then spliced in the circuits $110, 111, 101, 100, 110$ and $010, 011, 001, 000, 010$ of the two components of the remaining graph.

Solution 4: A graph Γ has an Euler circuit if and only if Γ is connected and every vertex of Γ has even degree.

The specified statement is true. If some vertex x of G has even degree, then x has odd degree in $\Delta(G)$. Hence, by the theorem, $\Delta(G)$ has no Euler circuit.

Conversely suppose that every vertex of G is of odd degree. Plainly, $\Delta(G)$ is connected and every vertex of G has even degree as a vertex of $\Delta(G)$. Finally, since the sum of the degrees of the vertices of G is even, G has an even number of vertices. So the vertex v of $\Delta(G)$ has even degree. Therefore, by the theorem, $\Delta(G)$ has an Euler circuit.

Solution 5: Part (a), $n - e + f = 2$.

Part (b). We have $n' = n - 1$ and $e' = e - d(x)$ and $f' = f - d(x) + 1$. Since $n' < n$, the minimality of G implies that $n' - e' + f' = 2$. But

$$n - e + f = (n' + 1) - (e' + d(x)) + (f' + d(x) - 1) = n' - e' + f' = 2\ .$$

This contradicts the assumption that G is a counter-example, as required.

Comment: If this were reformulated as a proof by induction, the ugly sting in the tail "this contradicts the assumption that G is a counter-example" would not be necessary.

Solution 6: Let n, e, f be the number of vertices, edges and faces, respectively. We have $e = 90$. Let f_3 and f_{10} be the number of faces with 3 edges and 10 edges, respectively. There are $60 = 3f_3$ pairs (ϵ, F) where ϵ is an edge on a face F such that F has 3 edges. There are $60 + 2.30 = 10f_{10}$ pairs (ϵ, F) where ϵ is an edge on a face F such that F has 10 edges. So $f_3 = 20$ and $f_{10} = 12$ and $f = 20 + 12 = 32$. Therefore, $n = e - f + 2 = 90 - 32 + 2 = 60$.

MATH 210: Finite and Discrete Mathematics. Midterm 2

Time allowed: 110 minutes. Please put your name on EVERY sheet of your manuscript. The use of telephones and other electronic devices is prohibited. The use of very faint pencils is prohibited too. You may take the question-sheet home.

LJB, 17 April 2015, Bilkent University.

1: 15% Solve the recurrence relation $x_{n+2} - 2\sqrt{2}x_{n+1} + 2x_n = 0$ with initial conditions $x_0 = 1$ and $x_1 = 0$.

2: 15% Solve the recurrence relation $5x_{n+2} = 6x_{n+1} - 5x_n$ with initial conditions $x_0 = 10$ and $x_1 = 6$.

3: 30% A partial ordering $\leq$ on a set X is said to have height 2 provided there exist elements $x, y \in X$ such that $x < y$ but there do not exist elements $x, y, z \in X$ such that $x < y < z$. (The statement $x < y$ means $x \leq y$ and $x \neq y$.)

(a) Up to isomorphism, how many partial orderings of height 2 are there on a set with size 3?

(b) How many partial orderings of height 2 are there on a set with size 3?

(c) Up to isomorphism, how many partial orderings of height 2 are there on a set with size 4?

4: 25% (a) State (without proof) a recurrence relation and initial conditions for the Stirling numbers of the second kind.

(b) How many ways are there of putting 7 coloured (distinguishable) balls into 5 plain (indistinguishable) boxes such that no box is left empty? (Evaluate the answer explicitly.)

(c) How many ways are there of putting 7 coloured balls and 4 plain balls into 5 plain boxes such that every box contains at least one coloured ball?

5: 15% Let a, b, c, d be complex numbers with $a \neq 0$ and $d \neq 0$. Let x_0, x_1, ... be a sequence of complex numbers such that

$$a x_{n+3} + b x_{n+2} + c x_{n+1} + d x_n = 0$$

for all natural numbers n. Suppose that there exist complex numbers α, β, γ such that

$$at^3 + bt^2 + ct + d = a(t - \alpha)(t - \beta)(t - \gamma)$$

for all complex numbers t. Suppose, also, that there exist complex numbers A, B, C such that

$$x_0 = A + B + C \ , \quad x_1 = A\alpha + B\beta + C\gamma \ , \quad x_2 = A\alpha^2 + B\beta^2 + C\gamma^2 \ .$$

Prove, by induction, that $x_n = A\alpha^n + B\beta^n + C\gamma^n$ for all n. (Advice: remember to clearly tell the reader what your inductive assumption is.)

