Archive of documentation for

# MATH 132, Discrete and Combinatorial Mathematics

Bilkent University, Fall 2017, Laurence Barker

version: 24 January 2018

Source file: arch132fall17.tex

# Course Specification

See STARS for detailed information.

**Main course text:** Ralph Grimaldi, "Discrete and Combinatorial Mathematics: an Applied Introduction", 5th edition, Pearson, 2014, (New International Edition, 2013).

**Syllabus:** The format below is, *Week number; Monday date; Subtopics and textbook section numbers*. The numbering $m.n$ indicates Chapter $m$ Section $n$ in the Grimaldi textbook.

**1: 18 Sep:** Principles of Counting: Rules of Sum and Product, Permutations, Combinations: The Binomial Theorem (1.1, 1.2, 1.3).

**2: 25 Sep:** Principles of Counting: Combinations with Repetitions, Catalan Numbers (1.4, 1.5)

**3: 2 Oct:** Set Theory: Sets and Subsets, Set Operations, Laws of Set Theory, Venn Diagrams, Probability (3.1, 3.2, 3.3, 3.4, 3.5).

**4: 9 Oct:** Properties of Integers: Well-Ordering Principle, Induction, Recursion (4.1, 4.2, 4.3).

**5: 16 Oct:** Properties of Integers: Division Algorithm, Euclidean Algorithm, Fundamental Theorem of Arithmetic (4.4, 4.5).

**6: 23 Oct:** Relations, Functions (Injections, Surjections, Composition, Inverse), Pigeonhole Principle, Function Composition, Inverse Functions.

**7: 30 Oct:** Inclusion-Exclusion Principle, Equivalence Relations, Stirling Numbers.

**8: 6 Nov:** Partial Orderings, Hasse diagrams.

**9: 13 Nov:** Derangements, Rook Polynomials, Arrangements with Forbidden Positions.

**10: 20 Nov:** Generating Functions, Partitions of Integers, Exponential Generating Functions (9.1, 9.2, 9.3, 9.4).

**11: 27 Nov:** First Order Linear Recurrence Relations, Second Order Linear Homogeneous Recurrence Relations, Nonhomogeneous Recurrence Relations (10.1, 10.2, 10.3, 10.4).

**12: 4 Dec:** Graph Theory, Subgraphs, Complements, Graph Isomorphism (11.1, 11.2).

**13: 11 Dec:** Handshaking Lemma, Euler Trails and Euler Circuits, 11.3.

**14: 18 Dec:** Planar Graphs, 11.4.

**Assessment:**

- Quizzes and Homework, 16%,
- Midterm I, 28%,
- Midterm II, 28%,
- Final, 28%.

75% attendance is compulsory. Attendance will be assessed through quiz returns.

# Homeworks and Quizzes

MATH 132, *Discrete and Combinatorial Mathematics*, Fall 2017

Laurence Barker, Mathematics Department, Bilkent University,
version: 27 December.

**Office Hours:** Fridays, 08:40 – 09:30, Office Room Fen A 129.

# Homeworks

Homeworks may be handed in to either teacher at the end of class, or may be placed in the box at room Fen A 106 on the day of the deadline. Any homeworks placed under a door or in a mailbox, then discovered the following Monday morning, will not be marked.

## Homework 1, due 17:45, Friday, 20 October.

**1.1:** **(a)** Find the number of nonnegative integer solutions of $x_1 + x_2 + x_3 + x_4 + x_5 = 230$ with $x_1, x_2, x_3 \geq 0$, $x_4 \geq 3$ and $x_5 \leq 153$.
**(b)** How many binary strings of size 16 consisting of six zeroes and ten ones has no consecutive zeroes?

**1.2:** Find the probability of being dealt each of the following hands if 5 cards are dealt from an ordinary 52-card deck:
**(a)** a flush (5 cards of the same suit),
**(b)** at least one card from each suit.

**1.3:** For each positive integer $n$, the $n$-th Catalan number is defined to be

$$c_n = \binom{2n}{n} - \binom{2n}{n+1}.$$

**(a)** Show that $(n+1)c_n = (4n-2)c_{n-1}$.
**(b)** Show that $c_n < 2^{2n-1}/(n+1)$ for all $n \geq 2$.

**1.4:** A fair coin is thrown $2n - 1$ times. What is the probability that:
**(a)** After the last throw, the number of heads is greater than the number of tails?
**(b)** After the last throw, the number of heads is exactly $n + 1$?
**(c)** After every throw, the number of heads is greater than the number of tails and, after the last throw, the number of heads is exactly $n + 1$?

## Homework 2, due 17:40, Friday, 24 October.

**2.1:** **(a)** Let $A_1$, $A_2$, $A_3$ be sets such that

$$|A_1| = |A_2| = |A_3| = 8\,, \quad |A_2 \cap A_3| = |A_1 \cap A_3| = |A_1 \cap A_2| = 4\,, \quad |A_1 \cap A_2 \cap A_3| = 1\,.$$

Evaluate $|A_1 \cup A_2 \cup A_3|$.

**(b)** For a positive integer $n$, let $A_1$, ..., $A_n$ be sets such that

$$|A_{i_1} \cap A_{i_2} \cap ... \cap A_{i_k}| = 2^{n-k}$$

for all $1 \le k \le n$ and $1 \le i_1 < i_2 < ... < i_k \le n$. Evaluate $|A_1 \cup A_2 \cup ... \cup A_n|$.

