Archive for

MATH 132, Discrete and Combinatorial Mathematics, Fall 2022

Bilkent University, Laurence Barker, 10 January 2023.

Source file: arch132fall22.tex

MATH 132, Sections 001, 002

Discrete and Combinatorial Mathematics, Fall 2022

Course specification

Laurence Barker, Bilkent University. Version: 15 September 2022.

Classes:
Section 001: Tuesdays 15:30 - 16:20, Fridays 10:30 - 12:20, room G 236.
Section 002: Tuesdays 10:30 - 11:20, Thursdays 15:30 - 17:20, room G 236.

Office Hours:
Both sections: Tuesdays 11:30 - 12:20, 16:30 - 17:20 in office room, SAZ 129.

For all students, those doing well and aiming for an A, those doing badly and aiming for a C, Office Hours is an opportunity to come and ask questions. (If you think you are likely to fail, and if you have been unable to grasp even some basics that ought to have been mastered weeks earlier, then you should first make sure you have made an effort yourself. If that has not worked, then come and ask me. If you can tell me what parts you are struggling with, then I might be able to help.)

Instructor: Laurence Barker
e-mail: barker at fen nokta bilkent nokta edu nokta tr.

Course Texts: The primary course text is:

> Kenneth H. Rosen, "Discrete Mathematics and its Applications", 8th Edition, (McGraw–Hill 2019).

Also recommended, is the following free downloadable PDF book, written in a different style:

> Oscar Levin, "Discrete Mathematics: An Open Introduction". It is available at http://discrete.openmathbooks.org/

The internet has a vast supply of text and videos on the material convered. Fully academic study involves consultation of multiple sources. One has not properly grasped material until one can follow disparate accounts by different authors who may sometimes use different notation and terminology.

Homework:
As the online platform Connect will be used for the homework in this course, please purchase your textbook only from Meteksan. The books purchased from Meteksan will come with a Connect code, which you will use to register for the Connect course. Alternatively, you can choose to purchase only a Connect code for this title from Meteksan. Those who will purchase the textbook are kindly invited to do so as soon as possible, as unsold textbooks will be returned shortly after the beginning of the semester.

Each student must purchase an individual textbook/code; codes cannot be shared, they are one time use only. Ensure that you purchase your textbook/code and complete your registration in the first 2 weeks of the semester. The registration URL is:

https://connect.mheducation.com/class/n-sahin-fall-2022

Full details on how to register to the Connect site are on the MATH 132 Moodle page.

Course Documentation: As the course progresses, further documentation will appear on the course Moodle site and my homepage.

Syllabus: *Below is a tentative course schedule.* The format of the following details is *Week number: Monday date: Subtopics (Section numbers).*

1: 12 Sept: (Classes only on Thursdays and Fridays.) Sets, functions, cardinalities. (2.1-2.5)

2: 19 Sept: Modular arithmetic, primes, greatest common divisors. (4.1-4.3)

3: 8 Feb: Congruences, applications. (4.4)

4: 26 Sept: Induction and well-ordering. (5.1-5.2)

5: 3 Oct: Counting, Pigeonhole principle. (6.1-6.2)

6: 10 Oct: Counting. (6.3-6.4)

7: 17 Oct: Generalized permutations and combinations. (6.5-6.6)

8: 24 Oct: (No classes on Thursday or Friday.) Further counting techniques. (8.1-8.2)

9: 31 Oct: Further counting techniques. (8.4)

10: 7 Nov: Further counting techniques. (8.5-8.6)

11: 14 Nov: Relations. (9.1)

12: 21 Nov: Relations. (9.5-9.6)

13: 28 Nov: Graphs. (10.1-10.3)

14: 12 Dec: Graphs. (10.4-10.6)

15: 19 Dec: Graphs. (10.7-10.8).

Assessment:

• Homework, 10%,
• Midterm, 45%,
• Final, 45%.

A Midterm score of least 20% (of the available Midterm marks) is needed to qualify to take the Final Exam, otherwise an FZ grade will be awarded.

75% attendance is compulsory.

Asking questions in class is very helpful. It makes the classes come alive, and it often improves my sense of how to pitch the material. The rule for talking in class is: if you speak, then you must speak to everyone in the room.

MATH 132: Discrete and Combinatorial Mathematics. Midterm

Examiners: Nil Şahin & LJB. Exam date: 4 November 2022

This is a reformatted record of the paper. The original was in fill-in-the-spaces form.

No books, notes or calculators of any kind are allowed.

You must give detailed mathematical explanations for all your conclusions in order to receive full credit. Each problem is worth 20 points.

The duration of the examination is 120 minutes.

1: Which of the following sets are countable?

(a) The set of real numbers x such that $1000x$ is an integer.

