Archive for

MATH 132, Discrete and Combinatorial Mathematics, Spring 2023

Bilkent University, Laurence Barker, 20 June 2023.

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MATH 132, Section 4

Discrete and Combinatorial Mathematics, Spring 2023

Course specification

Laurence Barker, Bilkent University. Version: 24 March 2023.

Classes: Tuesdays 10:30 - 11:20, Thursdays 15:30 - 17:20, room SAZ 18.

Office Hours: Tuesdays 11:30 - 12:20, SAZ 18.

For all students, those doing well and aiming for an A, those doing badly and aiming for a C, Office Hours is an opportunity to come and ask questions.

Instructor: Laurence Barker
e-mail: barker at fen nokta bilkent nokta edu nokta tr.

Course Texts: The primary course text is:

> Kenneth H. Rosen, "Discrete Mathematics and its Applications", 8th Edition, (McGraw–Hill 2019).

Also recommended, is the following free downloadable PDF book, written in a different style:

> Oscar Levin, "Discrete Mathematics: An Open Introduction". It is available at http://discrete.openmathbooks.org/

The internet has a vast supply of text and videos on the material convered. Fully academic study involves consultation of multiple sources. One has not properly grasped material until one can follow disparate accounts by different authors who may sometimes use different notation and terminology.

Homework:
As the online platform Connect will be used for the homework in this course, please purchase your textbook only from Meteksan. The books purchased from Meteksan will come with a Connect code, which you will use to register for the Connect course. Alternatively, you can choose to purchase only a Connect code for this title from Meteksan. Those who will purchase the textbook are kindly invited to do so as soon as possible, as unsold textbooks will be returned shortly after the beginning of the semester.

Each student must purchase an individual textbook/code; codes cannot be shared, they are one time use only. Ensure that you purchase your textbook/code and complete your registration in the first 2 weeks of the semester. The registration URL is:

> https://connect.mheducation.com/class/n-sahin-fall-2022

Full details on how to register to the Connect site are on the MATH 132 Moodle page.

Course Documentation: As the course progresses, further documentation will appear on the course Moodle site and my homepage.

Syllabus: *Below is a tentative course schedule.* The format of the following details is *Week number: Monday date: Subtopics (Section numbers).*

1: 30 Jan: Sets, functions, cardinalities. (2.3, 2.5)

2: 6 Feb: Divisibility and modular arithmetic (4.1)

3: 13 Feb: Primes and greatest common divisors (4.3)

4: 20 Feb: Congruences, applications. (4.4)

5: 27 Feb: Induction and well-ordering. (5.1-5.2)

6: 6 Mar: (No classes Tuesday.) The Pigeonhole Principle. (6.3-6.4)

7: 13 Mar: Counting. (6.3-6.4)

8: 20 Mar: Generalized permutations and combinations. (6.5)

9: 27 Mar: Further counting techniques. (8.2)

10: 3 Apr: Further counting techniques. (8.5-8.6)

11: 10 Apr: Relations. (9.1)

12: 17 Apr: (No classes Thursday.) Relations. (9.5-9.6)

13: 24 Apr: Graphs. (10.1-10.3)

14: 1 May: Graphs. (10.4-10.6)

15: 8 May: Graphs. (10.7-10.8).

Assessment:

- Homework, 25%,
- Midterm 1, 35%. in Midterm Week,
- Final, 40%.

A Midterm score of least 20% (of the available Midterm marks) is needed to qualify to take the Final Exam, otherwise an FZ grade will be awarded.

75% attendance is compulsory.

Asking questions in class is very helpful. It makes the classes come alive, and it often improves my sense of how to pitch the material. The rule for talking in class is: if you speak, then you must speak to everyone in the room.

MATH 132: Discrete and Combinatorial Mathematics. Midterm

Examiners: Laurence Barker, Alex Degtyarev, Nil Şahin. Exam date: 12 April 2023

This is a reformatted record of the paper. The original was in fill-in-the-spaces form.

No books, notes, phones or electronic calculators of any kind are allowed.

You should show your work to get full credit. Each problem is worth 20 points.

The duration of the examination is 120 minutes.

1: **(a)** Let A and B be sets with $|A| = |B| = |\mathbb{R}|$. Show that $|A \cup B| = |\mathbb{R}|$.

(b) A ternary string is a string $z_1, ..., z_n$ where each $z_i \in \{0, 1, 2\}$. How many ternary strings z of length 9 are there such that:

- **i.** Exactly 5 of the digits of z are 0?
- **ii.** At least 7 of the digits of z are 0?

(You should evaluate your answers numerically.)