Solutions to Midterm 2

1: The quadratic equation $t^2 - 2\sqrt{2}t + 2 = 0$ has unique solution $t = \sqrt{2}$. So $x_n = (C + nD)\sqrt{2}^n$ for some C and D. We have $1 = x_0 = C$ and $0 = x_1 = (C + D)\sqrt{2}$. Hence $D = -1$. Therefore $x_n = (1 - n)2^{n/2}$.

2: We have $5x_{n+2} - 6x_{n+1} + 5x_n = 0$. The quadratic equation $5t^2 - 6t + 5 = 0$ has solutions α and β where

$$\alpha = \frac{6 + \sqrt{36 - 100}}{10} = \frac{3 + 4i}{5} \,, \qquad \beta = \frac{3 - 4i}{5} \,.$$

So $x_n = A\alpha^n + B\beta^n$ for some A and B. Putting $n = 0$ and $n = 1$ yields $10 = A + B$ and $6 = A(3 + 4i)/5 + B(3 - 4i)/5$. Therefore $A = B = 5$ and

$$x_n = 5\left(\frac{3 + 4i}{5}\right)^n + 5\left(\frac{3 - 4i}{5}\right)^n \,.$$

3: Part (a). The Hasse diagrams are as shown. In particular, there are 3 isomorphism classes.

[Cue three Hasse diagrams.]

Part (b). For the disconnected Hasse diagram, the number of partial orderings is 6. For the other two Hasse diagrams, the number of partial orderings is 3. So the total is $6 + 3 + 3 = 12$.

Part (c). There are 8 isomorphism classes, with the illustrated Hasse diagrams.

[Cue eight Hasse diagrams.]

4: Part (a), $S(m, 1) = S(m, m) = 1$ for all $1 \leq m$ and $S(m + 1, n) = S(m, n - 1) + nS(m, n)$ for all $1 < n \leq m$.

Part (b). Using part (a), we obtain the following table of Stirling numbers.

[Cue table of Stirling numbers.]

The number of arrangements is $S(7, 5) = 140$.

Part (c). After the coloured balls have been placed, the boxes become distinguishable. For each arrangement of the coloured balls there are $\binom{5 - 1 + 4}{4} = 8.7.6.5/4.3.2 = 70$ arrangements of the plan balls. So the total number of arrangements is $140.70 = 9800$.

5: Let $y_n = A\alpha^n + B\beta^n + C\gamma^n$. We shall show, by induction, that $y_n = x_n$ for all n. It is given that $y_n = x_n$ for all $n \leq 2$. Now suppose that $y_n = x_n$ and $y_{n+1} = x_{n+1}$ and $y_{n+2} = x_{n+2}$. Write $f(t) = at^3 + bt^2 + ct + d$. We have

$$ay_{n+3} + by_{n+2} + cy_{n+1} + dy_n = Af(\alpha) + Bf(\beta) + Cf(\gamma) = 0 = ax_{n+3} + bx_{n+2} + cx_{n+1} + dx_n \,.$$

Cancelling summands and then dividing by a, we deduce that $y_{n+3} = x_{n+3}$. □

Comment 5.1: The inductive assumption in the above proof is "Suppose that $y_n = x_n$ and $y_{n+1} = x_{n+1}$ and $y_{n+2} = x_{n+2}$." This is a vital part of the proof. If you set the argument up as "simple induction", say, with the inductive assumption "$n \geq 1$ and $y_{n-1} = x_{n-1}$", then you would not have been able to deduce that $y_n = x_n$. If you did not state any inductive assumption, then you argument is incomplete because you failed to inform the reader of the premise behind your deduction.

Comment 5.2: In the case where α, β, γ are mutually distinct, it can be shown that there always exist A, B, C as specified. So, in that case, the solution to the recurrence relation always has the specified form. But when α, β, γ are not mutually distinct, such A, B, C might not exist.

MATH 210: Finite and Discrete Mathematics. Makeup

Time allowed: 110 minutes. Please put your name on EVERY sheet of your manuscript. The use of telephones and other electronic devices is prohibited. The use of very faint pencils is prohibited too. You may take the question-sheet home.

LJB, 13 May 2015, Bilkent University.

1: 20% (a) Let $y_n = n$. Show that

$$y_{n+2} - y_{n+1} - y_n = 1 - n \ .$$

(b) Let x_0, x_1, ... be an infinite sequence such that

$$x_{n+2} - x_{n+1} - x_n = 1 - n$$

and $x_0 = x_1 = 1$. Give an explicit formula for x_n in terms of n. (Hint: first find a recurrence relation for z_0, z_1, ... where $z_n = x_n - y_n$.)

