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MATH 132, Discrete and Combinatorial Mathematics,

Bilkent University, Fall 2099, Laurence Barker

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MATH 132, Discrete and Combinatorial Mathematics, Fall 2015

Course specification

Laurence Barker, Bilkent University, version: 29 December 2015

Course Aims: To supply an introduction to some concepts and techniques associated with discrete mathematical methods in engineering and information technology; in particular, to provide experience of the art of very clear deductive explanation.

Course Description: The terms *combinatorics* and *discrete mathematics* have similar meanings. The former refers to an area of pure mathematics concerned with mathematical objects that do not have very much topological, geometric or algebraic structure. The latter refers to an area of applicable mathematics that rose to prominence with the advent of electronic computers and information technology. Of course, the two cultures overlap considerably and cannot be clearly distinguished from each other.

In the 1950s and 60s, pioneers of computing and computer science found that the established styles of applied mathematics were unsuitable for the new kinds of problem that were appearing. Unlike the classical applied fields such as differential calculus, linear algebra and statistics, the new kind of mathematics was not conducive to formalism, that is to say, methods of calculation based on manipulation of written symbols. Applied mathematicians found that they needed to adopt a conceptual approach which had previously been mostly confined to pure mathematics and some areas of physics.

In discrete mathematics, as opposed to classical applied mathematics, solutions to problems tend to comparatively unsystematic, though certain fundamental ideas do tend to be used quite frequently. For that reason, the study of discrete mathematics depends heavily on the art of *very clear deductive explanation*, which will be emphasized throughout the course.

The course is intended for students who have little or no previous experience of this kind of mathematics. There are no course prerequisites, in fact, proficiency at formal methods of symbolic manipulation will confer no advantage.

We shall be studying three main areas, separate but with some interactions: (1) graph theory; (2) relations and enumerative combinatorics; (3) coding theory.

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Text: R. P. Grimaldi, "Discrete and Combinatorial Mathematics", 5th Ed. (Pearson, 2004).

Some notes will be supplied, on my webpage, for some of the syllabus material.

Classroom hours: All the classes are in room G 236.
Section 1, Monday 15:40 - 16:30, Thursday 13:40 - 15:30.
Section 2, Tuesday 13:40 - 14:30, Thursday 15:40 - 17:30.

Office Hours: in the classroom, G 236, or in my office, room SAZ 129 of Fen A Building.

Section 1, Monday 16:40 - 15:30.
Section 2, Tuesday 14:40 - 15:30.

Office Hours is not just for the stronger students. If you cannot do the easy questions, and if you do not even understand very much of the course material then, (provided you have at least thought about it and have something to talk about), come and see me during office hours. If you think the best grade you can get is a C, then I will help you get that C (and maybe, just maybe, you might surprise yourself and get better than that). I will not be annoyed about your dreadful weakness at this kind of mathematics, because I already know that there are always many students who are dreadfully weak at it; I cannot be of much help to them if they do not come to see me!

You may also come to Office Hours for help with the homework. I will not solve the problem for you, but I can give you guidance on how to do it.

Class Announcements: All students, including any absentees from a class, will be deemed responsible for awareness of announcements made in class.

Revision Aid: Some past exams, with solutions, can be found in discretepastpapers.pdf, on my homepage.

Assessment

Homeworks: The only way to pick up skill at mathematics is through lots of practise.

You may not copy homeworks and you may not do paraphrase rewrites of homeworks by other people. If you break this rule, then you will not catch up in time for the Midterm and Final exams. Just as you cannot learn to swim by watching other people do it, you cannot aquire mathematical skill by just writing out arguments produced by others, not even if you feel that you are "understanding" it as you copy.

You should discuss the homework with each other. In this way, you will teach each other. If you cannot do a homework question, ask another student or ask me during office hours.

Miniproject Essay: A two-page essay on a topic in discrete mathematics. (A short theoretical mathematical essay may consist of, for instance, some definitions, a theorem and a proof. A short procedural mathematical essay might consist of some definitions, a general description of a procedure, and an illustration of the procedure carried out for an example. Do not write a humanities essay, say, "The anthropology of the number 6 from a neostructuralist perspective".) The rules are as follows:

• The topic may be anything in discrete mathematics, but the material must not have been covered during the lectures for this course.
• The discussion must be comprehensible to competent students in the class. It may not assume any knowledge that lies outside the course syllabus. If you make use of concepts beyond the syllabus, you must explain them clearly.
• You must use the notation and terminology of the course.