**2.2:** Calculate your answers numerically, not as an expression for the reader to evaluate (say, as a number such as 10549, not as an expression involving binomial coefficients).
**(a)** How many ways are there of giving 6 differently colored candies to 4 distinguishable children?
**(b)** How many ways are there as in (a) and such that no child receives both the blue candy and the red candy?

**2.3:** For $U = \{1, 2, 3, 4\}$ and $T = \{1, 4\}$, define the relation $\Re$ on the power set of $\mathcal{P}(U)$ by $A \Re B$ provided $A \cup T = B \cup T$.
**(a), 8 marks.** Verify that $\Re$ is an equivalence relation on $\mathcal{P}(U)$.
**(b), 8 marks.** Find the equivalence classes $[\emptyset]$ and $[\{1, 2\}]$.
**(c), 9 marks** Write down the other equivalence classes of $\Re$.

**2.4:** Let $b(n)$ be the number of ways to distribute $n$ pennies to four people with the youngest person receiving either none or two pennies, the oldest person receiving at most 3 pennies, the other two people receiving an odd number of pennies each.
**(a), 13 marks.** Write down the generating function for $\{b(n)\}$. Simplify it as much as possible.
**(b), 12 marks.** Using part (a), compute $b(5)$.

# Homework Solutions

Of course, there is no such thing as a model solution. Each individual has his or her own style.

**Sol 1.1:** Part (a). The answer is $a - b$, where $a$ and $b$ are, respectively, the numbers of natural number solutions to $y_1 + ... + y_5 = 227$ and $z_1 + ... + z_5 = 74$. By a standard formula, $a = \binom{227+4}{4}$ and $b = \binom{73+4}{4}$. So the answer is $a - b = \binom{231}{4} - \binom{77}{4}$.

Part (b). The answer is $\binom{11}{6}$, because this is the number of ways of choosing 6 of the 11 possible gaps between the 1 digits.

**Sol 1.2:** Part (a). The answer is the number of suits, times the number of flushes in a given suit, divided by the number of possible hands. That equals

$$4\binom{13}{5} \Big/ \binom{52}{5} = \frac{4\,.\,13!\,.\,5!\,.\,47!}{5!\,.\,8!\,.\,52!} = \frac{13!\,.\,47!}{2\,.\,7!\,.\,52!} = \frac{13.12.11.10.9}{2.52.51.50.49.48}\,.$$

Part (b). Since there must be exactly 2 cards from one suit and exactly 1 card from each of the other three suits, the answer is

$$4\binom{13}{2}\binom{13}{1}^3 \Big/ \binom{52}{5} = 24.13^4.\binom{52}{5}^{-1}\,.$$

**Sol 1.3:** Part (a). From the definition, $c_n = \binom{2n}{n}\left(1 - \frac{n}{n+1}\right) = \frac{1}{n+1}\binom{2n}{n}$. So

$$(n+1)c_n = \binom{2n}{n} = \frac{2n(2n-1)}{n^2}\binom{2(n-1)}{n-1} = (4n-1)c_{n-1} \; .$$

Part (b). Let $d_n = 2^{2n-1}/(n+1)$. We shall show, by induction, that $c_n < d_n$ for all $n \geq 2$. First note that $c_2 = 2 < 8/3 = d_2$. Now suppose that $n \geq 3$ and that $c_{n-1} < d_{n-1}$. Then $c_n < d_n$ because

$$\frac{c_n}{c_{n-1}} = \frac{4n-2}{n+1} < \frac{4n}{n+1} = \frac{2^{2n-1}/(n+1)}{2^{2n-3}/n} = d_n/d_{n-1} \; .$$

*Alternative for part (b):* (I found this unexpected solution in the script of Bural KünÇü. Since most of the scripts were marked by the assistant, I guess others in the class may have found the same solution.) Applying a formula in part (a), then Pascal's Formula, then the Binomial Theorem,

$$(n+1)c_n = \binom{2n}{n} = \binom{2n-1}{n-1} + \binom{2n-1}{n} < (1+1)^{2n-1} = 2^{2n-1}$$

with the strict inequality holding because $2n-1 \geq 3$.

**Sol 1.4:** Part (a). By reversing the heads and tails (so that, for example, $HHTHTTHT$ becomes $TTHTHHTH$), each outcome with more heads than tails can be partnered with an outcome with more tails than heads. So exactly half of the possible outcomes have more heads than tails. All the outcomes are equally likely. So the probability is 1/2.

Part (b). The answer is $\binom{2n-1}{n+1}/2^{2n-1}$, because there are exactly $\binom{2n-1}{n+1}$ taxi-cab paths from $(0,0)$ to $(n+1, n-2)$.

Part (c). The first throw must be a head. The number of paths $(1,0)$ to $(n+1, n-2)$ not touching the line of points $(x,x)$ must be equal to the number of paths $(0,0)$ to $(n, n-2)$ not passing above that line. The paths that do pass above that line can be partnered with the paths from $(0,0)$ to $(n-3, n+1)$ by reversing the values of all the throws beyond the transgression. So the number of arrangements is

$$\binom{2n-2}{n-2} - \binom{2n-2}{n-3} = \binom{2n-2}{n} - \binom{2n-2}{n+1} = \frac{3}{n+1}\binom{2n-2}{n}$$

and the required probability is $3\binom{2n-2}{n}\Big/ 2^{2n-1}(n+1)$.