(b) The set of real numbers that have the form $a/10^n$ where a is an integer and n is a positive integer.

(c) The set of real numbers that have the form a/π^n where a and n are integers?

2: Find all the integers a such that $a \equiv 5 \mod 7$ and $a \equiv 6 \mod 9$ and $a \equiv 6 \mod 11$.

3: Show that $\left(\begin{array}{c} 2n \\ n \end{array} \right) < 2^{2n-2}$ for all integers $n \geq 5$.

4: Find the number of integer solutions to $x_1 + x_2 + x_3 + x_4 = 17$ where each $x_i \geq i$.

5: **(a)** Let n be a positive integer and let S be a subset of $\{1, 2, ..., 2n\}$ such that $|S| = n + 1$. Show that 2 of the elements of S have sum $2n + 1$.

(b) Express the greatest commmon divisor of 21 and 55 as a linear combination of these integers.

Solutions to Midterm

1: Part (a). Countable. It is a standard result that $\mathbb{Z}$ is countable. There is an injection f from the specified set to $\mathbb{Z}$ given by $f(x) = 1000x$.

Part (b). Countable. Let $s : \mathbb{Z} \to \mathbb{N}$ be an injection. Given an element s in the specified set, and writing $s = a/10^n$ with $a \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$ with n taken to be as small as possible, the formula $g(a/10^n) = 2^{s(a)}3^n$ defines an injection g from the specified set to $\mathbb{N}$.

Part (c). Countable. The formula $h(a/\pi^n) = 2^{s(a)}3^{s(n)}$ defines an injection h from the specified set to $\mathbb{N}$.

Comment 1: There are many alternative justifications for the conclusions in this question. For instance, the sets in (a) and (b) are countable because they are subsets of the countable set $\mathbb{Q}$.

Comment 2: A large proportion of students were unable to coherently specify mathematical objects or to deduce their properties. Consider the following hybrid of various scripts:

• We can think of $a/10^n$ as $\mathbb{Z} \times \mathbb{Z}^+$, which is countable, so $a/10^n$ is countable.

To the reader struggling to understand of that argument, it does not help that $a/10^n$, being a number, is not in the class of things that can be countable or uncountable. A bit better is:

• We can think of the set $S = \{a/10^n : a \in \mathbb{Z}, n \in \mathbb{Z}^+\}$ as $\mathbb{Z} \times \mathbb{Z}^+$, which is countable.

But *how* does one "think of S ... as $\mathbb{Z} \times \mathbb{Z}^+$"? One must relate the two sets by means of some function. And one must say what that function is. Much better is:

• Let $S = \{a/10^n " a \in \mathbb{Z}, n \in \mathbb{Z}^+\}$. There is a surjection $f : \mathbb{Z} \times \mathbb{Z}^+ \to S$ given by $f(a, n) = a/10^n$. Since the product of two countable sets is countable, $\mathbb{Z} \times \mathbb{Z}^+$ is countable. Therefore, S is countable.

2: Since $a \equiv_1 16$, we have $a = 11r + 6$ for some $r \in \mathbb{Z}$. The congruence $a \equiv_9 6$ is $11r + 6 \equiv_9 6$, in other words, $2r \equiv_9 0$, hence $r = 9s$ for some $s \in \mathbb{Z}$. Hence $a = 99s + 6$.

The congruence $a \equiv_7 5$ is $99s + 6 \equiv_7 5$, in other words, $s + 1 \equiv_7 0$. So $s = 7t + 6$ and $a = 99(7t + 6) + 6 = 693t + 600$.

In conclusion, the integers a are those having the form $a = 693t + 600$ where $t \in \mathbb{Z}$.

3: Let $A_n = \binom{2n}{n}$ and $B_n = 2^{2n-2}$. We have

$$A_5 = \frac{10.9.8.7.6}{5.4.3.2.1} = 9.4.7 = 252 < 256 = 2^8 = B_5 \, .$$

Now suppose, inductively, that $n \geq 6$ and $A_{n-1} < B_{n-1}$. We have

$$\frac{A_n}{A_{n-1}} = \frac{2n(2n-1)}{n^2} < 4 = \frac{B_n}{B_{n-1}} \, .$$

Applying the inductive assumption, we deduce that $A_n < B_n$.

Alternative: A few students discovered the following cunning alternative way of carrying out the inductive step. By two applications of Pascal's Formula,

$$\binom{2n+2}{n+1} = \binom{2n+1}{n} + \binom{2n+1}{n+1} = \binom{2n}{n-1} + 2\binom{2n}{n} + \binom{2n}{n+1} \geq 4\binom{2n}{n} \, .$$

Hence, in the notation above, $A_{n+1}/A_n < B_{n+1}/B_n$.