• It may be typed or handwritten. If handwritten with large handwriting, it may be three pages long. If the length is more than two pages, or three in the case of large handwriting, then the examiner will not read the extra pages.

You may use any textbook or internet source. However, because of the latter two rules above, you will have to adapt the material that you find. It would be a mistake just to copy from another source.

You are advised not to cover anything that is very advanced or very complicated. The essay will be marked according to how clearly it would communicate something with some non-trivial content to other members of the class.

Participation: This will be a mark awarded to the whole section for collective academic behaviour and participation. Asking questions is usually very helpful. All communication should be addressed to the whole class. (Making a distracting noise by murmuring to your neighbour is not proper academic behaviour.)

Principle of marking: In mathematics, marks for written work are not awarded according to guesses about what the student might have had in mind. They are awarded according to *how helpful the written explanation would be to other students in the class.*

Grading percentages:

- Quizzes, participation and homework, 20%,
- Midterm 35%, (Wednesday, 18 November, time to be arranged.)
- Essay project, 5%
- Final, 40%.

Letter Grades: This is done by the "curve method". A grade C requires an understanding of the concepts and reasonable attempts at the easiest exam questions. That fulfills the aim of the course: a competent grasp at an introductory level.

Some of the exercises and exam questions will be quite difficult. It has to be that way, not only for the benefit of the strongest students, but also because, without difficult questions, it would be hard to see the purpose of the art of *very clear deductive explanation.* However, students aiming for a grade C need not worry about being unable to do the more difficult questions.

FZ Grades: These will be awarded to students satisfying at least one of the following conditions: (a) Very poor attendance (less than 50%); (b) very poor Midterm mark (incompetance at most of the easy routine questions); (c) poor attendance and poor Midterm mark.

Attendance: A minimum of 75% attendance is compulsory. Attendance will be measured by random attendance calls and quizzes.

Syllabus

Sylabus Vote: On Thursday 1st October, a vote was taken on whether the last subject area should be (A) coding theory, or (B) generating functions. Only students attending class on that day were eligible to vote. The vote was overwhelmingly in favour of (A).

Week number: Monday date: Subtopics. Section numbers.

1: 7 Sept: (Classes only on Thursday. Classes start 9 Sept.) Discrete methods and information technology. Examples of problems in discrete mathematics. Sketch of the use of mathematical induction, 4.1.

2: 14 Sept: (The class will be taken by Ergün Yalçın, because I will be recovering from a hospital operation.) Recursive definitions and mathematical induction, 4.2. Second order recurrence relations as an application of induction, 10.2.

3: 21 Sept: (No classes, Kurban Bayram.)

4: 28 Sept: (Four hours as makeup for previous week.) Graphs. Sum of degrees formula. Circuits and Trees, 11.1, 11.2. Criteria for existence of Euler paths or Euler circuits, proved by mathematical induction, 11.3.

5: 5 Oct: Euler's characteristic formula for planar graphs, proved by mathematical induction. The non-planarity of the graphs K_5 and $K_{3,3}$, 11.4.

6: 12 Oct: Graph Colouring, 11.5. Second order recurrence relations as an application of induction, 10.2.

7: 19 Oct: Permutations, combinations, the Binomial theorem, 1.2, 1.3, 1.4.

8: 26 Oct: (No class on Thursday.) Inclusion-Exclusion Principle, 8.1.

9: 2 Nov: Sets and correspondences. Functions. Injections, surjections and bijections, 5.1, 5.2, 5.3, 5.6.

10: 9 Nov: (No class on Tuesday.) Relations. Incidence matrices. Reflexive, irreflexive, symmetric, antisymmetric and transitive relations. Enumeration of relations using incidence matrices, 7.1, 7.2.

11: 16 Nov: (Classes only on Thursday.) Midterm on Wednesday, 18 November. Exam postmortem. Partial ordering relations, 7.3.

12: 23 Nov: Hasse diagrams. Chains and antichains, 7.3.

13: 30 Nov: Equivalence relations and Stirling numbers of the Second Kind, 5.3, 7.4.

14: 7 Dec: Coding theory, Hamming metric, 16.5, 16.6. Hamming bound and Gilbert bound, 16.8.

15: 14 Dec: Parity-check and generator matrics, decoding using syndromes and coset leaders, 16.7, 16.8.