2: (20 points.) Find all the integers a such that $a \equiv 2 \mod 5$ and $a \equiv 5 \mod 6$ and $a \equiv 3 \mod 7$.

3: **(a)** Find $d := \gcd(17, 113)$ and express d as a linear combination of 17 and 113.

(b) If possible, solve the congruence $17x \equiv 15 \mod 113$.

4: Let $n \in \mathbb{Z}^+$ and define S_n by the formula $S_n = \dfrac{1}{2!} + \dfrac{2}{3!} + \dfrac{3}{4!} + \cdots + \dfrac{n}{(n+1)!}$.

(a) Calculate S_1, S_2, S_3 and try to guess a general formula for S_n.

(b) Prove your conjecture in part (a).

5: How many pairwise distinct integral points $A_i := (x_i, y_i, z_i) \in \mathbb{R}^3$ should be chosen to guarantee that there is a pair $A_i \neq A_j$ such that the midpoint of the segment $[A_iA_j]$ has integer coordinates? (You should justify both claims: the number announced does suffice whereas any smaller number may not do.)

Solutions to Midterm

1: Part (a). We make use of the well-known equality $|\mathbb{R}| = |(x,y)_\mathbb{R}|$ for any $x, y \in \mathbb{R}$ with $x < y$, where $(x,y)_\mathbb{R} = \{z : x < z < y\}$. Plainly, $|\mathbb{R}| = |A| \leq |A \cup B|$. Let $f : A \to (0,1)_\mathbb{R}$ and $g : B \to (1,2)_\mathbb{R}$ be injections. We define $h : A \cup B \to (0,2)_\mathbb{R}$ such that $h(a) = f(a)$ for $a \in A$ and $h(b) = g(b)$ for $b \in B - A$. Since h is injective, $|A \cup B| \leq |(0,2)_\mathbb{R}| = |\mathbb{R}|$.

Part (b i). The answer is $\binom{9}{5} 2^4 = \dfrac{9.8.7.6}{4.3.2} 16 = 9.7.2.16 - 126.16 = 2016.$

Part (b ii). The answer is $\binom{9}{7} 2^2 + \binom{9}{8} 2 + \binom{9}{9} = 9.8.2 + 9.2 + 1 = 163.$

Alternative for part (a): There is an injection sending each element a of A to a binary representation of a real number $...a_2 a_1 a_0.a_{-1}a_{-2}...$. There is also an injection sending each element $b \in B{-}A$ to a similar binary representation $...b_2 b_1 b_0.b_{-1}b_{-2}$. We have an injection $k : A{\cup}B \to \mathbb{R}$ sending each a to $...a_1 a_0 000.a_{-1}...$ and each b to $...b_1 b_0 010.b_{-1}...$. So $|A \cup B| \leq \mathbb{R}$. But clearly $|A \cup B| \geq |\mathbb{R}|$. Therefore $|A \cup B| = |\mathbb{R}|$.

2: We have $a = 7r + 3$ for some integer r. Then $7r + 3 \equiv_6 5$, that is, $r \equiv_6 2$, so $r = 6s + 2$ for some integer s. We have $a = 7(6s+2) + 3 = 42s + 17$. Then $42s + 17 \equiv_5 2$, that is, $2s \equiv_5 0$, so $s = 5t$ for some integer t. We have $a = 210t + 17$. In conclusion, $a \in \{210t + 17 : t \in \mathbb{Z}\}$.

3: Part (a). We have

$$113 = 6.17 + 11\,, \quad 17 = 1.11 + 6\,, \quad 11 = 1.6 + 5\,, \quad 6 = 1.5 + 1\,.$$

So $d = 1$. We have

$$1 = 6 - 5 = 6 - (11 - 6) = 2.6 - 11 = 2(17 - 11) - 11 = 2.17 - 3.11$$

$$= 2.17 - 3(113 - 6.17) = 20.17 - 3.113\,.$$

Part (b). Modulo 113, we have $20.17 \equiv 1$, so $x \equiv 20.15 = 300 = 2.113 + 74$. So $x \equiv 74$.

4: We have $S_1 = 1/2$ and $s_2 = (3+2)/3.2 = 5/6$ and $S_3 = (4.3 + 4.2 + 3)/4.3.2 = 23/24$. We shall show that

$$S_n = 1 - 1/(n+1)!\,.$$

Let $T_n = 1 - 1/(n+1)!$. Arguing by induction, we shall show that $S_n = T_n$. Trivially, $S_1 = T_1$. Suppose $n \geq 2$ and that $S_{n-1} = T_{n-1}$. We have

$$T_n - T_{n-1} = 1/n! - 1/(n+1)! = n/(n+1)! = S_n - S_{n-1}\,.$$

Cancelling, we deduce that $S_n = T_n$.