Comment: As usual in this course, many students failed to present the argument in a deductive way.

In an inductive argument, it is necessarly to be clear about what is being assumed, what is being deduced, what is being claimed prior to deduction. Even in simple induction, we sometimes assume the case indexed by $n-1$ and deduce the case indexed by n (as in the first above solution to Question 3), but we sometimes assume the case indexed by n and deduce the case indexed by $n+1$ (as in the above alternative solution to the question). In some applications of induction, the inductive hypothesis can be quite complicated. So it is always necessary to be clear about what exactly the inductive hypothesis is. Some students failed to do so for this question.

A common mistake in this particular question was to present an argument that appeared to assume $A_{n-1} < B_{n-1}$ and $A_n < B_n$, then to deduce a trivial assertion, such as $2n(2n-1)/n^2 < 4$. Such an approach can be made to work provided, at each point in the argument, statements not yet established are clearly distinguished from assumptions and deductions. That can be done using phrases such that "it is enough to show that" or "we shall show that".

The marking scheme for this mistake was as follows: when the argument came over clearly anyway, no marks were subtracted. When it was fairly easy for the reader to figure out where to insert phrases such as "it suffices to prove that", 2 marks were subtracted. When it was difficult for the reader to disentangle the correct flow of deductions, more marks were subtracted.

4: Making the substitution $y_i = x_i - i$, the answer is the number of natural number solutions to $y_1 + y_2 + y_3 + y_4 = 7$, which is

$$\binom{7+4-1}{7} = \binom{10}{7} = \frac{10.9.8}{3.2.1} = 10.3.4 = 120\ .$$

5: Part (a). We apply the pigeonhole principle, taking the pigeons to be the elements of S and the pigeonholes to be the sets $\{1, 2n\}$, $\{2, 2n-1\}$, ..., $\{n, n+1\}$. Since there are $n+1$ pigeons and n pigeonholes, at least one of the pigeonholes must own 2 of the pigeons.

Part (b). We have $55 = 2.21 + 13$, and $21 = 1.13 + 8$ and $13 = 1.8 + 5$. Also $8 = 1.5 + 2$ and $5 = 1.3 + 2$ and $3 = 1.2 + 1$. So $\gcd(21, 55) = 1$. Now

$$1 = 3 - 2 = 3(5-3) = 2.3 - 5 = 2(8-5) - 5 = 2.8 - 3.5 = 2.8 - 3(13-8) = 5.8 - 3.13$$

$$= 5.(21-13) - 3.13 = 5.21 - 8.13 = 5.21 - 8(55 - 2.21) = 21.21 - 8.55\ .$$

MATH 132: Discrete and Combinatorial Mathematics. Final

Examiners: Nil Şahin & LJB. Exam date: 6 January 2023

This is a reformatted record of the paper. The original was in fill-in-the-spaces form.

No books, notes or calculators of any kind are allowed.

You must give detailed mathematical explanations for all your conclusions in order to receive full credit. Each problem is worth 25 points.

The duration of the examination is 120 minutes.

1: Solve $a_{n+2} - 6a_{n+1} + 9a_n = 3.2^n$ with $a_0 = 4$ and $a_1 = 15$.

2: Let $S = \{1, ..., 8\} \times \{1, ..., 8\}$ and let $\sim$ be the relation on S such that $(a_1, b_1) \sim (a_2, b_2)$ if and only if $a_1 - b_1 \equiv a_2 - b_2 \pmod 5$.

(a) Show that $\sim$ is an equivalence relation.

(b) Determine the equivalence classes $[(1, 1)]$ and $[(1, 8)]$.

(c) How many equivalence classes does $\sim$ have?

3: Let n be a positive integer. Let G be the graph with vertices a_1, ..., a_n, b_1, ..., b_n, c_1, ..., c_n and edges $\{a_i, b_j\}$ and $\{b_j, c_i\}$ where $1 \le i \le n$ and $1 \le j \le n$.

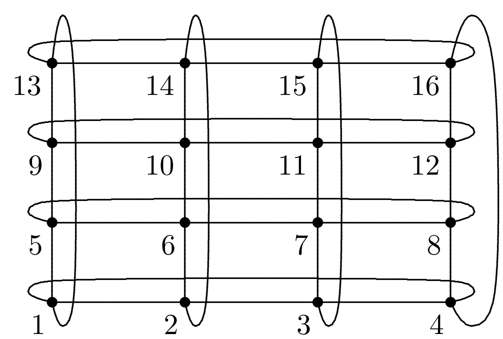
(a) For which values of n does G have an Euler path?

(b) For which values of n does G have an Euler circuit?

(c) When G does not have an Euler circuit, what is the mimumum number of edges that need to be added to produce a graph with an Euler circuit?

