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MATH 132, Discrete and Combinatorial Mathematics,

Bilkent University, Fall 2099, Laurence Barker

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MATH 132, Discrete and Combinatorial Mathematics, Spring 2014

Course specification

Laurence Barker, Bilkent University, version: 20 March 2014

Course Aims: To supply an introduction to some concepts and techniques associated with discrete mathematical methods in engineering and information technology; in particular, to provide experience of the art of very clear deductive explanation.

Formal Course Description: The two terms *discrete mathematics* and *combinatorics* mean the same thing, except that the former has the flavour of applicable mathematics, the latter has the flavour of pure mathematics. The terms refer to a branch of mathematics which arose with the advent of electronic computers and information technology. It is the study of mathematical objects which do not have very much topological, geometric or algebraic structure.

Unlike calculus, linear algebra, statistics, it does not have deep theory. Solutions tend to comparatively unsystematic, though certain fundamental ideas do tend to be used quite frequently. For that reason, the study of discrete mathematics depends heavily on the art of *very clear deductive explanation*, which will be emphasized throughout the course.

The course is intended only as an introduction, for students who have little or no previous experience of this kind of mathematics.

We shall be studying four main areas, separate but with some interactions: graph theory; relations and enumerative combinatorics; coding theory; Boolean algebra and gating networks.

Instructor: Laurence Barker, Office SAZ 129,
e-mail: barker at fen dot bilkent dot edu dot tr.

Assistants: Hatice Mutlu, Mehmet Akif Erdal, Bengi Ruken Yavuz.

Text: R. P. Grimaldi, "Discrete and Combinatorial Mathematics", 5th Ed. (Pearson, 2004).

Some notes will be supplied, on my webpage, for some of the syllabus material.

Classroom hours:
Section 1, Wednesday 9:30 - 10:20, Friday 10:30 - 12:20.
Section 2, Tuesday 10:30 - 12:20, Friday 9:30 - 10:20.
Section 3, Monday 9:30 - 10:20, Wednesday 10:30 - 12:20.

Office Hours: For all sections, 08:30 - 09:20 Monday, Wednesday, Friday in room SAZ 129 of Fen A Building. (During Week 8, before the Midterm, Office Hours will be held in the classroom G-236.)

Office Hours is not just for the stronger students. If you are having difficulty with the course, then you must come to me for help with the mathematics. In fact, I need students at all levels to come, to make sure that I pitch the classes appropriately.

Class Announcements: All students, including any absentees from a class, will be deemed responsible for awareness of announcements made in class.

Assessment

Homeworks: The only way to pick up skill at mathematical communication is through lots of practise. Because of the continuous assessment system, natural communication between students is subject to a constraint: *You may discuss homeworks solutions amongst yourselves, but you may not copy and you may not do paraphrase rewrites of work by others.*

I encourage you to discuss the homeworks with me during office hours. I will tend to give help to students who have at least some thoughts or questions of their own about the homework. Just to be clear: you will not lose any marks for any help I may give you with the homework.

However, before asking anyone else for help, *first do your best with the homework on your own.* If you get stuck on problems and then find out the solutions, the ideas will sink in. If you just lazily wait for other people to give you the solutions, then you will not learn how to do mathematics.

Participation: Each section will have different quizzes. The total mark for quizzes will incorporate a mark for the academic participation performance of the whole class: for example, addressing questions to the teacher, not making a distracting noise by murmuring just to your neighbours.

Principle of marking: In mathematics, marks for written work are not awarded according to guesses about what the student might have had in mind when writing out the solution. They are awarded according to *how helpful the explanation would be to other students in the class.*

Grading percentages:

- Quizzes, participation and homework, 20%,
- Midterm 30%, (Wednesday, 2 April, at 18:00).
- Final, 50%.

Letter Grades: This is done by the "curve method". A grade C requires an understanding of the concepts and reasonable attempts at the easiest exam questions. That fulfills the aim of the course: a competent grasp at an introductory level.

Some of the exercises and exam questions will be quite difficult. It has to be that way, not only for the benefit of the strongest students, but also because, without difficult questions, it would be hard to see the purpose of the art of *very clear deductive explanation.* However, students aiming for a grade C need not worry about being unable to do the more difficult questions.

Attendance: A minimum of 75% attendance is compulsory. Exceptions may be made for students with very good exam results. However, less than 50% attendance as measured by quizzes will result in an FZ grade.

Syllabus

Week number: Monday date: Subtopics. Section numbers

1: 3 Feb: (Classes start 5 Feb.) Discrete methods and information technology. Examples of problems in discrete mathematics. Sketch of the use of mathematical induction, 4.1.

2: 10 Feb: Recursive definitions and mathematical induction, 4.2. Second order recurrence relations as an application of induction, 10.2.

3: 17 Feb: Graphs. Sum of degrees formula. Circuits and Trees, 11.1, 11.2.

4: 24 Feb: Criteria for existence of Euler paths or Euler circuits, proved by mathematical induction, 11.3.

5: 3 Mar: Euler's characteristic formula for planar graphs, proved by mathematical induction. The non-planarity of the graphs K_5 and $K_{3,3}$, 11.4.

6: 10 Mar: Review of mathematical induction and graph theory. More practise at exercises.

7: 17 Mar: Permutations, combinations, the Binomial theorem, 1.2, 1.3, 1.4.

8: 24 Mar: Sets and correspondences. Functions. Injections, surjections and bijections, 5.1, 5.2, 5.3, 5.6.

9: 31 Mar: Relations. Incidence matrices. Reflexive, irreflexive, symmetric, antisymmetric and transitive relations. Enumeration of relations using incidence matrices, 7.1, 7.2.

Midterm: Wednesday, 2 April, at 18:00.

10: 7 Apr: Partial ordering relations and Hasse diagrams, equivalence relations, 7.3, 7.4.

11: 14 Apr: Coding theory, Hamming metric, 16.5, 16.6.

12: 21 Apr: Parity-check and generator matrics, decoding using syndromes and coset leaders, 16.7, 16.8.

13: 28 Apr: Logic, evaluation of compound statements using truth tables or laws of logic, 2.1, 2.2.

14: 5 May: Logic, Boolean algebra, gating networks, 5.1, 15.2.

15: 12 May: (Classes end 16 May.) Review of enumerative combinatorics, coding theory and Boolean algebra. More practise at exercises.

Homeworks

The due date for homeworks is two weeks after being set in class.

Homework 1

Due: Wednesday 19th February for Section 3; Friday 21st February for Sections 1, 2.

The following two exercises are, of course, practise in the art of very clear deductive explanation. You must try to produce an argument that would persuade other people in the class.

1: Suppose we have a chessboad and 32 dominoes pieces which are of such a size that each dominoes piece covers exactly two squares of the chessboard. If is not hard to see that that the 64 squares of the chessboard can be covered using the dominoes pieces. Now suppose that two opposite corners of the chessboard are removed. Is it possible to cover the remaining 62 squares using 31 dominoes pieces? (Hint: the answer is no.)
2: Let G be a graph, and let r be the number of vertices of G that have odd degree. Show that r is even.

Homework 2

Due: Friday 7 March for Sections 1 and 2, Monday 10 March for Section 3.

1: Section 4.1 Question 14 (about $n!$ and 2^n).
2: Section 4.2 Question 14 (about Lucas numbers).
3: Show that $1^3 + 2^3 + ... + n^3 = n^2(n+1)^2/4$. for all positive integers n. (This appears part of Section 4.2 Question 19, but that question has some other parts. A direct way to do this question is to adapt the argument we gave for $1^2 + 2^2 + ... + n^3$.)
4: Section 10.2 Question 24 (about a 2 times n chessboard and dominoes).

Homework 3

Due: Monday 24th March for Section 3, Tuesday 25th for Section 2, Wednesday 26th for Section 1.

1: Prove Theorem 11.4 in the book. (Euler circuits for directed graphs.)
2: 11.3.4. (Directed graphs and Euler paths again. Hint: Use Theorem 11.3.)
3: 11.4.18. (To show $n \geq 82$.)
4: 11.4.20. (Euler's characteristic formula generalized for all planar graphs.)

Homework 4

Due: Monday 28th April for Section 3, Tuesday 29th for Section 2, Wednesday 30th for Section 1.

1: Section 5.3 Question 4 (about counting functions of varoius kinds).
2: Section 5.3 Question 6 (conceptual interpretation of Stirling numbers).
3: Chapter 5 Supplementary Question 2 (abstract functions, seven parts, proof our counter-example).
4: Chapter 5 Supplementary Question 16 (three people with ten chores).