# Quizzes

**Q1:** There are 10 different cakes. How many ways are there of giving Ali, Burcu, Cen, Deniz one cake each, wioth six cakes remaining"

**Q2:** How many 6-step journeys are there from one corner to the opposite corner of a $3 \times 3$ square lattice. [A diagram was supplied.]

**Q3:** Find $\gcd(120, 196)$ using the Euclidian algorithm.

**Q4:** How many bijections $\{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ are there?

**Q5:** Let $C$ be the "chessboard" formed from an $n \times n$ board by deleting the squares not on the top-left to bottom-right diagonal. Thus, the number of undeleted squares is $n$.
**(a)** Find the rook polynomial of $C$.
**(b)** For $0 \leq k \leq n$, how many arrangements with $k$ rooks are there?

**Q6:** Find an Euler circuit for the depicted graph. [A graph was depicted.]

# Quiz Solutions

**Sol 1:** The answer is $10.9.8.7 = 5040$.

**Sol 2:** The answer is $\big(6//3\big) = 6.5.4/3.2.1 = 20$.

**Sol 3:** We have

$$196 = 120 + 76 \; , \quad 120 = 76 + 44 \; , \quad 76 = 44 + 32 \; , \quad 44 = 32 + 12 \; , \quad 32 = 2.12 + 8 \; .$$

Finally, $12 = 8 + 4$ and $8 = 2.4$. So $\gcd(120, 196) = 4$.

**Sol 4:** The answer is $4! = 24$.

**Sol 5:** Part (a), $(1 + x)^n$. Part (b), $\begin{pmatrix} n \\ k \end{pmatrix}$.

**Sol 6:** [The solution required a diagram.]

# MATH 132: Discrete and Combinatorial Mathematics. Midterm 1

Hamza Yeşilyurt & LJB, 22 October 2017, Bilkent University.

*This is a reformatted record of the paper. The original was in fill-in-the-spaces form.*

Time allowed: 2 hours. Please show all your work in legibly written, well-organized sentences. Simplify as far as possible unless otherwise stated.

**1: 12 marks.** How many 7-digit positive integers can we for using the digits 3, 4, 5, 5, 6, 7 such that $n > 5\,000\,000$? (Use 3 once, 4 twice, 5 twice, 6 once, 7 once.)

**2: 22 marks.** Five unbiased dice are thrown. For each die, the six faces are labelled with the numbers from 1 to 6. After the throw, the sum of the numbers appearing on the five upper faces is recorded as the score.
**(a)** How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 11$ where each $x_i$ is an integer and each $x_i \geq 1$?
**(b)** How many solutions are there to (a) if we also require that each $x_i \leq 6$?
**(c)** What is the probability that the score is 17?

**3: 22 marks.** A coin is thrown $2n$ times, and the sequence of heads and tails is recorded.
**(a)** Give a formula for the number of sequences such that, throughout the trial, the number of heads is never greater than the number of tails, and the final number of heads is equal to the number of tails.
**(b)** Give a formula for the number of sequences such that, throughout the trial, except before the first throw and after the last throw, the number of heads is less than the number of tails and the final number of heads is equal to the final number of tails.
**(c)** List all possible throws in (a) and (b) when $n = 3$.

**4: 22 marks.** Let $a_0 = a_1 = 1$ and $a_{n+2} = 3a_{n+1} + a_n$. Prove, by induction, that $a_n \leq (7/2)^n$ for all integers $n$ with $n \geq 0$.

**5: 22 marks.** For an integer $n$ with $n \geq 1$, we consider the binary strings with length $n$, that is, the sequences $z_1 z_2 ... z_n$ where each $z_i$ is 0 or 1.
**(a)** Supposing $n \geq 4$, how many such strings are there with exactly 2 occurances of 01? (For instance, in the case $n = 6$, the strings 010010 and 100101 satisfy that condition but the strings 101111 and 010101 do not satisfy the condition.)
**(b)** Supposing $n \geq 6$, how many such strings are there with exactly 3 occurances of 01?
**(c)** Give a combinatorial proof that, for all integers $n$ with $n \geq 1$, we have

$$2^n = \begin{cases} \binom{n+1}{1} + \binom{n+1}{3} + ... + \binom{n+1}{n+1}, & \text{if } n \text{ is even,} \\ \binom{n+1}{1} + \binom{n+1}{3} + ... + \binom{n+1}{n}, & \text{if } n \text{ is odd.} \end{cases}$$

# Solutions to Midterm 1

**1:** The number of choices with first digit 7 is $6!/2!2!$, likewise the number of choices with first digit 6. The number of choices with first digit 5 is $6!/2!$. So the total number of choices is

$$\frac{6!}{2!2!} + \frac{6!}{2!2!} + \frac{6!}{2!} = 6!\left(\frac{1}{4} + \frac{1}{4} + \frac{1}{2}\right) = 6! = 720\ .$$

**2:** Part (a). Substituting $y_i = x_i - 1$, we are to count the integer solutions to $y_1 + ... + y_5 = 6$ with each $y_i \geq 0$. The answer is

$$\binom{4+6}{6} = \binom{10}{6} = \frac{10.9.8.7}{4.3.2} = 10.3.7 = 210\ .$$

Part (b). The solutions are as in part (a), but with the further constraint that each $y_i \leq 5$. Exactly 5 of the solutions in (a) fail this constraint. So the answer is $210 - 5 = 205$.