5: The midpoint of $[A_i A_j]$ is $((x_i{+}x_j)/2, (y_i{+}y_j)/2, (z_i{+}z_j)/2)$, which has integral coordinates if and only if x_i, y_i, z_i have the same partities as x_j, y_j, z_j, respectively. Let $\overline{A_i} = (\overline{x_i}, \overline{y_i}, \overline{z_i})$ where $\overline{n}$ is 0 or 1 depending on whether n is even or odd, respectively. We are to find the minimum number of points A_i such that the function $A_i \mapsto \overline{A_i} \in \mathbb{Z}/2{\times}\mathbb{Z}/2{\times}\mathbb{Z}/2$ cannot be injective.

By the Pigeonhole Principle, the minimal number of points is 9, since $|\mathbb{Z}/2{\times}\mathbb{Z}/2{\times}\mathbb{Z}/2| = 8$ and, if there are only 8 points, we can take them to be $(0,0,0)$, $(0,0,1)$, $(0,1,0)$, $(0,1,1)$, $(1,0,0)$, $(1,0,1)$, $(1,1,0)$, $(1,1,1)$.

MATH 132: Discrete and Combinatorial Mathematics. Final

Examiners: Laurence Barker, Alex Degtyarev, Nil Şahin. Exam date: 10 June 2023

This is a reformatted record of the paper. The original was in fill-in-the-spaces form.

No books, notes, phones or electronic calculators of any kind are allowed.

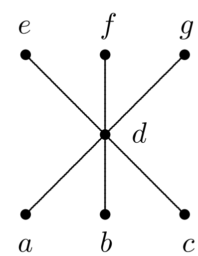
You should show your work to get full credit. Each problem is worth 20 points.

The duration of the examination is 120 minutes.

1: In how many ways can the digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ be arranged so that no even digit is in its original position?

2: **(a)** Draw a Hasse diagram for divisibility on the set $\{1, 2, 3, 6, 12, 24, 36, 48\}$

(b) Find the number of all compatible total orderings for the depicted poset.



3: Let S be a set with n elements and let a and b be distinct elements of S. How many relations are there on S such that:

(a) $(a, b) \in R$?

(b) $(a, b) \notin R$?

(c) No ordered pair in R has a as its first element?

(d) At least one ordered pair in R has a as its first element?

(e) No ordered pair in R has a as its first element or b as its second element?

(f) At least one ordered pair in R has a as its first element or b as its second element?

4: Let a_n be the number of $n \geq 0$ binary strings without substring 11. Let b_n be the number of length n binary strings without substring 10. Find recurrence relations, initial conditions and closed formulas for a_n and b_n.

5: Does there exist a connected simple (i.e., without loops or multiple edges) planar graph with all vertices of degree

(a) 3?

(b) 4?

(c) 5?

(d) 6?

(Justify your answers, i.e., provide examples or rigorous proofs of non-existence.)

Solutions to Final

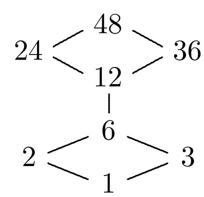
1: For $i \in \{0, 2, 4, 6, 8\}$, let A_i be the set of permutations of $\{0, ..., 9\}$ that fix i. For $1 \leq d \leq 5$, the intersection of d of those sets has size $(10-d)!$. So, by the inclusion-exclusion formula,

$$|\bigcap_i A_i| = \binom{5}{1} 9! - \binom{5}{2} 8! + \binom{5}{3} 7! - \binom{5}{4} 6! + \binom{5}{5} 5! \, .$$

Therefore, the answer is

$$10! - |\bigcap_i A_i| = 10! - 5.9! + 10.8! - 10.7! + 5.6! - 5! \, .$$

2: Part (a). A Hasse diagram for the poset is as shown.



Part (b). The specified total orderings have the form $u < v < w < d < x < y < z$ where $\{u, v, w\} = \{a, b, c\}$ and $\{x, y, z\} = \{e, f, g\}$. The number of them is $(3!)^2 = 36$.

3: The total number of relations on S is 2^{n^2} since there are 2 choices for each element of the incidence matrix. For each of the following parts, we count to number of possible incidence matrices.

Part (a). The constraint on the incidence matrix is that the (a, b) entry is 1. So the answer is 2^{n^2-1}.

Part (b). By much the same argument, the answer is 2^{n^2-1}.

Part (c). The constraint is that the a-th row of the incidence matrix must be full of 0 digits, so the answer is $2^{n(n-1)}$.

Part (d). By part (c), the answer is $2^n - 2^{n(n-1)}$.