<u>Homework Solutions</u>

Solutions to all homework questions were discussed in class. Below, we record solutions only for the first homework.

Solutions to Homework 1

Solution 1: Recall that the 64 squares of a chessboard can be coloured in such a way that each square is black or white, each square coloured differently from its 2 or 3 or 4 nearest neighbours. Half of the squares are black, the other half white. The two removed squares are of the same colour. So the mutilated board has more squares of one colour than of the other colour. But each dominoes piece would cover one black square and one white square. So the answer to the question is: No. □

Comment: The response "No", without any justification, would be of no help at all to your reader, since you would have supplied her with no good reason for believing that your answer is correct.

Solution 2: Let e be the number of edges of G. Cutting each edge at the middle, then each vertex x now has $d(x)$ half-edges attached to it. The number of half-edges is

$$2e = \sum_{x \in V(G)} d(x) \ .$$

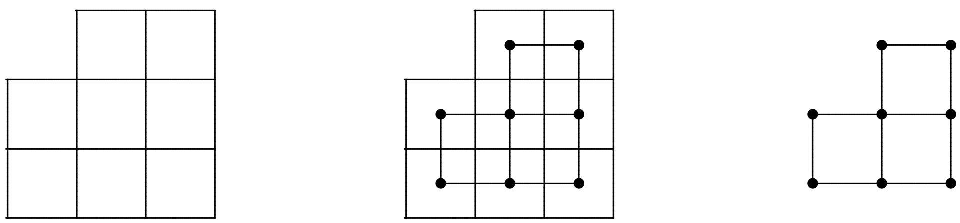
The sum is even number, and r is the number of vertices x such that $d(x)$ is odd, so r must be even. □

Section 1 Quiz Questions

Section 1, Quiz 1, set Wednesday 12 February:

One version of the problem is as follows. Consider the diagram below on the left, with 8 squares and 10 boundaries between adjacent squares. We are to move a chess-piece from square to square, always moving horizontally or vertically from one square to an adjacent square. Can we do this in such a way that each boundary is crossed exactly once and the chess-piece finishes on the square where it started?

Another way of expressing the problem is as follows. Replacing each square with a vertex and replacing each boundary with an edge, as indicated in the middle diagram, we obtain the graph shown on the right. The problem becomes: can we make a tour of the graph using each edge exactly once, finishing at the vertex where we started?



Section 1, Quiz 2, set Friday 28 February:

Solve the recurrence relation $x_{n+2} - 4x_{n+1} + 4x_n = 0$ with $x_0 = 1$ and $x_1 = 2$.

Section 1, Quiz 3, set Friday 7 March:

For the depicted graph, complete the depicted circuit to form an Euler circuit. [The depicted graph was a four-by-four grid in the real projective plane.]

Section 1, Quiz 4, set Friday 21 March:

Solve $x_{n+2} - 4x_{n+1} + 4x_n = 0$ with initial conditions $x_0 = x_1 = 2$. (Set as continuation of discussion of Quiz 2.)

Section 1, Quiz 5, set 18 April: Give an example of a set X and a surjection $X \to X$ that is not a bijection.

Section 1, Quiz 6, set 25 April: How many reflexive antisymmetric relations are there on $\{1, 2, 3, 4\}$?

Section 1, Quiz 7, set 2 May: Evaluate the encoding $E(000)$ for the given generating matrix. (Context in class. The answer was 000000.)

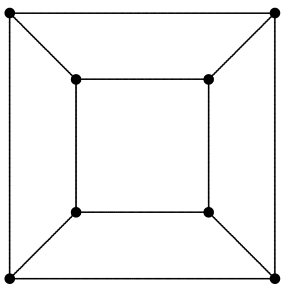
Section 1, Quiz 8, set 9 May: Decode 100100 using the given coset leaders. (Context in class.)

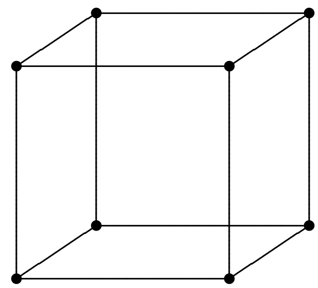
Section 1, Quiz 9, set 16 May: Evaluate the binomial coefficient $\binom{9}{3}$. (Trivial, for relief.)

Section 2 Quiz Questions

Section 2, Quiz 1, set Friday 14 February:

The following two diagrams are two different depictions of the same graph. The graph has 8 vertices and 12 edges. Show that there is no way of adding two more edges so as to produce a graph with a path that uses every edge exactly once.





Section 2, Quiz 2, set Tuesday 25 February:

When n is odd and $n \geq 3$, what is the minimum number of edges that must be added to an n-cube to obtain a graph with an Euler path?

Section 2, Quiz 3, set Friday 7th March:

For the depicted graph, find an Euler path. [The depicted graph had 14 vertices and 27 edges. Exactly two of the vertices had odd degree.]

Section 2, Quiz 4, set Tuesday 18th March:

Solve the recurrence relation $x_{n+2} - 8x_{n+1} + 16x_n = 0$ with $x_0 = 2$ and $x_2 = 17$.
(Second attendence check at end of two-hour class.)

Section 2, Quiz 5, set 15 April:

For sets X_0, ..., X_n and injections $f_i : X_{i-1} \to X_i$, show that the composite $f_n \circ ... \circ f_1$ is injective.

Section 2, Quiz 6, set 25 April:

How many transitive relations on $\{1, 2\}$ are there?

Section 2, Quiz 7, set 2 May:

Evaluate $\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ as a product of matrices over $\mathbb{Z}_2$.

Section 2, Quiz 8, set 13 May:

For the coding scheme under discussion, what is the rate? (Context in class. The message words and received words were of lengths 3 and 6, respectively.)

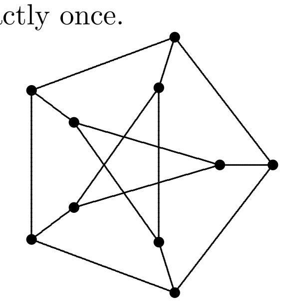
Section 2, Quiz 9, set 16 May:

Evaluate the binomial coefficient $\binom{8}{4}$. (Trivial, for relief.)

Section 3 Quiz Questions

Section 3, Quiz 1, set Monday 17 February:
Let G_1 and G_2 be graphs which do not have any vertices in common. Suppose that G_1 has a path such that the start is the same as the finish and every edge is used exactly once. Suppose that G_2 has the same property. Let x_1 and y_1 be distinct vertices of G_1. Let x_2 and y_2 be distinct vertices of G_2. Let G be the graph consisting of the vertices and edges of G_1 and G_2 together with two further edges, one of those edges joining x_1 to x_2, the other joining y_1 to y_2. Show that G does not have a path using every edge exactly once.

Section 3, Quiz 2, set Wednesday 19 February:
The following diagram depicts a graph called the **Peterson graph**. Show that there is no way of adding 3 edges to this graph so as to obtain a graph with a path that uses every edge exactly once.



Section 3, Quiz 3, set Wednesday 5 March:
Find an Euler circuit for the complete graph K_7. [The graph was depicted.]

Section 3, Quiz 4, set Monday 17 March:
Consider the recurrence relation $x_{n+2} + x_{n+1} + x_n = 0$.
(a) Show that $x_{n+3} = x_n$ for all n.
(b) Write down a formula for x_n in terms of the complex numbers $\omega = (-1 + i\sqrt{3})/2$ and $\omega^2 = (-1 - i\sqrt{3})/2$.

Section 3, Quiz 5, set 21 April: If $f : Y \leftarrow X$ and $g : X \leftarrow Y$ are functions such that $g \circ f$ is bijective, must $f \circ g$ be bijective?

Section 3, Quiz 6, set 28 April: How many reflexive transitive relations on $\{1, 2\}$ are there?

Section 3, Quiz 7, set 5 May: Review the proof that, given an equivalence relation and intersecting equivalence classes $[y]$ and $[x]$, then $[y] \subseteq [x]$. (Notation from discussion in class immediately prior to quiz.)

Section 3, Quiz 8, set 12 May: Decode the received word 10101 using the decoding table on the board. (Context in class.)

Section 3, Quiz 9, set 14 May: Evaluate the binomial coefficient $\binom{15}{2}$. (Trivial, for relief.)

Quiz Solutions

All quiz solutions were presented in class. Initial acclimatizion to the style being vital, we record solutions to some early quizzes.

Section 1 Quiz 1 Response without using Euler Path Theorem: If such a path exists then, for each vertex x, the number of entries to x is equal to the number of exits from x. So the degree of every vertex must be even. But two of the vertices have degree 3. So no such path exists.