Part (c). The probability is $n/6^5$, where $n$ is the number of integer solutions to $x_1 + ... + x_5 = 17$ with $1 \leq x_i \leq 6$ for each $x_i$. By the method in part (a), $n$ is the number of integer solutions to $y_1 + ... + y_5 = 12$ with $0 \leq y_i \leq 5$.

The number of integer solutions $z_1 + ... + z_5$ satisfying $0 \leq z_i$ is

$$\binom{16}{12} = \frac{16.15.14.13}{4.3.2} = 2.5.14.13 = 10.182 = 1820\ .$$

Now $5\binom{10}{6}$ of those solutions satisfy $6 \leq z_i$ for at least one $i$. Also $\binom{5}{2}$ of the solutions satisfy $6 \leq z_i$ for at least two of the indices $i$. There are no solutions satisfying $6 \leq z_i$ for three or more of the $i$. So

$$n = 1820 - 5\binom{10}{6} + \binom{5}{2} = 1820 - 5.210 + 5.4/2 = 1820 - 1050 + 10 = 780\ .$$

Therefore, the probability is $\frac{n}{6^5} = \frac{130}{6^5} = \frac{65}{3.36.36} = \frac{65}{108.36} = \frac{65}{3600 + 288} = \frac{65}{3888}$ .

**3:** Part (a). Regarding heads as upward moves $(x, y) \to (x, y+1)$ and tails as rightward moves $(x, y) \to (x+1, y)$, we are to count the integer lattice paths with length $2n$ from $(0, 0)$ to $(n, n)$ that never pass above the line between those two points. That number is the Catalan number

$$c_n = \binom{2n}{n} - \binom{2n}{n+1} = \frac{1}{1+n}\binom{2n}{n}\ .$$

Part (b). For any such sequence, the first throw must be tails and the last must be heads. So the answer is the number of paths with length $2n - 2$ from $(0, 1)$ to $(1, 0)$ that never pass above the line between those two points. The answer is the Catalan number

$$c_{n-1} = \binom{2n-2}{n-1} - \binom{2n-2}{n} = \frac{1}{n}\binom{2n-2}{n-1}\ .$$

Part (c). For (a), $TTTHHH$, $TTHTHH$, $TTHHTH$, $THTTHH$, $THTHTH$ where $T$ indicates tails, $H$ indicates heads. For (b), $TTTHHH, TTHTHH$.

**4:** The required inequality is clear in the cases $n = 0$ and $n = 1$. Now suppose, inductively, that the inequality holds for $a_n$ and $a_{n+1}$. Then

$$a_{n+2} \leq 3(7/2)^{n+1} + (7/2)^n = (7/2)^n(3.7/2 + 1) = (7/2)^n 23/2 < (7/2)^n 49/4 = (7/2)^{n+2} \ .$$

**5:** Generally, for any integer $k$ in the range $0 \leq k \leq n/2$, consider the binary strings with length $n$ that have exactly $k$ occurrences of the substring 01. For any such string, let $x_0$ be the number of 1 digits before the first 01. Let $y_0$ be the number of 0 digits before the first 01. Let $x_1$ be the number of 1 digits between the first and second 01. Let $y_1$ the be the number of 1 digits between the first and second 01. We define $x_i$ and $y_i$ in the same way. Finally, $x_k$ is the number of 1 digits after the last 01, and similarly for $y_k$. The definition is illustrated thus:

$$\underbrace{1...1}_{x_0}\ \underbrace{0...0}_{y_0}\ 01\ \underbrace{1...1}_{x_1}\ \underbrace{0...0}_{y_1}\ 01 \ldots 01\ \underbrace{1...1}_{x_k}\ \underbrace{0...0}_{y_k}\ .$$

The number of such strings is the number of integer solutions to

$$x_0 + y_0 + x_1 + y_1 + ... + x_k + y_k = n - 2k$$

with each $x_i \geq 0 \leq y_i$. So the number of such strings is

$$\binom{(2k+1) + (n-2k)}{2k+1} = \binom{n+1}{2k+1} \ .$$

Part (a). Putting $k = 2$, the answer is $\binom{n+1}{3} = (n^2 - 1)n/6$.

Part (b). The answer is $\binom{n+1}{5}$ by a similar argument.

Part (c). In view of the preamble, the total number of binary strings with length $n$ is

$$2^n = \sum_k \binom{n+1}{2k+1}$$

where $k$ runs over the integers such that $0 \leq k \leq n/2$.

MATH 132: Discrete and Combinatorial Mathematics. Makeup 1

Hamza Yeşilyurt & LJB, 18 December 2017, Bilkent University.

Time allowed: 2 hours. Please show all your work in legibly written, well-organized sentences. Justify your answers. Simplify as far as possible unless otherwise stated.

**1: 25 marks.** How many ways are there of arranging 5 blue marbles and 5 red marbles in order such that no four consecutive marbles are all red?