Part (e). The constraint is that the a-th row and b-th column must be full of 0 digits, so the answer is $2^{(n-1)^2}$.

Part (f). By Part (e), the answer is $2^{n^2} - 2^{(n-1)^2}$.

4: For the strings with no substring 11, a valid length n string is obtained either by appending 0 to any valid length $(n-1)$ string (a_{n-1} possibilities) or, else, appending 1 to a valid length $(n-1)$ string *ending in* 0, i.e., appending 01 to any valid length $(n-2)$ string (a_{n-2} possibilities). Thus, $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$.

The solutions to $t^2 - t - 1 = 0$ are $\gamma = (1+\sqrt{5})/2$ and $\delta = (1-\sqrt{5})/2$. So $a_n = A\gamma^n + B\beta^n$ for some A and B. Putting $n = 0$ and $n = 1$, we obtain $A + B = 1$ and $A\gamma + B\delta = 2$. So $A = (3+\sqrt{5})/2\sqrt{5}$ and $B = (-3+\sqrt{5})/2\sqrt{5}$. In conclusion,

$$a_n = \left(\frac{3+\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n \, .$$

The strings of length n with no substring 10 consist of r digits 0 followed by $n-r$ digits 1, where $0 \leq r \leq n$. So $b_n = n+1$. A recurrence relation is $b_n = b_{n-1} + 1$ for $n \geq 1$, with initial condition $b_0 = 1$.

Comment: An alternative way of obtaining the recurrence relation for a_n is as follows. We have $a_0 = 1$ and $a_1 = 2$ and $a_2 = 3$. Let c_n be the number of strings of length n, ending with 0 and with no substring 11. Let d_n be the number of strings of length n, ending with 1 and with no substring 11. We have $c_{n+1} = c_n + d_n$ and $d_{n+1} = c_n$. Therefore $c_{n+2} = c_{n+1} + c_n$. It follows that, for $n \geq 1$, we have $d_{n+2} = d_{n+1} + d_n$. Since $a_n = c_n + d_n$, we have $a_{n+2} = a_{n+1} + a_n$ for $n \geq 1$. From the initial conditions above, we observe that this recurrence relation also holds when $n = 0$.

5: Part (a). Yes, an example of such a graph being the graph of a tetrahedron.

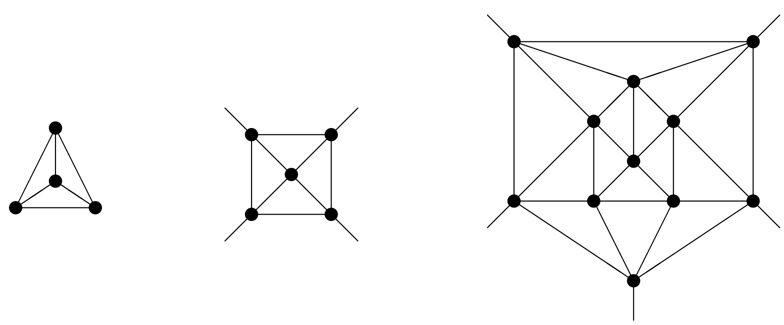
Part (b). Yes, an example is the graph obtained from a cube by introducing a suitable diagonal for each of the four vertical faces of the cube.

Part (c). Yes, an example being the graph of an icosahedron.

Part (d). No. It is a standard theorem that, for any finite planar graph of order n and size e, we have $e \leq 1$ or $e \leq 3n - 6$. But $2e$ is the sum of the degrees of the vertices. So the average degree is less than 6 and some vertex of the graph must have degree 5 or less.

Alternative for part (d): The corollary obtained above, asserting that the minimum degree is at most 5, is itself a standard result. Full marks we obtained by those who appealed directly to that corollary.

Comment: If you did not learn about the applications of planar graphs to Platonic solids, the examples for the first three parts can easily be drawn. Assuming that each vertex has degree d, so that $2e = dn$ by the handshake lemma, the inequality $e \leq 3n - 6$ takes the form $(6 - d)n \geq 12$. For $d = 3, 4, 5$, we conclude that the *minimal* number of vertices is $n = 4, 6, 12$, respectively. Furthermore, for this minimal number, the inequality turns into an equality, which is possible if and only if all regions are *triangles.* (Recall that you are also supposed to know the proofs of the major statements.) Thus, starting the drawing from any vertex ("central" in the figures below), we essentially have no choice, keeping "closing the triangles" and arriving at the skeletons of a tetrahedron, octahedron, or icosahedron, respectively:



(In the last two figures, the d "hanging" outer edges are supposed to be joined at a common extra vertex, which is not shown for a technical reason.)