Quicker Response using Euler Path Theorem: Two of the vertices have odd degree so, by the Euler Path Theorem, there is no Euler circuit.

Section 1 Quiz 2: By theorem, $x_n = (A + nB)2^n$. Solving, $A = 1$ and $B = 0$. So $x_n = 2^n$.

Section 1 Quiz 3: [Diagrammatic solution.]

Section 1 Quiz 4: Similar to Quiz 2. Answer $x_n = (2 - n)2^n$.

Section 1 Quiz 5: Letting $X = \mathbb{N}$, there is a surjection given by $0 \mapsto 0$ and $n + 1 \mapsto n$ for $n \in \mathbb{N}$.

Section 1 Quiz 6: For elements $i < j$ of $\{1, 2, 3, 4\}$, the (i, j)-entry and (j, i)-entry of the incidence matrix cannot both be 1. So there are 3 choices associated with (i, j). There are $1 + 2 + 3 = 6$ such pairs (i, j), so the answer is $3^6 = 243$.

Section 1 Quiz 7: Answer 000000.

Section 1 Quiz 8: Syndrom 001, coset leader 001000, codeword 101100, message word 101.

Section 1 Quiz 9: Answer $9.8.7/3.2 = 84$.

Section 2 Quiz 1 Response without using Euler Path Theorem: If a graph has such a path then, for every vertex x which is not a start or a finish, the number of times we enter x is equal to the number of times we exit x, hence the degree of x must be even. In particular, the graph cannot have more than two vertices with odd degree. But, after adding two edges to the graph depicted above, at least four of the vertices must have odd degree.

Quicker Response using Euler Path Theorem: After adding the edges, the number of vertices with odd degree is at least 4. So, by the Euler Path Theorem, there is no Euler path.

Section 2 Quiz 2: There are 2^n vertices, all with odd degree. Joining each vertex to exactly one other (its opposite, for instance) would require 2^{n-1} new edges. Omitting one of those new edges yields a graph with exactly 2 vertices with odd degree. Omitting more than one of those new edges yields a graph with at least 4 vectices with odd degree. So the answer is $2^{n-1} - 1$.

Section 2 Quiz 3: [Diagrammatic solution.]

Section 2 Quiz 4: Have $x_n = (A + nB)2^n$ by theorem. Solving, $x_n = (2 - 15n/32)4^n$.

Section 2 Quiz 5: The case $n = 1$ is trivial. The case $n = 2$ was established just prior to the quiz. For $n \geq 3$, inductively assuming the case for $n - 1$, we deduce that the composite $f_n \circ \dots \circ f_1 = f_n \circ (f_{n-1} \circ \dots \circ f_1)$ is injective.

Section 2 Quiz 6: Let $\sim$ be a non-transitive relation on $\{1, 2\}$. Then there must exist $x, y, z \in \{1, 2\}$ such that $x \sim y$ and $y \sim z$ and $x \not\sim z$. It follows that $x \neq y$ and $y \neq z$, hence $x = z$. It is now clear that the two non-diagonal entries of the incidence matrix must be 1, and one or both of the other two entries must be 0. Thus, there are exactly 2 non-transitive relations, and $16 - 2 = 14$ transitive relations.

Section 2 Quiz 7: Answer [0 1 1 0].

Section 2 Quiz 8: Answer $6/3 = 1/2$.

Section 2 Quiz 9: Answer $8.7.6.5/4.3.2 = 70$.

Section 3 Quiz 1 Response without using Euler Path Theorem: Every vertex of G_1 has an even number of edges in G_1 because of the existence of a path in G_1 as specified. Indeed, for each vertex, the number of times the path enters the vertex must be equal to the number of times the path exits. Similarly, every vertex of G_2 has an even number of edges in G_2. It follows that, x_1, y_1, x_2, y_2 all have an odd number of edges in G. If G has a path using every edge exactly once then, by the argument we have just given, x_1, y_1, x_2, y_2 must all be starts or finishes. But that is impossible.

Quicker Response using Euler Path Theorem: By the Euler path theorem, all of the vertices of G_1 have even degree in G_1, likewise for G_2. So exactly 4 of the vertices of G have odd degree in G, namely x_1, y_1, x_2, y_2. By the Euler Path Theorem, G has no Euler path.

Section 3 Quiz 2 Response without using Euler Path Theorem: After adding 3 edges, at least 4 of the vertices will have odd degree. If a path as specified exists, then every vertex which is not a start or finish must have even degree, since the number of entries into that vertex must be the same as the number of exits. So such a path cannot exist.

Quicker Response using Euler Path Theorem: The argument is the same as for Section 1 Quiz 1.

Section 3 Quiz 3: [Diagrammatic solution.] (Summary: It is not hard to find three separate cycles. The Euler circuit can be formed by concaternating those cycles.)

Section 3 Quiz 4: Part (a) holds by subsituting $x_{n+2} = -(x_{n+1} + x_n)$ into $x_{n+3} = -(x_{n+2} + x_{n+1})$. Alternatively, part (b) yields part (a) immediately. For part (b), $x_n = A\omega^n + B\omega^{2n}$ for some A and B, since ω and ω^2 are the solutions to $t^2 + t + 1 = 0$.

Section 3 Quiz 5: Any example with $|X| = 1 < |Y|$ is a counter-example.

Section 3 Quiz 6: All 4 reflexive relations on $\{1, 2\}$ are transitive. So answer is 4.

Section 3 Quiz 7: We use the symmetry condition throughout the argument. Choose $z \in [y] \cap [x]$. Then $y \equiv z$ and $z \equiv x$, hence $y \equiv x$. For all $w \in [y]$, we have $w \equiv y$. Transitivity yields $y \equiv z$, in other words, $w \in [z]$.

Section 3 Quiz 8: Answer 001.

Section 3 Quiz 9: Answer is $15.14/2 = 105$.

Practise Midterm

Please make sure that:
• Your name is on every sheet of your script.
• Your answers to each of the four questions are on separate sheets of paper. You may use more than one sheet for a question. Sheets carrying answers to parts of two separate questions may not be fully marked.
• Your handwriting is legible and not very faint. Do not use a red pen.

1: 20% (a) Show that $\sum_{i=1}^{n} i(i+1) = n(n+1)(n+2)/3$ for all $n \geq 1$.

b: Guess a formula for $\sum_{i=1}^{n} i(i+1)(i+2)$ and then prove it by induction.

2: 30% (a) Let $y_n = n^3$ for $n \geq 0$. Show that $y_{n+2} - 4y_{n+1} + 4y_n = n^3 - 6n^2 + 4$.

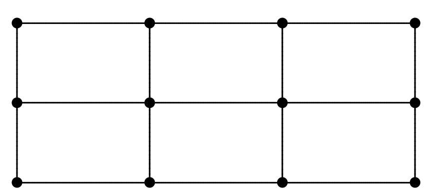
(b) Find the general solution to the recurrence relation $z_{n+2} - 4z_{n+1} + 4z_n = 0$.

(c) Using parts (a) and (b), solve the recurrence relation $x_{n+2} - 4x_{n+1} + 4x_n = n^3 - 6n^2 + 4$ with initial conditions $x_0 = 2$ and $x_1 = 7$.

3: 30% (a) State and prove Euler's theorem concerning the number of edges n, the number of edges e and the number of faces f for a connected planar graph.

(b) Let n be an integer with $n \geq 4$. Suppose that G is a connected planar graph with n vertices and $3n - 6$ edges. Show that every face has precisely 3 edges.

4: 20% For integers $1 \leq m \leq n$, let $G_{m,n}$ be the graph defined as follows. The vertices are $v_{i,j}$, where i and j are integers such that $1 \leq i \leq m$ and $1 \leq j \leq n$. There is an edge between two vertices $v_{i,j}$ and $v_{i',j'}$ if and only if $|i - i'| + |j - j'| = 1$. For example, the graph $G_{3,4}$ has a diagram as shown.



(a) For which m and n does $G_{m,n}$ have an Euler circuit? (Say what general criterion you are using.)
(b) For which m and n does $G_{m,n}$ have an Euler path that is not a circuit? (Say what general criterion you are using.)
(c) What is the minimum number of edges that can be removed from $G_{3,4}$ to produce a graph that has an Euler path that is not a circuit?

Further Exercises for Midterm

(Solutions to the Practise Midterm, and also solutions to some of the following exercises, were discussed in class, selection according to requests from the students.)

1: Show that, for all natural numbers m and n, we have $2F_{m+n} = F_m L_n + F_n L_m$, where F_0, F_1, ... are the Fibonacci numbers and L_0, L_1, ... are the Lucas numbers.