**2: 25 marks.** Consider the integer solutions to the equation $a_1 + a_2 + a_3 + a_4 + a_5 = 25$.
**(a)** How many solutions are there such that each $a_i \leq i$?
**(b)** How many solutions are there such that each $a_i \geq i$ and $a_1 \leq 5$?

**3: 25 marks.** In every tennis match between Ayşe and Barış, there is always a winner, never a tie.
**(a)** Suppose they play 21 tennis and that, in each game, Ayşe has a 1/2 chance of winning. What is the probability that Ayşe wins 11 matches without scoring more matches than Bariş at any point until the end of the 21st match?
**(b)** Now suppose the probability of Ayşe winning is slightly more than 1/2. Is the new answer to part (a) greater than the answer you gave, or less?

**4: 25 marks.** In the game of nim, there are some piles of matches. There are two players, who take moves alternately. Each player takes one or matches from a pile. The player who takes the last match wins. Suppose there are an even number of piles and all the piles start off having the same number of matches. Under correct play, which player wins, the first or the second?

MATH 132: Discrete and Combinatorial Mathematics. Midterm 2

Hamza Yeşilyurt & LJB, 26 November 2017, Bilkent University.

*This is a reformatted record of the paper. The original was a fill-in-the-spaces form.*

Time allowed: 2 hours. Please show all your work in legibly written, well-organized sentences. Simplify as far as possible unless otherwise stated.

**1: 25 marks.** **(a)** For finite sets $A_1$, $A_2$, $A_3$, $A_4$, write down a formula for $|A_1 \cup A_2 \cup A_3 \cup A_4|$ in terms of intersections of sets. (No proof is required.)

**(b)** Let $m$ be an integer with $m \geq 4$. For $i$ in the range $1 \leq i \leq 4$, let $A_i$ be the set of functions $f : \{1, 2, ..., m-1, m\} \to \{1, 2, 3, 4\}$ such that $i$ is not a value of $f$. Using part (a), evaluate $|A_1 \cup A_2 \cup A_3 \cup A_4|$.

**(c)** Using part (b), give a formula for $S(m, 4)$. (No marks will be awarded for using a different method.)

**(d)** Using part (c), evaluate $S(6, 4)$ and check that the answer agrees with part (e) below.

**(e)** State a recurrence relation for the Stirling numbers (of the second kind) $S(m, n)$, and use it to write down a $6 \times 4$ table showing all the values of $S(m, n)$ for $1 \leq m \leq 6$ and $1 \leq n \leq 4$. (No marks will be awarded for calculating the values by a different method.)

**2: 25 marks.** **(a)** Given a set $X$ and equivalence relations $\equiv_1$ and $\equiv_2$ on $X$, let $\sim$ be the relation on $X$ such that, for any $x, y \in X$, we have $x \sim y$ if and only if $x \equiv_1 y$ or $x \equiv_2 y$. Must $\sim$ always be an equivalence relation? (Give either a general proof or else give a single clear counter-example.)

**(b)** Given a set $X$ and equivalence relations $\equiv_1$ and $\equiv_2$ on $X$, let $\simeq$ be the relation on $X$ such that, for any $x, y \in X$, we have $x \simeq y$ if and only if $x \equiv_1 y$ and $x \equiv_2 y$. Must $\simeq$ always be an equivalence relation? (Again, proof or counter-example.)

**3: 25 marks.** **(a)** Let $a(n)$ be the total number of ways one can select, in order, $n$ marbles from a large supply of blue, red, white and yellow marbles (all of the same size) with the selection including an odd number of blue ones and at most 3 red ones. Write down the generating function for $a(n)$. Simplify as much as possible.

**(b)** Use the Euclidean algorithm to find the greatest common divisor of 1392 and 504. Use Euclidean algorithm to express the greatest common divisor of 1392 and 504 as an integer linear combination of 1392 and 504. (No credit will be given if the Euclidean algorithm is not used).

**4: 25 marks.** Use generating functions to solve the following problem. Carol is collecting money from her cousins to have a party for her aunt. If 6 of the cousins promise to give \$2, \$3 or \$4 each , and two others each give \$5 or \$6, what is the probability that Carol will collect exactly \$32? (One sample collection is 2+4+3+4+4+4+5+6.)

# Solutions to Midterm 2

**1:** Part (a). We have

$$
\begin{aligned}
|A_1 \cup A_2 \cup A_3 \cup A_4| = {} & |A_1| + |A_2| + |A_3| + |A_4| - \\
& |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_3 \cap A_4| + \\
& |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4| - \\
& |A_1 \cap A_2 \cap A_3 \cap A_4| .
\end{aligned}
$$

Part (b). We have $A_1 \cap A_2 \cap A_3 \cap A_4 = \emptyset$. Also,

$$|A_1| = 3^m \,, \qquad |A_1 \cap A_2| = 2^m \,, \qquad |A_1 \cap A_2 \cap A_3| = 1$$

and similarly for the other terms in the right-hand side of part (a). So

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = 4.3^m - 6.2^m + 4$$

Part (c). We have $4!S(m,4) = 4^m - |A_1 \cup A_2 \cup A_3 \cup A_4|$ because there are $4^m$ functions $\{1, ..., m\} \to \{1, 2, 3, 4\}$. Therefore,

$$S(m,4) = (4^m - 4.3^m + 6.2^m - 4)/24 \,.$$

Part (d). In particular, $24S(m,4) = 4^6 - 4.3^6 + 6.2^6 - 4 = 4096 - 4.729 + 6.64 - 4$, so

$$S(m,4) = \frac{4096 - 2916 + 384}{24} = \frac{1180 + 380}{24} = \frac{1560}{3.8} = \frac{520}{8} = 65,$$

which is in agreement with the $(6,4)$ entry of the table in the next part.