16: 21 Dec: (Classes end 24 December.) Review of enumerative combinatorics and coding theory. More practise at exercises and past exam questions.

Midterm Syllabus

The numberings are chapter and section numbers in Grimaldi.

• Mathematical Induction, 4.1, 4.2.

Test: Do you know what the term *inductive assumption* means? When writing out induction arguments, can you state the inductive assumption? (If not, ask me. Many people find this difficult.)

• Introduction to graph theory, 11.1.

Major result: Letting e be the number of edges, then $2e$ is equal to the sum of the degrees.

• Trees, 12.1.

Definition: A tree is a connected graph with no cycles.
Major result: Given a tree with n vertices and e edges, then $e = n - 1$.

• Euler paths, 11.3.

Euler's Path Theorem: Let r be the number of odd-degree vertices of a connected graph G. Then G has an Euler path if and only if $r = 0$ or $r = 2$. Also, G has an Euler circuit if and only if $r = 0$.

Test: Do you know how to find Euler paths for given graphs?

• Planar graphs, 11.4.

Euler's Characteristic Theorem: Given a connected planar graph G with n vertices and e edges, supposing some planar diagram of G has f faces, then $n - e + f = 2$.

Corollary: Given a connected graph G that is not a tree, and an integer c with $c \geq 3$, supposing that every cycle in G has length at least c, then $e \leq c(n-2)/(c-2)$.

• Second order recurrence relations, 10.1, 10.2.

Assuming $a \neq 0$ and $c \neq 0$, then formula for the solutions to $ax_{n+2} + bx_{n+1} + c = 0$ depends on whether or not the quadratic equation $aX^2 + bX + c = 0$ has a repeated solution.

• Binomial coefficients, binomial theorem. Formal and conceptual proofs of Pascal's relation and other relations for binomial coefficients. The number of ways of putting m plain balls in n coloured boxes is $\binom{m+n-1}{m} = \binom{m+n-1}{n-1}$, proved using binary strings. 1.2, 1.3, 1.4.

Test: How many natural number solutions are there to $x_1 + x_2 + x_3 + x_4 = 10$? Note that this is the same as the number of ways of putting 10 plain balls into 4 coloured boxes.

• Functions, injections, surjections, bijections, 5.1, 5.2, beginning of 5.3.

Test: Given finite sets A and B, express, in terms of the sizes $|A|$ and $|B|$,
(a) the number of functions $A \to B$,
(b) the number of injections $A \to B$.

Not examinable in the Final or Midterm: Russell's Paradox, the Halting Problem, graphs on a torus, composition of functions, invertibility of bijections.

Examinable in the Final but not in the Midterm: graph isomorphism, 11.2; surjections and Stirling numbers 5.3. (We have delayed these because they are best explained in connection with equivalence relations.)

See hw132fall15.pdf for 12 past paper questions recommended for midterm practise.

Syllabus for Final Exam

The numberings are chapter and section numbers in Grimaldi.

Note: All of the Midterm syllabus material is also on the Final Syllabus. The questions in the Final exam will primarily be on the following topics, but they may also presuppose knowledge of the Midterm material.

• Composition of functions, inverses of bijections, 5.6.

Remark: Given bijections $f : X \to Y$ and $g : Y \to Z$, then $g \circ f : X \to Z$ is bijective and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Easy Test: Let $f : X \to Y$ be an injection and let $g : Y \to Z$ be a bijection. Answer each of the following questions by giving a proof or a counter-example. Must $g \circ f$ be an injection? Must $g \circ f$ be a bijection? Must $g \circ f$ be a surjection?

• Relations on a set. Reflexive, anti-reflexive, symmetric, anti-symmetric, transitive relations. (An anti-reflexive relation on a set X is a relation $\sim$ on X such that $x \not\sim x$ for all $x \in X$. The book uses the misnomer "irreflexive" for this property.) Counting relations using incidence matrices, 7.1, 7.2.

• Partial ordering relations. Hasse diagrams. Recall, a **total ordering** on a set x is a partial ordering $\leq$ on X such that $x \leq y$ or $y \leq x$ for all $x, y \in X$. 7.3.

• Equivalence relations, equivalence classes, 7.4.

Theorem: Given an equivalence relation $\equiv$ on a set X, then every element of X belongs to exactly one equivalence class of $\equiv$. In other words, X is the disjoint union of the equivalence classes.