2: (a) Find the real numbers e and f such that the recurrence relation $x_{n+2} + ex_{n+1} + fx_n = 0$ has a solution $x_n = 123456789(\sqrt{2}+1)^n - 987654321(\sqrt{2}-1)^n$.

(b) Give an example of two non-real complex numbers α and β and four real numbers g, h, A, B such that the recurrence relation $x_{n+1} + gx_{n+1} + hx_n = 0$ has solution $x_n = A\alpha^n + B\beta^n$ and each x_n is a real number.

3: Let $x_1, x_2, ...$ be an infinite sequence of complex numbers such that $x_1 = x_2 = 0$ and

$$x_{n+2} - 2ix_{n+1} - x_n = 1 - in\,.$$

Find a formula for x_n. (Hint: First try $x_n = (n-1)/2$; but note that, unfortunately, this does not satisfy the initial conditions.)

4: Solve the recurrence relation $x_n = 1 + x_0 + x_1 + ... + x_{n-1}$ with the initial condition $x_0 = 1$. (Hint: work out the first few values of x_n, guess the solution, then prove it by induction. Make sure you are clear about what the inductive assumption is.)

5: (a) The Fibonacci numbers are defined by the recurrence relation $F_{n+2} = F_{n+1} + F_n$ with $F_0 = 0$ and $F_1 = 1$. Find a general formula for F_n.

(b) Solve the recurrence relation $x_{n+2} = x_{n+1} + x_n + 2n(n+2)(n!)$ with the initial condition $x_0 = 0$ and $x_1 = 1$. (Hint: First experiment with the trial solution $x_n = 2(n!)$.)

6: Show that $\sum_{i=0}^{n}(2i+3)3^i = (n+1)3^{n+1}$.

7: (a) State (without proof) Euler's criterion for a connected graph to have (1) an Euler path, (2) an Euler circuit.

(b) Let G be a connected graph with at least one edge and such that the vertices of G all have the same degree and G has an Euler path but no Euler circuit. How many vertices does G have?

(c) A four-dimensional geometric figure called the **24-cell** has 24 vertices, 96 edges, 96 two-dimensional faces and 24 three-dimensional faces. The vertices and edges form a connected graph such that all the vertices have the same degree. Does this graph have an Euler circuit?

8: We define a **multigraph** in the same way that we define a graph, except that we no longer require there to be at most one edge between any two given points. We say that a multigraph is **planar** provided it can be drawn on the plane with no intersecting edges. For example, the

Seven Bridges of Königsberg Problem can be expressed in terms of a planar multigraph with 4 vertices, 7 edges and 5 faces. Let G be a connected multigraph with n vertices, e edges and f faces. Prove that $n - e + f = 2$ by adapting the well-known inductive proof of this equation for ordinary graphs. You may assume the preliminary lemma which asserts that $n = e + 1$ for a tree.

9: (a) State and prove a result about the minimum degree of a vertex of a planar graph. (You may assume Euler's formula for planar graphs.)

(b) On desert planet, all the land is divided up into countries (each country being one continuous region.) Show that, with six colours, it is (mathematically) possible to paint the sand so that each country has sand of just one colour, and there is a change of colour at every border.

10: Let G be a graph with n vertices, e edges and c components. (A graph with c components is a disjoint union of c connected graphs.) Prove that G is a forest with c components if and only if $n = e + c$. (A forest, recall, is a disjoint union of trees.)

12: Let G be a connected planar graph. Suppose that it is possible to remove m edges from G so that the remaining graph is a forest with r connected components. (A forest is a graph such that every connected component is a tree.) Express the number of faces of G in terms of m and r. Justify your answer, clearly stating any standard results that you use.

13: (a) State Euler's Criterion for a connected graph to have an Euler path. (You do not need to give a proof of this.)

(b) Give a similar criterion for a connected graph to have a path that uses each edge an odd number of times. Justify your answer using part (a).

14: Let G be a connected planar graph such that every vertex has the same degree. Suppose that G can be drawn on the plane in such a way that every face has four edges.

(a) Show that $2f = e$.

(b) Show that $n = 4$ or $n = 8$, where n is the number of vertices.

(c) 5% Give an example of such a graph with $n = 4$ and an example with $n = 8$.

Summary of syllabus material for Final revision

Laurence Barker, Bilkent University
version: 11 June 2014

While writing this draft of the syllabus for the Final exam, I do not have the Grimaldi textbook to hand, so I omit the section numbers. Actually, looking up topics in the index and table of contents of a book is a vital academic skill, so it is best for me to leave that to you anyway.

Final syllabus Midterm prerequisites: The following topics are prerequisite. Some questions in the Final may involve this material, though not as a main focus.

• Mathematical induction.
• Recurrence relations (especially homogeneous second order recurrence relations for infinite sequences).
• Graph theory (general basics, Euler paths, planar graphs).

Final syllabus main topics: The Final questions will concentrate on the following topics, which were not tested in the Midterm.

• Binomial coefficients
• Functions
• Relations
• Coding theory

Below are some revision notes for the final syllabus. You will notice that I have written "(Why?)" in many places. Memorizing the statements would be useless. Memorizing the answers to the questions "(Why?)" would be useless. Memorizing the definitions would be better than nothing, but still not very useful. To do well in the exam, you need to *digest* the definitions. (Yes, as in digesting food.) A good way of digesting the definitions is to make sure you fully *understand* the proofs of the answers to the "(Why?)" questions below. Understanding means knowing it well enough to be able to explain it without looking at the textbook. Of course, if you understand the concepts, then you can answer other questions about them. Moreover, with understanding, there is no need to memorize anything.

Binomial coefficients

Recall, for an integer r and a natural number n, we define

$$\binom{n}{r} = \begin{cases} n!/r!(n-r)! & \text{when } 0 \le r \le n, \\ 0 & \text{otherwise.} \end{cases}$$

As another characterization, given a set S with size n, then $\binom{n}{r}$ is the number of subsets of S that have size r. (Why?)

Pascal's Formula: given $1 \le r \le n$, we have $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$. (Conceptual proof. Another proof by calculation.)

Binomial theorem. (Conceptual proof. Another proof by induction.)

Given positive integers m and n, then the number of ways of putting m indistinguishable balls into n distinguishable boxes is $\binom{m+n-1}{m} = \binom{m+n-1}{n-1}$. (Why?)

How many natural number solutions x_1, ..., x_m are there to $x_1 + ... + x_m = n$? When $m \geq n$, how many positive integer solutions x_1, ..., x_m are there?

Functions

Injective, surjective and bijective functions. Composition of functions preserves injectivity, surjectivity, bijectivity. (Why?)

Let M and N be finite sets with sizes m and n, respectively. How many functions $M \to N$ are there? When $m \leq n$, how many injective functions $M \to N$ are there?

For $m \geq n$, the Stirling number $S(m, n)$ is defined to be the number of ways of putting m distinguishable balls into n distinguishable boxes such that each box has at least one ball. We can calculate $S(m, n)$ using the recurrence relation $S(m, 1) = S(m, m) = 1$ and

$$S(m, n) = S(m-1, n-1) + nS(m-1, n)$$

for $1 \leq n < m$. (Why?)

When $m \geq n$, the number of surjective functions $M \to N$ is $n!S(m, n)$. (Why?)

Relations

Reflexive, symmetric, antisymmetric, transitive relations. Counting relations using the incidence matrix.

Partial orderings. Hasse diagrams. Proving that a given relation is a partial ordering.

Equivalence relations. Proving that a given relation is an equivalence relation.

For M and N as above, the number of equivalence relations on M that have N equivalence classes is $S(m, n)$. (Why?)

Coding theory

The Hamming metric on the set $\mathbb{Z}_2^m$ with length m. The rate of a code. The number of single-digit errors that can be detected or corrected.

Given a word w in $\mathbb{Z}_2^m$, then $|\{x \in \mathbb{Z}_2^m : d(w, x) = r\}| = \binom{m}{r}$. (Why?)

Linear codes. The number of errors that can be detected is one less than the weight of a nonzero codeword. (Why?)

Decoding tables for linear codes. Technique for decoding using syndromes.

Practise Exercises for Final

1: Find a positive integer n such that, given any set N with size $|N| = n$, then the number of reflexive relations on N is equal to the number of symmetric relations on N. (You may use any standard results, provided you are clear about which results you are using.)