Part (e). We have $S(m,n) = S(m-1, n-1) + nS(m-1, n)$ and $S(m,1) = 1 = S(m,m)$, which yield the following table.

| $S(m,n)$ | 1 | 2 | 3 | 4 | $n$ |
|---|---|---|---|---|---|
| 1 | 1 | | | | |
| 2 | 1 | 1 | | | |
| 3 | 1 | 3 | 1 | | |
| 4 | 1 | 7 | 6 | 1 | |
| 5 | 1 | 15 | 25 | 10 | |
| 6 | 1 | 31 | 90 | 65 | |
| $m$ | | | | | |

*Comment:* I subtracted small numbers of marks, usually 1 or 2, for gross failures to simplify. One popular but absurd answer to (c) was

$$S(m,4) = \frac{1}{4!}\left( \binom{4}{0} 4^m - \binom{4}{1} 3^m + \binom{4}{2} 2^m - \binom{4}{1} 1^m + 0 \right) \,.$$

Candidates who made a mistake in the simplification lost one mark. Candidates who failed to attempt that part of the question also lost one mark.

**2:** Part (a). No. For a counter-example, put $X = \{0, 1, 2\}$ and let $\equiv_1$ and $\equiv_2$ have equivalence classes $\{0, 1\}$ and $\{0, 2\}$, respectively. Then $1 \sim 0$ and $0 \sim 2$ but $1 \not\sim 2$.

Part (b). Yes. Let $x, y, z \in X$. Suppose $x \simeq y$ and $y \simeq z$. Then $x \equiv_1 y$ and $y \equiv_1 z$, so $x \equiv_1 z$. Likewise, $x \equiv_2 z$. We deduce that $x \simeq z$. We have shown transitivity. Similar and easier arguments show that $\simeq$ is reflexive and symmetric: $x \simeq x$; if $x \simeq y$ then $y \simeq x$.

*Comment 1:* The argument in part (b) is a complete proof, not a sketch. A mathematical proof is a communication to a human, and it is approriate to use words such as *likewise* and *similarly* as a way of avoiding repetitious text which no mathematically competent human would bother to read.

*Comment 2:* A great many candidates lost many marks in part (b). There were 2 marks for getting the correct answer, 10 marks for the proof. Recall, a proof should be clear and should progress step by step: clear step 1, clear step 2, and so on. The following flaws were very frequent: failure to distinguish clearly between what was being assumed and what was being deduced; creation of an artwork of mere statement-blobs, distributed all over the page, sometimes accompanied by networks of mysterious arrows; unexplained notation, random letters suddenly appearing in the middle of tracts of other symbols.

To avoid such flaws, reasoning should be presented in sentences that are grammatically correct or at least grammatically understandable.

**3:** Part (a). The number $a(n)$ is the coefficient of $x^n/n!$ in the exponential feneralting function

$$\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...\right)\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\right)\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\right)^2$$

$$= \frac{e^x + e^{-x}}{2}\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right)e^{2x} = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right)\frac{e^{3x} + e^x}{2}$$

$$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right)\sum_{k=0}^{\infty}\frac{(3^k - 1)x^k}{2k!} .$$

Part (b). We have

$$1392 = 2.504 + 384 , \quad 504 = 1.384 + 120 , \quad 384 = 3.120 + 24 , \quad 120 = 5.24 .$$

So $\gcd(1392, 504) = 24 = 384 - 3.120 = 384 - 3(504 - 384) = 4.384 - 3.504$

$$= 4(1392 - 2.504) - 3.504 = 4.1392 - 11.504 .$$

**4:** The probability is $a/3^6.2^2 = a/729.4 = a/2916$ where $a$ is the coefficient of $x^{32}$ in

$$(x^2 + x^3 + x^4)^6(x^5 + x^6)^2 .$$

Thus, $a$ is the coefficient of $x^{10}$ in

$$(1 + x + x^2)^6(1 + 2x + x^2) .$$

Let $b_k$ be the coefficient of $x^k$ in $(1 + x + x^2)$. Then

$$a = b_8 + 2b_9 + b_{10} .$$

From $2+2+1+1+1+1$, $2+2+2+1+1+0$, $2+2+2+2+0+0$, we have

$$b_8 = \frac{6!}{2!4!} + \frac{6!}{3!2!1!} + \frac{6!}{4!2!} = 15 + 60 + 15 = 90 \ .$$

From $2+2+2+1+1+1$, $2+2+2+2+1+0$, we have

$$b_9 = \frac{6!}{3!3!} + \frac{6!}{4!1!1!} == 20 + 30 = 50 \ .$$

From the partitions $2+2+2+2+1+1$, $2+2+2+2+2+0$, we have

$$b_{10} = \frac{6!}{4!1!1!} + \frac{6!}{5!1!} = 15 + 6 = 21 \ .$$

So $a = 90 + 2.50 + 21 = 211$, and the probability is $211/2916$.