• Isomorphism of partial orderings. Isomorphism of graphs. Recall, the isomorphism relation is an equivalence relation. Roughly covered by 7.3, 7.4, 11.2.

• Surjections, equivalence relations and Stirling numbers (of the second kind), 5.3, 7.4.

Recall, the Stirling number $S(m, n)$ is:

(a) The number of ways of putting m coloured balls into n plain boxes with no box empty.
(b) The number of equivalence relations with n equivalence classes on a set of size m.

The number $n!S(m, n)$ is:

(a) The number of ways of putting m coloured balls into n coloured boxes with no box empty.
(b) The number of surjections from a set of size m to a set of size n.

One way to calculate Stirling numbers is to build up a table using the recurrence relation in the following theorem.

Theorem: For integers $1 \leq n \leq m$ we have $S(m, 1) = 1 = S(m, m)$ and

$$S(m + 1, n) = S(m, n - 1) + nS(m, n) .$$

• General principles of coding theory, the Haming metric, 16.5, 16.6.

Remark: Let C be a code in $\mathbb{Z}_2^n$. Let d be the minimum distance between distinct codewords of C. We can detect s errors of transmission when $s + 1 \leq d$. We can correct t errors of transmission provided $2t + 1 \leq d$.

Remark: Let C and d be as above. If C is a linear code, then d is the minimum weight of a non-zero codeword.

• The generating matrix $G = [I|A]$. Constructing the decoding table. The parity-check matrix $H = [A^T|I]$. Decoding using coset leaders and syndromes. 16.7, 16.8.

Recall, for an encoding scheme $\mathbb{Z}_2^m \to \mathbb{Z}_2^n$ with generating matrix G as above, the process for decoding a received word r is:
(1) Calculate the syndrome Hr^T.

(2) Find the coset leader s with the same syndrome $Hr^T = Hs^T$.
(3) Calculate the codeword $c = r + s$.
(4) Determine the message word w just by reading off the first m digits of c.

The following item, advertised in the weekly syllabus, is not on the Final Exam syllabus because we did not cover it: Week 12 "Chains and antichains". (This was intended as a reference to Dilworth's Theorem, which asserts the equivalence of two definitions of the *width* of a partial ordering on a finite set. Unfortunately, Dilworth's Theorem is not in the textbook but, at the time of writing, proofs can be found in the Wikipedia article called "Dilworth's Theorem". Let us just state it: A **chain** in a poset P is a totally ordered subposet of P. A poset is said to be **discrete** provided it has no non-trivial chains. An **antichain** is a discrete subposet of P. Dilworth's Theorem asserts that, when P is finite, the maximal size of an antichain is equal to the minimal m such that P is a union of m chains. We call m the **width** of P.)

MATH 132, Discrete and Combinatorial Mathematics
Homeworks and Quizzes, Fall 2015

Laurence Barker, Mathematics Department, Bilkent University,
version: 20 January 2016.

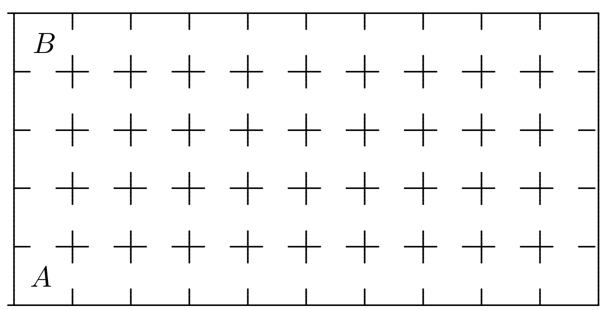
Office Hours: Section 1: Mondays 16:40 - 17:30. Section 2: Tuesdays 14:40 - 15:30. The Office Hours are to be held in the classroom, G 236, or in my office, room SAZ 129 of Fen A Building.

Office Hours would be a good time to ask me for help with the homeworks.

Homework 0 Due 1 October.

This warm-up homework has only half the credit of a normal homework.

Question 0: A prisoner is trapped in a dungeon consisting of 50 cells arranged in a 10 by 5 grid. She is free to pass from cell to cell via the doorways indicated in the diagram. Can she make a tour of the dungeon, visiting each cell exactly once, starting in the bottom-left cell labelled A, finishing in the top-left cell labelled B? (Hint: consider the number of up moves, the number of down moves, the number of left moves, the number of right moves.)