2: Let m and n be positive integers with $m \leq n$. Let M and N be sets with sizes $|M| = m$ and $|N| = N$.
(a) How many injective functions $M \to N$ are there? (Explain your reasoning.)
(b) Let $f : M \to N$ and $g : N \to M$ be functions such that $g(f(x)) = x$ for all $x \in M$. Show that f is injective and g is surjective.
(c) Let f be an injective function $M \to N$. How many functions $g : N \to M$ are there such that $g(f(x)) = x$ for all $x \in M$?

3: Let n be any positive integer and let N be a set with size $|N| = n$.
(a) Explain why, for any integer m in the range $0 \leq m \leq n$, the number of subsets of size m in N is equal to the binomial coefficient $\binom{n}{m} = \dfrac{n!}{m!(n-m)!}$.
(b) Let $n_1, n_2, ..., n_k$ be natural numbers such that $n = n_1 + n_2 + ... + n_k$. Show that there are exactly $n!/n_1!n_2!...n_k!$ ways of choosing subsets N_1, N_2, ..., N_k such that each $|N_i| = n_i$ and $N_1 \cup N_2 \cup ... \cup N_k = N$.
(c) Give a counter-example to the following statement: *There are exactly $n!/k!\,n_1!\ldots n_k!$ equivalence relations $\equiv$ on N such that $\equiv$ has exactly k equivalence classes and the equivalence classes have sizes n_1, n_2, ..., n_k.*

4: Let a be any positive integer, let $n = 3a$ and let N be a set with size $|N| = n$.
(a) How many equivalence relations $\equiv$ on N are there such that all the equivalence classes of $\equiv$ have size a?
(b) Let d be the Hamming metric on the set $\mathbb{Z}_2^n$ (the set of binary strings with length n). How many triples (x, y, z) of elements $x, y, z \in \mathbb{Z}_2^n$ are there such that $d(x, y) = d(y, z) = 2a$?
(c) How many of the triples (x, y, z) satisfy $d(x, y) = d(y, z) = d(x, z) = 2a$?

5: Consider the encoding function with generating matrix $G = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$.
(a) Write down the codewords for each of the 8 message words.
(b) Find the parity-check matrix H, and find the syndromes for each of the 8 strings

000000 , 100000 , 010000 , 001000 , 000100 , 000010 , 000001 , 000011 .

(c) Explain why the 8 strings in part (b) can be used as the coset leaders. (Do not forget to explain why the minimal weight condition for selecting coset leaders is satisfied.)
(d) Taking the 8 strings in part (b) as the coset leaders, use the method of calculating syndromes to decode the received words 111111, 011111, 001001.
(e) Write down only the following part of the decoding table: the top line of consisting of the message words, the next line consisting of the codewords, and the last line (which begins with the coset leader 000011).
(f) Instead of 000011, which other strings could be used as the last coset leader?

(g) For this code, how many single-digit errors of transmission can be detected? How many single-digit errors of transmission can be corrected?

6: For positive integers m and n with $m \geq n$, we define the Stirling number $S(m, n)$ be the number of equivalence relations on $\{1, 2, ..., m\}$ that have n equivalence classes. All your arguments in this question must be deduced from that definition. (You may not use any general formulas without proof.)

(a) Show that $S(m+1, n) = S(m, n-1) + nS(m, n)$ for all integers $m \geq n \geq 1$.

(b) Briefly explain why $S(m, 1) = 1 = S(m, m)$ for all integers $m \geq 1$.

(c) Using mathematical induction and parts (a) and (b), show that $S(n+1, n) = n(n+1)/2$ for all integers $n \geq 1$.

(d) Directly from the above definition of $S(m, n)$, without using mathematical induction, give another proof that $S(n+1, n) = n(n+1)/2$.

(e) Using parts (a) and (b), evaluate $S(7, 4)$.

Solutions to Practise Exercises for Final

Reminder: Of course, there are no "model solutions" to mathematical questions. Often, a conclusion can be justified in many different ways, always, an argument can be expressed in many different styles.

Solution 1: On N, the number of reflexive relations is $2^{n(n-1)}$. The number of symmetric relations is $2^{n(n+1)/2}$. Those two numbers are equal when $n = 3$.

Solution 2: Part (a). Enumerate $M = \{x_1, ..., x_n\}$. To choose an injection f, there are n choices for $f(x_1)$, then $n-1$ choices for $f(x_2)$, and so on. Finally, there are $n-m+1$ choices for $f(x_m)$. So there are $n(n-1)...(n-m+1) = n!/(n-m)!$ injections $M \to N$.

Part (b). Given $x_1, x_2 \in M$ such that $f(x_1) = f(x_2)$, then $x_1 = g(f(x_1)) = g(f(x_2) = x_2$. So f is injective. Each element $x \in M$ is the image of $f(x)$ under g, so g is surjective.

Part (c). To choose such g, the values at elements of the image of f are already determined but, for each of the $n-m$ elements y of M that are not in the image, there are m choices for $g(y)$. So the number of choices for g is m^{n-m}.

Solution 3: Part (a). To choose, in order, m mutually distinct elements x_1, ..., x_m of N, there are n choices for x_1, then $n-1$ choices for x_2, and so on, finally $n-m+1$ choices for x_m. So there are $n(n-1)...(n-m+1) = n!/(n-m)!$ choices altogether. By the same argument, for each subset M of N with size m, there are $m!$ was of ordering the elements of M. So there are $n!/m!(n-m)!$ choices of M.

Part (b). By part (a), there are $\begin{pmatrix} n \\ n_1 \end{pmatrix}$ choices for N_1, then $\begin{pmatrix} m_1 \\ n_2 \end{pmatrix}$ choices for N_2, generally, $\begin{pmatrix} m_{j-1} \\ n_j \end{pmatrix}$ choices for N_k, where $m_j = n - n_1 - ... - n_j$. We have

$$\begin{pmatrix} n \\ n_1 \end{pmatrix} \begin{pmatrix} m_1 \\ n_2 \end{pmatrix} ... \begin{pmatrix} m_{k-2} \\ n_{k-1} \end{pmatrix} \begin{pmatrix} m_{k-1} \\ n_k \end{pmatrix} = \frac{n!}{n_1!m_1!} \cdot \frac{m_1!}{n_2!m_2!} \cdots \frac{m_{k-2}!}{n_{k-1}!m_{k-1}!} \cdot \frac{m_{k-1}!}{n_k!m_k!} = \frac{n!}{n_1!...n_k!} \ .$$

Part (c). Put $k = 2$ and $n_1 = 1$ and $n_2 = 2$. Then the number of equivalence classes is 3, whereas $n!/k!n_1!...n_k! = 3!/2!1!2! = 3/2$.

Solution 4: Part (a). As a special case of the formula stated in part (a) of Question 3, the number of ways of choosing the three equivalence classes, N_1, N_2, N_3, in some order, is $(3a)!/(a!)^3$. For each of the equivalence relations $\equiv$, there are $3! = 6$ ways of ordering the three equivalence classes. So there are $(3a)!/3!(a!)^3$ equivalence relations $\equiv$ as specified.

Part (b). We write $x = x_1...x_n$ where each $x_i \in \{0, 1\}$. Recall, $d(x, y) = |\{i : x_i \neq y_i\}|$. There are 2^n choices for y. For each y, the number of choices for x is $\begin{pmatrix} n \\ 2a \end{pmatrix}$, because that is the number of ways of choosing the $2a$ indices i for which $x_i \neq y_i$. Similarly, the number of choices for z is $\begin{pmatrix} n \\ 2a \end{pmatrix}$. So the number of choices for (x, y, z) is

$$2^n \begin{pmatrix} n \\ 2a \end{pmatrix}^2 = 2^{3a}((3a)!/a!(2a)!)^2 \ .$$

Part (c). There are 2^n choices for x. For each x, choosing y and z amounts to choosing the three sets $\{i : x_i \neq y_i = z_i\}$ and $\{i : x_i \neq y_i \neq z_i\}$ and $\{i : x_i = y_i \neq z_i\}$. Each of those three sets has size a, they are mutually disjoint and their union is the whole set of indices i.

The argument in part (a) shows that there are $(3a)!/(a!)^3$ ways of choosing those three sets (in order, of course). So the number of choices for (x, y, z) is now

$$2^n \binom{n}{2a} \binom{2a}{a} = 2^{3a} \frac{(3a)!}{(a!)^3} \,.$$

5: Part (a). Each message word w has codeword wG. The message words 000, 001, 010, 011, 100, 101, 110, 111 have corresponding codewords 000000, 001101, 010111, 011010, 100110, 101011, 110001, 111100.

Part (b), $H = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$, syndromes 000, 110, 111, 101, 100, 010, 001, 011.