*Comment:* Sure enough, the problem is easier to solve directly, by the same method, without mentioning generating functions. The question illustrates a technique that becomes genuinely useful in more sophisticated applications.

# MATH 132: Discrete and Combinatorial Mathematics. Makeup 2

Hamza Yeşilyurt & LJB, 18 December 2017, Bilkent University.

Time allowed: 2 hours. Please show all your work in legibly written, well-organized sentences. Simplify as far as possible unless otherwise stated.

**1: 25 marks.** How many distinguishable ways are there of putting 6 coloured balls into 8 plain boxes? (Some of the boxes may be empty.)

**2: 25 marks.** Let $X$ be the set of finite sequences $(x_1, ..., x_9)$ where each $x_i \in \{1, 2, 3\}$. We define a relation $\equiv$ on $X$ such that $(x_1, ..., x_9) \equiv (y_1, ..., y_9)$ if and only if there exists a bijection $\sigma : \{1, ..., 9\} \to \{1, ..., 9\}$ satisfying $x_{\sigma(i)} = y_i$ for all $i$.

**(a)** Show that $\equiv$ is an equivalence relation.

**(b)** How many equivalence classes under $\equiv$ are there?

**3: 25 marks.** Use the Euclidian algorithm to find integers $a$ and $b$ such that

$$1234a + 4321b = \gcd(a, b) \ .$$

(No marks will be awarded for using other methods.)

**4: 25 marks.** Use generating functions and a formula for $(1 - x)^{-1}$ to find the number of solutionsto the equation

$$x_1 + x_2 + x_3 + x_4 + x^5 = 25$$

where each $x_i$ is an integer in the range $0 \leq x_i \leq 10$.

MATH 132: Discrete and Combinatorial Mathematics. Final

Hamza Yeşilyurt & LJB, 6 January 2018, Bilkent University.

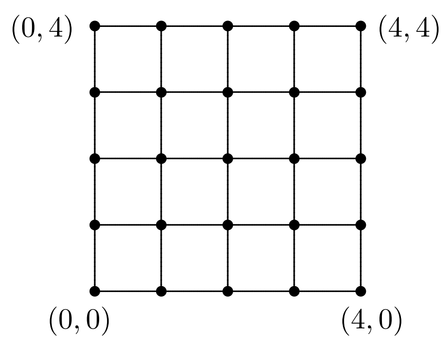
*This is a reformatted record of the paper. The original was a fill-in-the-spaces form.*

Time allowed: 2 hours. Please show all your work in legibly written, well-organized sentences. Simplify as far as possible unless otherwise stated.

**1: 25 marks. (a)** Consider ternary strings (strings over the alphabet $\{0, 1, 2\}$) that do not contain the substring 20, for instance 0010221021 or 1111011102. Let $a_n$ be the number of such strings of length $n$. Find a recurrence relation for $a_n$.
**(b)** Consider quaternary strings (strings over the alphabet $\{0, 1, 2, 3\}$). Let $b_n$ be the number of such strings of length $n$ containing an odd number of 1 digits. Find and solve a recurrence relation for $b_n$.

**2: 25 marks. a)** Let $k$ be a positive integer. Prove that there exits a positive integer $n$ such that $k|n$ and all the digits of $n$ are 0 and 3.
**b)** Write the generating function for the given sequences (do *not* simplify).
**(i)** a(n) is the number of nonnegative integers solutions of $2x + 4y + 6z + 9t = n$,
**(ii)** b(n) is the number of nonnegative integers solutions of $x + y + z + t = n$ with $x$ odd, $y$ is even, $z$ is at least 4 and $t$ is at most 6.
**(iii)** $c(n)$ is the number of quaternary strings of length $n$ containing an even number of 1 digits.

**3: 25 marks.** Consider the graph whose vertices are the integer coordinates $(x, y)$ where $0 \leq x \leq 4$ and $0 \leq y \leq 4$ and the edges are the horizontal and vertical lines as depicted.
**(a)** What is the minimum number of edges that must be deleted to form a graph that has an Euler circuit? (Remember to justify your answer.)
**(b)** Delete that number of edges and, for the resulting graph, find an Euler circuit. Specify the Euler circuit by listing, in order, the coordinates of the vertices of the circuit. (Marks will be subtracted for specifying the Euler circuit in any other way.)




**4: 25 marks.** Assume that $G$ is a planar graph with $n$ vertices such that:
- every vertex of $G$ has degree 3,
- $G$ can be drawn such that any two faces have the same number of edges,
- $G$ is connected and, upon removing any edge, the resulting graph is still connected.

What are the possible values of $n$? (Draw diagrams to show that each of those values is possible, and give a general argument to show that no other value is possible.)

# Solutions to Final

**1:** Part (a). We have $a_n = 3a_{n-1} - a_{n-2}$. Indeed, the integer $a_n + a_{n-2} = 3a_{n-1}$ is the number of ternary strings which do not have substring 20 except possibly at the end.