4: **(a)** Show that the depicted graph is non-planar.

(b) What is the minimum number of edges that need to be removed from that graph to produce a planar graph?



Solutions to Final

1: The auxiliary quadratic is is $t^2 - 6t + 9 = (t-3)^2$. So the general form for the solution to the homogenous equation is $a_n^{(h)} = (A + Bn)3^n$ where A and B are constants. Try $a_n^{(p)} = C2^n$ as a particular solution. Then

$$a_{n+2}^{(p)} - 6a_{n+1}^{(p)} + 9a_n^{(p)} = C2^n(2^2 - 6.2 + 9) = C2^n \ .$$

So we can put $C = 3$.

The general form of the solution is $a_n = a_n^{(h)} + a_n^{(p)} = (A + Bn)3^n + 3.2^n$. We have $4 = a_0 = A + 3$, so $A = 1$. We also have $15 = a_1 = (1 + B)3 + 6$, so $B = 2$. In conclusion, the solution is $a_n = (1 + 2n)3^n + 3.2^n$.

2: Part (a). The reflexivity and symmetry of $\equiv$ are obvious. Let $s_i = (a_i, b_i)$ for $i \in \{1, 2, 3\}$. Suppose $s_1 \equiv s_2$ and $s_2 \equiv s_3$. Then $a_1 - b_1 = a_2 - b_2 = a_3 - b_3$. So $s_1 \equiv s_3$. We have shown that $\equiv$ is transitive.

Part (b). We have

$$[1, 1] = \{(a, b) \in S : a \equiv b\} = \{(11, 22, 33, 44, 55, 66, 77, 88, 16, 27, 38, 61, 72, 83\}$$

where we write ab as an abbreviation for (a, b). We have

$$[1, 8] = \{(a, b) \in S : b \equiv a + 2\} = \{13, 18, 24, 35, 46, 57, 68, 31, 81, 42, 53, 64, 75, 86\} \ .$$

Part (c). The conjugacy classes are C_0, C_1, C_2, C_3, C_4 where $C_r = \{(a, b) \in S : a - b \equiv r\}$. It is now clear that the number of equivalence classes of 5.

Alternative approach:. It is a standard observation that, given a surjection $f : X \to Y$, then there is an equivalence relation $\equiv$ on X such that $x \equiv x'$ provided $f(x) = f(x')$. Part (a) follows by considering the function $f : S \to \{0, 1, 2, 3, 4\}$. Parts (b) and (c) can then be done by noting that the equivalence classes are the preimages $f^{-1}(y)$ where $y \in \{0, ..., 4\}$.

Comment: In part (a), it would be a waste of space to write out detailed mechanical verifications of the reflexivity and symmetry of $\equiv$. No reader would bother to read those details. (At a more advanced level, our mechanical verification of transitivity would also be superfluous and, in fact, it would be enough to state that $\equiv$ is obviously an equivalence relation.)

3: Parts (a) and (b). The graph G is connected and number of vertices of odd order is $2n$ when n is odd, 0 when n is even. So G has an Euler path if and only if $n = 1$ or n is even. Also, G has an Euler circuit if and only if n is even.

Part (c). When n is odd, we must add at least n edges to produce a graph with an Euler circuit. Adding the n edges having the form $\{a_i, b_j\}$ does have that effect. So The numimum number of edges to be added is n.

Comment: It is not always possible to add edges to a connected graph to produce a graph with an Euler circuit. A counter-example is a graph with order 2 and size 1. So, to obtain full marks for part (c), it is necessary to say which edges are to be added.

4: Part (a). For a contradiction, suppose the depicted graph is planar. Recall that, given a planar graph G of order n and size e, we have $e \leq (n-2)c/(c-2)$ where every cycle in G has length at least c. Taking G to be the depicted graph, then $n = 16$ and $e = 32$. (The edges are not hard to count directly, but an easy way to evaluate e is to observe that, since every vertex has degree 4, the Handshaking Lemma implies that $2e = 4n = 64$.) We can put $c = 4$. Then $32 \leq (16-2)(4-2)/4 = 28$, which is a contradiction, as required to show that the depicted graph is planar.

Part (b). Let G' be a planar graph of size e' obtained from G by removing m edges. Then, by a calculation above, $e' = 32 - m \leq 28 - m$. So $m \geq 4$. If we remove the 4 curved vertical edges, then the curved horizontal edges can be redrawn so as to avoid intersections. Therefore, the minimum possible m is 4.

Comment: The inequality $e \leq (n-2)c/(c-2)$ does not imply the the graph is planar. So, for full marks in part (b), it is necessary to say which 4 edges have to be removed, in order to show that there does exist a planar graph whose size is only 4 less than the size of the depicted graph.