Part (d). Decodings are 111, 010, 001. The calculation is expressed in the following table, where r denotes the received word, s the syndrome, v the coset leader, $c = r + v$ the codeword, w the message word.

r	s	v	c	w
111111	011	000011	111100	111
011111	101	001000	010111	010
001001	100	000100	001101	001

Part (e). The specified part of the decoding table is as follows.

message words	000	001	010	011	100	101	110	111
codewords	000000	001101	010111	011010	100110	101011	110001	111100
last line	000011	001110	010100	011001	100101	101000	110010	111111

Part (f). The only other possible strings for the last coset leader are 010100 and 101000. Indeed, all the other strings in the last line have greater weight.

Part (g). The minimal weight of a nonzero codeword is 3, so 2 errors are detectable, 1 error is correctable.

6: Part (a). To choose an equivalence relation on $\{1, ..., m+1\}$ with n equivalence classes, we can either choose an equivalence relation on $\{1, ..., m\}$ with $n-1$ classes, then put $m+1$ in a class of its own, or else we can choose an equivalence relation on $\{1, ..., m\}$ with n classes, then put $m+1$ in one of those classes. In the former case, there are $S(m, n-1)$ choices. In the latter case, bearing in mind that there are n possible classes in which to place $m+1$, there are $nS(m, n)$ choices.

Parts (b). There is only one equivalence relation on $\{1, ..., m\}$ such that all the elements of $\{1, ..., m\}$ are equivalent to each other, and there is only one equivalence relation such that no two distinct elements are equivalent.

Part (c). Let $s(n) = S(n+1, n)$. By parts (a) and (b), $s(1) = 1$ and $s(n) = s(n-1) + n$ for all $n \geq 2$. To prove that $s(n) = n(n+1)/2$, we shall argue by induction on n. The case $n = 1$ is trivial. Now suppose that $n \geq 2$ and that the required formula holds for $s(n-1)$. Thus, $s(n-1) = n(n-1)/2$. We deduce that $s(n) = n(n-1)/2 + n = n(n+1)/2$, as required.

Part (d). The integer $S(n+1, n)$ is the number of equivalence relations such that all the classes have size 1 except for one class of size 2. The number of ways of choosing the class of size 2 is $\binom{n+1}{n} = n(n+1)/2$.

Part (e). By straightforward recursive calculation, $S(7, 4) = 350$.

MATH 132: DISCRETE AND COMBINATORIAL MATHEMATICS.

Midterm, Spring 2014, Bilkent, LJB, 2 April 2014

Time allowed: two hours.
Please put your name and section number on *every* sheet of your manuscript.
Submit one sheet of paper for each question that you attempt (maximum five sheets).
No sheet may have solutions for two separate questions. (Misplaced parts will not be marked).
You may use extra sheets of paper for rough work (which does not need to be handed in).
Do not place sheets of written work in such a way that other candidates can see it.
All use of electronic devices is strictly prohibited. Please remember to silence your phone.
Please DO NOT hand in the question sheet! The examiner already knows the questions.

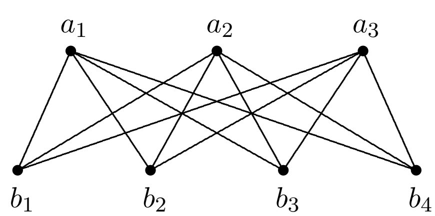
Question 1: 20% Show that, given any natural number n, then

$$1^2 + 3^2 + 5^2 + ... + (2n+1)^2 = (n+1)(2n+1)(2n+3)/3 .$$

Question 2: 20% (a) Let $y_n = n$. Show that $y_{n+2} - 6y_{n+1} + 9y_n = 4(n-1)$ for all natural numbers n.
(b) Solve $z_{n+2} - 6z_{n+1} + 9z_n = 0$ with initial conditions $z_0 = 1$ and $z_1 = 6$.
(c) Solve $x_{n+2} - 6x_{n+1} + 9x_n = 4(n-1)$ with initial conditions $x_0 = 1$ and $x_1 = 7$.

Question 3: 20% For a positive integer n, the graph K_n has n vertices and $n(n-1)/2$ edges. For which values of n is it possible to colour the edges of K_n, each edge coloured either red or blue, such that the graph with the red edges has no cycles and the graph with the blue edges has no cycles? (Do not forget to justify your answer with a very clear deductive explanation.)

Question 4: 20% For integers $1 \leq m \leq n$, the graph $K_{m,n}$ has vertices a_1, a_2, ..., a_m, b_1, b_2, ..., b_n and precisely mn edges, each edge joining a vertex a_i to a vertex b_j. The graph $K_{3,4}$ is depicted.



(a) Give a complete statement of a theorem about existence of Euler paths and Euler circuits.
(b) For which positive integers m and n, with $m \leq n$, does the graph $K_{m,n}$ have an Euler path? (The answer should be obvious from part (a). No further explanation is required.)
(c) Find an Euler path for the graph $K_{4,4}$. Specify the Euler path by listing the vertices in order. (If you try to specify the path by drawing the graph and labelling the edges, then the diagram may be difficult to read and marks may be subtracted.)

Question 5: 20% Let G be a connected planar graph. Suppose that every vertex has the same degree d and that, for every face, the edges of that face form a cycle with length c. (Helpful comment: these conditions ensure that every edge has two different faces on each side.)
(a) State, without proof, a formula relating the number of vertices n, the number of edges e, the number of faces f.
(b) By considering the pairs (ϵ, F), where ϵ is an edge on face F, explain why $2e = cf$. (The explanation may be very short, only one line. Just explain how you use those pairs.)
(c) State and prove an equation relating e amd n and d.
(d) Using parts (a), (b), (c), show that if $c \geq 6$ then $d = f = 2$.

Midterm Solutions

1: Let $S_n = 1^2 + 3^2 + ... + (2n+1)^2$ and $T_n = (n+1)(2n+1)(2n+3)/3$. We shall show, by induction, that $S_n = T_n$ for all natural numbers n. First observe that $S_0 = 1 = T_0$. Now suppose that $n \geq 1$ and that $S_{n-1} = T_{n-1}$. We deduce that $S_n = T_n$ because

$$3(T_n - T_{n-1}) = (2n+1)\big((n+1)(2n+3) - n(2n-1)\big) = 3(2n+1)^2 = 3(S_n - S_{n-1}) . \qquad \square$$

2: Part (a). We have $y_{n+2} - 6y_{n+1} + 9y_n = n + 2 - 6(n+1) + 9n = 4n - 4$.

Part (b). The quadratic equation $t^2 - 6t + 9 = 0$ has unique solution $t = 3$. So the general solution for the recurrence relation is $z_n = (A + nC)3^n$ for some A and C. Now $1 = z_0 = A$ and $6 = z_1 = 3(A + C)$, hence $C = 1$. Therefore $z_n = (1 + n)3^n$.

Part (c). Putting $z_n = x_n - y_n$ we see that the sequence $(z_0, z_1, ...)$ satisfies the recurrence relation and the initial conditions in part (b). So $x_n = y + n + z_n = n + (1+n)3^n$.

3: We shall show that such a colouring exists if and only if $1 \leq n \leq 4$. Any such colouring realizes K_n as the disjoint union of two forests, a red forest and a blue forest, each edge of K_n belonging to one or the other of the two forests.

We need only deal with the cases $n = 4$ and $n = 5$ because, if such a colouring exists for K_{n+1}, then such a colouring also exists for K_n. Such a colouring does exist when $n = 4$ because, letting a, b, c, d be the vertices of K_4, we can take ab, bc, cd to be the edges of the red forest. Such a colouring cannot exist for K_5 because a forest with 5 vertices has at most 4 edges, whereas K_5 has 10 edges.

4: Part (a). Let G be a connected graph and let r be the number of vertices with odd degree. Then $r = 0$ if and only if G has an Euler circuit. Also, $r = 2$ if and only if G has an Euler path that is not a circuit.

Part (b). The graph $K_{m,n}$, with $m \leq n$, has an Euler path if and only if m and n are both even or $m = 2$ or $(m, n) = (1, 1)$ or $(m, n) = (1, 2)$.

Part (c). An Euler circuit: $a_1,\ b_2,\ a_4,\ b_1,\ a_3,\ b_4,\ a_2,\ b_3,\ a_1,\ b_1,\ a_2,\ b_2,\ a_3,\ b_3,\ a_4,\ b_4,\ a_1$.

5: Part (a). We have $n - e + f = 2$.

Part (b). The number of such pairs (ϵ, F) is $2e = cf$.

Part (c). As a special case of the Handshaking Lemma, $2e = nd$.