Part (b). To form such a string of length $n$, we can either append 1 to one of the $4^{n-1} - b_{n-1}$ quaternary strings with an even number of 1 digits, or else we can append 0 or 2 or 3 to one of the $b_{n-1}$ quaternary strings of the specified kind with length $n$. Therefore, $b_n = 2b_{n-1} + 4^{n-1}$. Taking the solution to be of the form $b_n = \alpha 2^n + \beta 4^{n-1}$ and solving for the initial conditions $b_1 = 1$ and $b_2 = 6$, we find that $\alpha = -1/2$ and $\beta = 2$, hence $b_n = 2^{2n-1} - 2^{n-1}$.

**2:** Part (a). By the Pigeonhole Principle, two distinct terms of the infinite sequence 3, 33, 333, ... must be congruent modulo $k$. The difference of two such terms is divisible by $k$. We have shown, in fact, that $k$ divides infinitely many positive integers having decimal representation of the form 3...30...0.

Part (b). We have

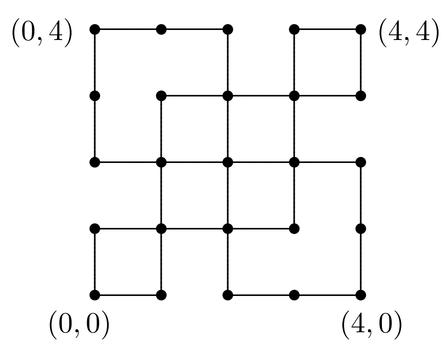
$$\sum_n a(n)x^n = (1 + x^2 + x^4 + ...)(1 + x^4 + x^8 + ...)(1 + x^6 + x^{12} + ...)(1 + x^9 + x^{18} + ...) \ ,$$

$$\sum_n b(n) = (x + x^3 + x^5 + ...)(1 + x^2 + x^4 + ...)(x^4 + x^5 + x^6 + ...)(1 + x + ... + x^5 + x^6) \ ,$$

In the notation of the previous question, $c(n) = 4^n - b_n = 2^{2n-1} + 2^{n-1}$. So

$$\sum_n c(n)x^n = \sum_n (2^{2n-1} + 2^{n-1})x^n \ .$$

**3:** Part (a). To form a graph with an Euler circuit, we must delete edges so as to ensure that the resulting graph is connected and every vertex of that graph has even degree. In the original graph, the 12 vertices of odd degree are: 3 at the top; 3 to the left; 3 to the right; 3 at the bottom. Plainly, we must remove at least 2 edges from each of those 4 sets of 3. We can do so as depicted. Therefore, the answer is 8.




Part (b). An Euler circuit for the depeicted modified graph is as follows:

$(0, 0)$, $(1, 0)$, $(1, 1)$, $(2, 1)$, $(3, 1)$, $(3, 2)$, $(2, 2)$, $(2, 3)$,
$(1, 3)$, $(1, 2)$, $(2, 2)$, $(2, 1)$, $(2, 0)$, $(3, 0)$, $(4, 0)$, $(4, 1)$,
$(4, 2)$, $(3, 2)$, $(3, 3)$, $(4, 3)$, $(4, 4)$, $(3, 4)$, $(3, 3)$, $(2, 3)$,
$(2, 4)$, $(1, 4)$, $(0, 4)$, $(0, 3)$, $(0, 2)$, $(1, 2)$, $(1, 1)$, $(0, 1)$, $(0, 0)$.

*Comment:* The above answer for (b) is complete. Since it can be directly checked that the specified path is an Euler circuit, no further justification is needed.

Just for the benefit of anyone mystified as to how one might find an Euler path, I mention that I carried out the method described in a standard proof of the Euler Path Theorem. The first circuit I found was:

$$\begin{array}{llllllllll} (0,0), & (1,0), & (1,1), & (2,1), & (2,0), & (3,0), & (4,0), & (4,1), \\ (4,2), & (3,2), & (3,3), & (4,3), & (4,4), & (3,4), & (3,3), & (2,3), \\ (2,4), & (1,4), & (0,4), & (0,3), & (0,2), & (1,2), & (1,1), & (0,1), & (0,0). \end{array}$$

Deleting the edges appearing in that circuit, the remaining graph, with only 8 edges, plainly has Euler circuit:

$$(3,1),\ \ (3,2),\ \ (2,2),\ \ (2,3),\ \ (1,3),\ \ (1,2),\ \ (2,2),\ \ (2,1),\ \ (3,1).$$

Splicing the last circuit into the penultimate one as described in the proof, we obtain the Euler circuit given in part (b).

**4:** We shall show that the possible values of $n$ are 4 and 8 and 20. Let $e$ be the number of edges, $f$ the number of faces. Euler's Characteristic Formula says that

$$n - e + f = 2\ .$$

By the Handshaking Lemma, $2e = 3n$. Let $c$ be the integer such that $G$ can be drawn with every face having $c$ edges. Of course, $c \geq 3$. The connectivity condition tells us that, in such a diagram, each edge is part of the boundary of two distinct faces. So $2e = cf$. Therefore

$$n(3/c - 1/2) = 2\ .$$

So $c \leq 5$. Taking $c$ to be 3 and 4 and 5, respectively, the values of $n$ are 4 and 8 and 20.

Those values of $n$ can all be realized, because they are the numbers of vertices of the tetrahedron, cube and dodecahedron, respectively.