2: 30% A **minimum element** of a partial ordering $\leq$ on a set X is an element $x_0 \in X$ such that $x_0 \leq x$ for all $x \in X$. Suppose that $|X| = 4$.

(a) How many partial orderings $\leq$ on X are there such that $\leq$ has a minimum element? (Hint: do part (b) first.)

(b) How many such partial orderings $\leq$ on X are there up to isomorphism?

3: 30% (a) Write out a table of Stirling numbers $S(m, n)$ for $8 \geq m \geq n$. What is the rule for working out the entries of the table?

(b) How many ways are there of putting 8 distinguishable objects into 3 red boxes and 2 blue boxes such that each box contains at least one ball? (The red boxes are indistinguishable from each other, likewise the blue boxes.)

4: 20% For positive integers r, m, n, let $S_r(m, n)$ be the number of ways of arranging m coloured balls in n plain boxes such that each box has at least r balls. Express $S_r(m+1, n+1)$ in terms of $S_r(m, n+1)$ and $S_r(m+1-r, n)$.

MATH 210: Finite and Discrete Mathematics. Final

Time allowed: 2 hours. Please put your name on EVERY sheet of your manuscript. The use of telephones and other electronic devices is prohibited. The use of very faint pencils is prohibited too. You may take the question-sheet home.

LJB, 25 May 2015, Bilkent University.

1: 40% Consider the coding scheme with encoding function $\mathbb{Z}_2^3 \to \mathbb{Z}_2^6$ given by generating matrix

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} .$$

(a) Write down all 8 message words and their corresponding codewords.
(b) How many errors of transmission can be detected? How many errors of transmission can be corrected? Briefly justify your answers.
(c) Let 000011 be one of the coset leaders. Without working out the decoding table, find all the other coset leaders and explain why those binary strings must be the coset leaders.
(d) Without working out the decoding table, decode the the received words 011111, 111110, 000000.

2: 20% (a) Find the number of isomorphism classes of trees T such that T has exactly 7 vertices and exactly 3 of the vertices have degree 1.

(b) Let V be a set with $|V| = 7$. How many trees T are there such such V is the set of vertices of T and exactly 3 of the vertices of T have degree 1?

3: 15% Evaluate $\displaystyle\sum_{j=1}^{n} \frac{n!}{(n-j)!} S(m,j)$ for all $m \geq n \geq 1$.

4: 25% Let m be a positive integer.

(a) Let n be a positive integer with $n \geq m$. Let C be a linear code in $\mathbb{Z}_2^n$ such that $|C| = 2^m$. How many injective linear encoding functions $E : \mathbb{Z}_2^m \to C$ are there?

(b) Now suppose that $n = m + 1$. Let B be the set of elements of $\mathbb{Z}_2^{m+1}$ that are not in C. Let $b \in B$. Show that

$$B = \{b + c : c \in C\} .$$

(c) Show that, given b_1, ..., $b_r \in B$, then $b_1 + ... + b_r \in C$ if and only if r is even.

(d) How many linear codes C in $\mathbb{Z}_2^{m+1}$ are there such that $|C| = 2^m$ and the minimum distance between two distinct codewords is 2?

(e) How many injective linear encoding functions $E : \mathbb{Z}_2^m \to \mathbb{Z}_2^{m+1}$ are there such that the minimum distance between two distinct codewords is 2?

Final Solutions

1: Part (a). For the message words

$$000\,,\quad 001\,,\quad 010\,,\quad 011\,,\quad 100\,,\quad 101\,,\quad 110\,,\quad 111\,,$$

the corresponding codewords are, respectively,

$$000000\,,\quad 001101\,,\quad 010111\,,\quad 011010\,,\quad 100110\,,\quad 101011\,,\quad 110001\,,\quad 111100\,.$$

Part (b). Since the minimum weight of a nonzero codeword is 3, we can detect 2 errors and correct 1 error.

Part (c). The party-check matrix is

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} .$$

The binary strings

$$000000\,,\quad 000001\,,\quad 000010\,,\quad 000100\,,\quad 001000\,,\quad 010000\,,\quad 100000\,,\quad 000011\,,$$

have respective syndromes

$$000\,,\quad 001\,,\quad 010\,,\quad 100\,,\quad 101\,,\quad 111\,,\quad 110\,,\quad 011\,.$$

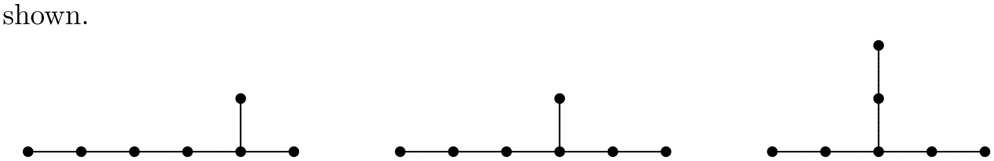
Since the received words with weight 0 or 1 have distinct syndromes, they all appear as coset leaders. Since none of those sydromes coincide with the syndrome 011 of 000011, the proposed coset leader 000011 satisfies the minimal weight condition.