Part (d). By parts (a) and (b), $n - e + 2e/c = 2$. Rearranging and also using part (c), we have $e(c-2)/c = n - 2 = 2e/d - 2$. Since $c \geq 6$, we have $2/3 \leq (c-2)/c < 2/d$. But d is a positive integer and plainly $d \neq 1$. So $d = 2$. From (c), $n = e$ and now, form (a), $f = 2$.

Alternative solutions to question 1: The required equality can be obtained without induction by using the equality $1^2 + 3^2 + ... + (2n+1)^2 = (1^2 + 2^2 + 3^2 + 4^2 + (2n+1)^2)$ or by using the equality $1^2 + 3^2 + ... + (2n+1)^2 = \sum_k (2k+1)^2 = 4\sum_k k^2 + 4\sum_k k + \sum_k 1$ summed over $0 \leq k \leq n$.)

Comments on Common Mistakes in Midterm

Please read these comments before seeing your exam scripts. You may have queries about my comments below. You may also wish to debate the way I have judged things. That would be fine: discussion is how we teach and learn. However, I would like the discussions to start with an awareness of the points I have made. Repeating the same point more than a hundred times would not be good for my sanity.

Actually, I intend to write more detailed comments later. I am having difficulty catching up with all the things that I wish to do this week.

General Mistake: *To prove a statement P, it is not enough to show that P implies something that is true.* For example, if P is the statement "$2+2=5$" then, multiplying by 0, we see that P implies the equality $0=0$, which is true. Nevertheless, $2+2$ is not equal to 5.

When experienced mathematicians see an argument that has the above form, they consider it to be unacceptable. That is because they have lots of experience of how such arguments can often contain subtle mistakes. See, for instance, the comment on Question 5 part (d) below.

I subtracted marks for that kind of mistake — quite often in Question 1 and in Question 2 part (a) — even though I did understand perfectly well what was meant by the argument. The reason is that, in the future, when you are presenting a mathematical argument to an audience or to a critical reader, you will not be allowed to get away with an argument which looks like just a check on consistency.

1: The following attempt has some serious defects in commication:

1) $1^2+3^2+...(2n+1)^2=(n+1)(2n+1)(2n+3)/3$
Step 1 for $n=0 \implies 1^2=1.1.3/3=1$
for $n=1 \implies 1^2+3^2=10=2.3.5/3=10$ deals with formula
Step 2 $A=1^2+3^2+...(2n+1)^2=(n+1)(2n+1)(2n+3)/3=A'=4n^3+12n^2+11n+3/3$
$B=1^2+3^2+...(2n+3)^2=(n+2)(2n+3)(2n+5)/3=B'=4n^3+24n^2+47n+30/3$
$A+(2n+3)^2=B \implies B-A=(2n+3)^2$
$B'-A'=(4n^3-24n^2+47n+30)/3-(4n^3-12n^2+11n+3)/3=4n^2+12n+9=(2n+3)^2$
So $B-A=B'-A'$
$1^2+3^2+...(2n+1)^2=(n+1)(2n+1)(2n+3)/3$

Although the candidate has given evidence that he or she can privately check assertions using mathematical induction, that is not what is being tested. The solution is of low value, because it would not communicate much to anyone who cannot already do the question. Only the stronger students in the class would be able to make sense of this argument; and those students could write clearer solutions anyway.

In the line with "Step 2" in it, the sign "=" is used in three completely different ways: the first and third use of "=" is to indicate the *definitions* of A and A'. The second use of "=" has the sense of *to be proved*. The fourth use of "=" indicates a *deduction* about the value of A'. But the mathematical skill that I had to draw upon to glean those interpretations is much greater than the mathematical skill needed simply to do the question.

To express mathematal arguments clearly: *use grammatically understandable sentences.* "We define...", "Assume...", "Therefore...". Incidentally, sentences end with full-stops. No marks are awarded for punctuation. However, if your proof has no full-stops, then that may be a sign that something is amiss.

2: I wonder what the correlation is between people who have been absent from class and people who were unable to do the basic routine bit of this question, part (b)?

3: We applied the usual method for finding Euler paths and Euler circuits. The above Euler circuit for $K_{4,4}$ was obtained by first considering the circuit a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, a_1. Deleteing the edges of that circuit, the remaining graph was easy to deal with because it is a circuit. There is no need to narrate the method to justify the answer, because the correctness of the answer is easy to check directly.

4: General: A circuit is a particular kind of path. An Euler circuit is a particular kind of Euler path. Despite the emphatic warnings I gave about this during classes, many people wrongly felt that an Euler circuit cannot be an Euler path. I think this is because the Euler Path Theorem is potentially confusing if one does not read it very carefully. Moral: mathematical definitions and theorems do have to be read very carefully!

Incidentally, in part (c), I deliberately asked for an Euler path that also happens to be an Euler circuit. I hoped that this would help to prevent people from getting confused about the terminology!

If you got this wrong, then you lost only one mark in part (a), but you lost more marks in part (b) because you will have missed the main cases where the condition on m and n is satisfied.

Part (a). Many people were unable to express the theorem clearly enough to be unambiguous. For instance, "An Euler path is zero or two vertices with odd-degree". Does that mean "If there is an Euler path then zero or two vertices have odd degree", or does it mean "There is an Euler path if and only if zero or two vertices have odd degree". I subtracted 2 marks when the "if and only if" was absent or unclear.

I subtracted one mark for failing to say that this is a theorem about *graphs*, another mark for failing to mention that the graph must be *connected*.

Part (b). Since the answer is complicated, it is especially important to say clearly what the answer actually is! If you bury the answer in the middle of a complicated justification, that makes it much harder for the reader to extract it.

Part (c) seemed fairly hard to me, though many people were able to do it. My method for finding an Euler path was to follow the procedure that we did in class. I first considered the circuit a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, a_1. Deleteing the edges of that circuit, the remaining graph was easy to deal with because it is, itself, just a circuit. However, there was no need to narrate the method to justify the answer, since the correctness of the answer is easy to check directly.

5: Part (b). I did not accept "every edge has two different faces on each side" because that is just a repetition of a phrase in the question. You needed to explain, in some form or another, that you were *counting* something in two different ways.

Part (d). Many people made the above General Mistake, arguing by *assuming* that $d = f = 2$, then showing that this is consistent with the other equalities. But that does no constitute a deductive argument to conclude that $d = f = 2$. After all, the equality $d = f = 2$ is still consistent with the other equalities when we forget the condition that $c \geq 6$. Yet, if we remove the condition $c \geq 6$, then the conclusion $d = f = 2$ can be false: a counter-example is the graph of a cube.

MATH 132: DISCRETE AND COMBINATORIAL MATHEMATICS.

Midterm Makeup, Spring 2014, Bilkent, LJB, 13 May 2014

Time allowed: two hours.
Please put your name and section number on *every* sheet of your manuscript.
Submit one sheet of paper for each question that you attempt (maximum five sheets).
No sheet may have solutions for two separate questions. (Misplaced parts will not be marked).
You may use extra sheets of paper for rough work (which does not need to be handed in).
Do not place sheets of written work in such a way that other candidates can see it.
All use of electronic devices is strictly prohibited. Please remember to silence your phone.
Please DO NOT hand in the question sheet! The examiner already knows the questions.

Question 1: 20% Find real numbers a and b such that

$$1^2 + 4^2 + 7^2 + ... + (3n-2)^2 = 3n^3 + an^2 + bn$$

for all positive integers n. (Do not forget to prove that your answer is correct.)

Question 2: 20% Solve the recurrence relation $x_{n+2} - 4x_{n+1} + 4x_n = 3^n$ with $x_0 = x_1 = 0$. (Hint: Try $x_n = 3^n$. Note, however, that this solution is not correct.)

Question 3: 20% Let G be a graph with n vertices, where $n \geq 10$. Suppose that G has $n-5$ edges, no cycles and no vertices with degree 0? What is the minimum possible number of vertices with degree 1? (You may assume that a tree with m vertices has exactly $m-1$ edges. Any other results about trees must be proved.)

Question 4: 20% Let n be a positive integer. Let C_n be the graph of an n-cube. (Recall that the vertices of C_n are binary strings with length n. Two binary strings have an edge between them provided all except one of their digits are the same.
(a) For which values of n does C_n have an Euler path?
(b) For all of those values of n where there is no Euler path, what is the minimum number of edges that must be added to produce a graph that does have an Euler path?

Question 5: 20% Let G be a planar graph with $n \geq 3$.
(a) Show that G has at most $2n-4$ faces.
(b) Show that it is possible to add edges to G so as to obtain a planar graph with exactly $2n-4$ faces.

Midterm Makeup Answers

The following is just an answer key, not an illustration of a satisfactory exam response.