Part (d). Letting r denote the received word, σ the syndrome, s the coset leader, c the codeword and w the corresponding message word, the decodings are as shown.

r	σ	s	c	w
011111	101	001000	010111	010
111110	010	000010	111100	111
000000	000	000000	000000	000

Comment: Some candidates decoded by simply noticing that each of the three given received words is at distance at most 1 from a codeword. This is valid because (as an oversight) the question did not demand that the decoding be done using the method of syndromes.

2: Part (a). For any tree with e edges and n vertices, the sum of the degress is $2e = n - 2$. So, for any tree with exactly 3 vertices of degree 1 all the other vertices have degree 2 except for one vertex with degree 3. It is now easy to see that the 3 isomorphism classes of trees T are as shown.



Part (b). The numbers of ways of assigning elements of V to the vertices of the three diagrams are, in order, $7!/2$ and $7!$ and $7!/3!$. The total is

$$7!(1/2+1+1/6) = 2520 + 5040 + 840 = 8400 \; .$$

3: The sum can be rewritten as $\sum_{j=1}^{n} \binom{n}{j} j!\, S(m,j)$. Since $\binom{n}{j}$ is the number of ways of choosing a subset $J \subseteq \{1, ..., n\}$ with size $|J| = j$ and since $j!S(m,j)$ is the number of surjections from $\{1, ..., m\}$ to J, the value of the sum is the number of functions from $\{1, ..., m\}$ to $\{1, ..., n\}$, which is n^m.

4: Part (a). Let e_i be the weight unity element of $\mathbb{Z}_2^m$ with i-th digit equal to 1. there are $2^m - 1$ choices for $E(e_1)$, then $2^m - 2$ choices for $E(e_2)$ and so on, finally $2^m - 2^{m-1}$ choices for $E(e_m)$. So the number of such E is

$$(2^m - 1)(2^m - 2)...(2^m - 2^{m-1}) = 2^{m(m-1)/2}(2^m - 1)(2^{m-1} - 1) \; ...(2-1) \; .$$

Part (b). Let $B' = \{b + c : c \in C\}$. We have $B' \subseteq B$ because, if $b + c$ were a codeword, then $b = (b+c) + c$ would be a codeword, contradicting the definition of b. But $|B'| = 2^m = |B|$, so $B' = B$.

Part (c). Writing $b_i = b + c_i$ where c_i is a codeword, we have $b_1 + ... + b_r = rb + c_1 + ... + c_r$. The condition follows because rb is 0 or b depending on whether r is even or odd, respectively.

Part (d). We apply part (c). Letting b_1, ..., b_r be words of weight 1 then, since they all belong to B, their sum is in C if and only if r is even. Therefore C is the set of words of even weight. In particular, there is only one such C.

Part (e). The code must be C as in part (d). So, by part (a), the answer is $(2^m - 1)(2^m - 2)...(2^m - 2^{m-1})$.

MATH 210: Finite and Discrete Mathematics. Retake Final

Time allowed: 2 hours. Please put your name on EVERY sheet of your manuscript. The use of telephones and other electronic devices is prohibited. The use of very faint pencils is prohibited too. You may take the question-sheet home.

LJB, 4 June 2015, Bilkent University.

1: 40% Let

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

be the generating matrix of a coding scheme with message words in $\mathbb{Z}_2^3$ and received words in $\mathbb{Z}_2^6$.

(a) Write down all the message words and their corresponding codewords.

(b) Write down the parity-check matrix.

(c) Find a set of coset leaders and, without working out the decoding table, explain why those binary strings can be used as the coset leaders.

(d) Using the method of syndromes, taking the coset leaders to be as in your answer to part (c), decode the the received words 010101, 101010, 111111.

(e) How many errors of transmission can be detected? How many errors of transmission can be corrected? Briefly justify your answers.

2: 20% Let X be a finite set.

(a) What do we mean when we say that two relations $\sim$ and $\approx$ on X are isomorphic? Give a precise definition.

(b) Suppose that $|X| = 7$. Find the number of isomorphism classes of equivalence relations on X.

3: 20% For $m \geq n \geq 1$, evaluate $\displaystyle\sum_{j=n}^{m} \binom{m}{j} S(j, n)$.

4: 20% Let $m \leq n$ be positive integers.

(a) What is a linear code in $\mathbb{Z}_2^n$? Give a precise definition.

(b) How many linear codes C in $\mathbb{Z}_2^n$ are there such that $|C| = 2^m$?

Solutions to Final Retake

1: Part (a). For the message words

$$000\,, \quad 001\,, \quad 010\,, \quad 011\,, \quad 100\,, \quad 101\,, \quad 110\,, \quad 111\,,$$

the corresponding codewords are, respectively,

$$000000\,, \quad 001111\,, \quad 010111\,, \quad 011000\,, \quad 100111\,, \quad 101000\,, \quad 110111\,, \quad 111111\,.$$

Part (b). The party-check matrix is

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Part (c). We can take the coset leaders to be

$$000000\,, \quad 000001\,, \quad 000010\,, \quad 000100\,, \quad 001000\,, \quad 000110\,, \quad 000101\,, \quad 000011\,.$$

Their respective syndromes

$$000\,, \quad 001\,, \quad 010\,, \quad 100\,, \quad 111\,, \quad 110\,, \quad 101\,, \quad 011$$

are mutually distinct. The minimal weight condition is satisfied because the omitted words with weight 1 have the same syndrome as 001000.

Part (d). Letting r denote the received word, σ the syndrome, s the coset leader, c the codeword and w the corresponding message word, the decodings are as shown.

r	σ	s	c	w
010101	010	000010	010111	010
101010	010	000010	101000	101
111111	000	000000	111111	111

Part (e). Since the minimum weight of a nonzero codeword is 2, we can detect 1 error and correct no errors.

Comment: Other answers to part (c) are possible, but the answers to part (d) will still be the same.

2: Part (a). We say that $\sim$ and $\approx$ are isomorphic provided there exists a bijection f on X such that, for all $x, y \in X$, we have $x \sim y$ if and only if $f(x) \approx f(y)$.

Sketch, part (b). Two equivalence relations on X are isomorphic provided their equivalence classes have the same sizes. Any equivalence relation $\equiv$ is determined, up to isomorphism, by the finite sequence of positive integers n_1, ..., n_r where r is the number of equivalence classes of $\equiv$ and n_i is the size of the i-th largest equivalence class of equiv. For $|X| = 7$, there are 15 isomorphism classes, because that is the number of ways of expressing 7 as a sum $n_1 + ... + n^r$ of positive integers with $n_1 \geq ... \geq n_r$, indeed,

$$7 = 6+1 = 5+2 = 5+1+1 = 4+3 = 4+2+1 = 4+1+1+1 = 3+3+1$$

$$= 3+2+2 = 3+2+1+1 = 3+1+1+1+1 = 2+2+2+1 = 2+2+1+1+1$$

$$= 2 + 1 + 1 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1 + 1 + 1 \ .$$

3: The sum can be rewritten as $\sum_{j=1}^{n} \binom{n}{j} j!\, S(m,j)$. Since $\binom{n}{j}$ is the number of ways of choosing a subset $J \subseteq \{1, ..., n\}$ with size $|J| = j$ and since $j!S(m,j)$ is the number of surjections from $\{1, ..., m\}$ to J, the value of the sum is the number of functions from $\{1, ..., m\}$ to $\{1, ..., n\}$, which is n^m.

4: Part (a). A linear code in $\mathbb{Z}_2^n$ is a non-empty subset C of $\mathbb{Z}_2^n$ such that, given $c_1, c_2 \in C$, then $c_1 + c_2 \in C$.

Part (c). Given $x_1, ..., x_r \in \mathbb{Z}_2^n$, let us write $\langle x_1, ..., x_r \rangle$ to denote the smallest code in $\mathbb{Z}_2^n$ containing x_1, ..., x_r. We can choose a code C in the following way. We first choose a non-zero word b_1. Having chosen b_i for $i < m$, we choose $b_{i+1} \in \mathbb{Z}_2^n - \langle b_1, ..., b_n \rangle$. We let $C = \langle b_1, ..., b_n \rangle$.

There are $2^n - 1$ choices for b_1, then $2^n - 2$ choices for b_2, and so on, finally $2^n - 2^{m-1}$ choices for b_m. But each C appears multiple times in that way. Given C, there are $2^m - 1$ choices for b_1, then $2^m - 2$ choices for b_2, and so on, finally $2^m - 2^{m-1}$ choices for b_m. So the number of possible C is:

$$(2^n - 1)(2^n - 2)...(2^n - 2^{m-1})/(2^m - 1)(2^m - 2)...(2^m - 2^{m-1}) \ .$$