1: Sum is $(6n^3 - 3n^2 - 1)/2$. That is, $a = -3/2$ and $b = -1/2$.

2: Answer, $x_n = 3^n - (2 + n)2^{n-1}$.

3: Answer is 10. (Each of the 5 components is a tree with exactly 2 vertices of degree 1.)

4: For odd n with $n \geq 3$, we must add $2^{n-1} - 1$ edges. (They can join opposite corners.)

5: Use $n - e + f = 2$. Also, for a maximal planar graph, $2e = 3f$.

MATH 132: DISCRETE AND COMBINATORIAL MATHEMATICS.

Final, Spring 2014, Bilkent, LJB, 28 May 2014

Time allowed: two hours.
Please put your name and section number on *every* sheet of your manuscript.
Submit one sheet of paper for each question that you attempt (maximum four sheets).
No sheet may have solutions for two separate questions. (Misplaced parts will not be marked).
You may use extra sheets of paper for rough work (which does not need to be handed in).
Do not place sheets of written work in such a way that other candidates can see it.
All use of electronic devices is strictly prohibited. Please remember to silence your phone.
Please DO NOT hand in the question sheet! The examiner already knows the questions.

Question 1: 5% How many ways are there of putting 6 indistinguishable balls into 3 distinguishable boxes? (Answers are to be explicit, in decimal notation, for example, as 625, not as 5^4. You may use a standard formula, without proof.)

Question 2: 35% Let A and B be finite sets with sizes $|A| = 4$ and $|B| = 6$. (Answers to the following questions are to be explicit, in decimal notation. You may use standard formulas, without proof.)
(a) How many functions $A \to B$ are there?
(b) How many injective functions $A \to B$ are there? (These are sometimes called "into functions".)
(c) How many surjective functions $A \to B$ are there? (These are sometimes called "onto functions".)
(d) How many injective functions $B \to A$ are there?
(e) How many surjective functions $B \to A$ are there?

Question 3: 20% For A and B as in Question 2, let F be the set of functions $A \to B$. Let $\equiv$ be the relation on F such that, given $f, g \in F$, then $f \equiv g$ if and only if there exists a bijection $h : B \to B$ such that $f = h \circ g$.
(a) Show that $\equiv$ is an equivalence relation.
(b) How many equivalence classes does $\equiv$ have?

4: 40% Let E be the encoding function with generating matrix $G = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$.
(Yes, two of the columns of G are the same. It is not a misprint.)
(a) Find the encodings $E(w)$ for each message word w.
(b) Find the syndromes for each of the received words

000000 , 000001 , 000010 , 000011 , 000101 , 000110 , 001000 .

(c) Find a received word t such that t together with the 7 received words in (b) can be used as the coset leaders. Explain why only one such received word t exists.
(d) Using those coset leaders, decode the received words 111111 and 110011 and 100001.
(e) For this code, how many errors of transmission can be detected?
(f) Suppose that there was at most error of transmission, in other words, the Hamming distance between the transmitted word $E(w) = q$ and the received word r is $d(q, r) = 1$. Suppose the decoded word was $D(r) = 111$, where D is the decoding function using the above coset leaders. What are the possible values of the original message word w?

Final Exam Solution Key

The following is just a sketch of solutions, not an illustration of a satisfactory exam response.

Marking: Questions 2, 3, 4 as $5+5+5+5+15$, $10+10$, $5+10+5+10+5+5$.

1: $\binom{6+3-1}{2} = 28.$

2: (a) $6^4 = 1296$, (b) $6!/2! = 360$, (c) 0, (d) 0, (e) $4!S(6,4) = 24.65 - 1560$.

3: (a) Routine, (b) Number of equivalence classes is equal to number of equivalence relations on A, which is $S(4,1) + S(4,2) + S(4,3_+S(4,4) = 1+7+6+1 = 15$.

4: Codewords for 000, 001, 010, 011, 100, 101, 110, 111 are

000000, 001111, 010111, 100001, 110110, 011000, 101110, 111001.

Syndroms of given received words are $000, 001, 010, 011, 101, 110, 111$. Only possibility for t is 000100 because it is the only received word with weight 1 that has the remaining syndrome 100. Letting r denote the received word, σ the syndrome, s the coset leader, $c = r + s = E(w)$, the decodings are as shown.

r	σ	s	c	w
111111	110	000110	111001	111
110011	101	000101	110110	110
100001	000	000000	100001	100

Only one error is detectable.

Finally, the codeword for 111 is 111001. Since all the coset leaders have weight at most 2, the received word must differ from 111001 by at most 2. There is at most one error of transmission, so the correct codeword c must differ from 111001 by at most 3. In other words, the sum $x = c + 111001$ is a codeword with weight at most 3. There are exactly 3 possibilities for x, hence exactly 3 possibilities for $c = x + 111001$. The first three digits of those c are the possible message words, 111 or 011 or 100.

MATH 132: DISCRETE AND COMBINATORIAL MATHEMATICS.

Final Retake, Spring 2014, Bilkent, LJB, 5 June 2014

Time allowed: two hours.
Please put your name and section number on *every* sheet of your manuscript.
Submit one sheet of paper for each question that you attempt (maximum four sheets).
No sheet may have solutions for two separate questions. (Misplaced parts will not be marked).
You may use extra sheets of paper for rough work (which does not need to be handed in).
Do not place sheets of written work in such a way that other candidates can see it.
All use of electronic devices is strictly prohibited. Please remember to silence your phone.
Please DO NOT hand in the question sheet! The examiner already knows the questions.

Question 1: 10% How many ways are there of putting 5 distinguishable balls into 5 indistinguishable boxes? (Some boxes may be left empty.)

Question 2: 30% For a natural number m, let B_m be the number of equivalence relations on a given set with size m. It is to be understood that $B_0 = 1$.
(a) Express B_m in terms of the Stirling numbers (of the second kind).
(b) Evaluate B_1, B_2, B_3, B_4, B_5.
(c) Simplify the expression $\sum_{n=0}^{m} \binom{m}{n} B_n$.

Question 3: 20% A reflexive transitive relation is called a **preorder**. How many preorders are there on the set $\{1, 2, 3\}$? (Do not forget to explain your reasoning clearly.)

4: 40% Consider the coding scheme with generating matrix $G = \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right]$.
(a) Find the corresponding codewords for each message word.
(b) Find the syndromes for each received word of weight 1.
(c) Explain why every set of coset leaders contains all the received words of weight 1.
(d) By considering syndromes, decode 0111110 and 1111100.
(e) For this code, how many errors of transmission can be corrected? How many errors can be detected?
(f) How many possibilities are there for the set of coset leaders (just as a set of received words, not as a sequence of received words).

Solution Key to Final Retake

Question 2 marked as $10 + 10 + 10$, Question 4 as $5 + 10 + 5 + 10 + 5 + 5$.

1: After calculating Stirling numbers $S(m, n)$ using $S(m, n) = S(m - 1, n - 1) + nS(m - 1, n)$ and $S(m, 1) = S(m, m) = 1$, we have $\sum_{n=1}^{5} S(5, n) = 1 + 15 + 25 + 10 + 1 = 52$.

2: (a) $B_m = \sum_{n=1}^{5} S(m, n)$.
(b) $(B_1, B_2, B_3, B_4, B_5) = (1, 2, 5, 15, 52)$.
(c) Sum equals B_{m+1} by considering the complement of the equivalence class of a given element.

3: Consider the associated partial ordering on equivalence classes. There are 19, 9, 1 preorders in the case of 3, 2, 1 equivalence classes respectively. So the total number of preorders is $19 + 9 + 1 = 29$.

4: The codewords associated with 000, 001, 010, 011, 100, 101, 110, 111 are

0000000, 0011101, 0101011, 011011, 1000111, 1011010, 1101100, 1110001 .

The syndromes for 0000001, 0000010, ..., 1000000 are the columns of the parity-check matrix, thus, 0001, 0010, 0100, 1000, 1101, 1011, 0111. Letting r denote the received word, σ the syndrome, s the coset leader, $c = r + s = E(w)$, the decodings are as shown.

r	σ	s	c	w
0111110	1000	0001000	0110110	011
1111100	1101	0010000	1101100	110

One error can be corrected, three detected. There are $2^3.3^4.4 = 864$ possible choices for the other coset leaders. (All coset leaders have weight 2 except for the one associated with syndrome 1110. A neat way of dealing with each of the 8 new syndromes is to examine the number of ways of expressing it as a sum of syndromes of received words with weight 1. A longer but more straightforward method is to find the entries in the last 8 lines of the decoding table.)