Solution: Each dominoes piece covers one black and one white square. If the mutilated board can be covered, then it has the same number of blacks and whites. The original board does have the same number of blacks and whites, but the two removed squares have the same colour, so the mutilated board does not have the same number of blacks and whites.

Another solution: No, such a tour is impossible. For a contradiction, suppose such a tour exists. Let U, D, L, R be the number of moves, from one cell to the next one, in the upwards, downwards, leftwards, rightwards directions, respectively. Then $U - D = 4$. So U and D and either both odd or else both even. It follows that $U + D$ is even. We also have $L = R$, so $L + R$ is even. But the total number of moves made in the tour is $U + D + L + R = 49$, which is odd. This is a contradiction, as required. □

Section 1, Quiz 1: *Monday 5th October.* Let $x_0, x_1, x_2, ...$ be an infinite sequence such that $x_0 = 2$ and $x_{n+1} = 2x_n$ for all integers $n \geq 0$. Give a formula for x_n.

Solution: Obviously, $x_n = 2^{n+1}$.

Section 2, Quiz 1: *Tuesday 6th October.* Let $x_0, x_1, x_2, ...$ be an infinite sequence such that $x_0 = 25$ and $x_{n+1} = 5x_n$ for all integers $n \geq 0$. Give a formula for x_n.

Solution: Obviously, $x_n = 5^{n+2}$.

Homework 1 Due Thursday, 8 October.

1.1: Show that $F_0 + F_1 + ...F_n = F_{n+2} - 1$ for all integers $n \geq 0$, where F_n denotes the n-th Fibonacci number (with $F_0 = 0$ and $F_1 = 1$).

1.2: Show that $L_1^2 + L_2^2 + ... + L_n^2 = L_n L_{n+1} - 2$ for all integers $n \geq 1$, where L_n denotes the n-th Lucas number (with $L_0 = 2$ and $L_1 = 1$).

1.3: Give an example of a graph G such that G and $\overline{G}$ are connected (where $\overline{G}$ denotes the complement of G).

1.4: Show that, if a graph G is not connected, then $\overline{G}$ is connected. (Note that the graph with only one vertex is understood to be connected.)

Section 1, Quiz 2: *Thursday, 8th February.* [The depicted graph is the direct product of a 4-cycle and a tree of order 2.] **(a)** What is the minimum number of edges that need to be deleted from the depicted graph to produce a graph with an Euler circuit? (b) Delete such edges and find an Euler circuit.

Answer: Delete 4 edges. Apply the method in the proof of the Euler Path Theorem.

Section 2, Quiz 2: *Thursday, 8th February.* [As in Section 1 Quiz 2, but with the depicted graph being the direct product of a 3-cycle and a linear tree with order 4.]

Solution: Delete 5 edges, then apply the method in the proof of the Euler Path Theorem.

Homework 2 Due Thursday, 22 October.

If you can do Questions 2.1 and 2.2, that is satisfactory. If you can also do 2.3, that is good, because it requires several lines of proof. I do not know whether people will find 2.4 very difficult, but maybe it will seem very difficult, because it requires you to discover a theorem.

For each positive integer n, let O_n be the graph with $2n$ vertices such that every vertex has degree $2n - 2$.

2.1: How many edges does O_n have?

2.2: For which values of n does O_n have an Euler path? Find an Euler path in the case $n = 4$.

2.3: Let G be a graph with an Euler circuit. Construct a new graph H by starting from G, adding two new vertices x and y and, for each vertex z of G, adding edges xz and yz. Show that H has an Euler path.

2.4: Let G be a connected graph and suppose that, for each edge ϵ of G, there is an associated positive integer $w(\epsilon)$, called the **weight** of ϵ. For a vertex x of G, we define the **weighted**

degree of x, denoted $\mathrm{wd}(x)$, to be the sum of the weights of the edges with end-point x. We define a **weighted Euler circuit** of G to be a circuit such that each edge ϵ is used $w(\epsilon)$ times. State and prove a theorem which expresses, in terms of the weighted degrees, a necessary and sufficient criterion for G to have a weighted Euler circuit.

Solution 2.1: The number of edges e is given by $2e = 2n(2n-2)$, hence $e = 2n(n-1)$.

Solution 2.2: The graph O_n has an Euler path if and only if $n \geq 2$. That is because, for all $n \geq 1$, every vertex has even order, furthermore, O_n is connected except in the case $n = 1$.

Solution 2.3: Since G has an Euler circuit, every vertex z of G has even degree $d_G(z)$. Reinterpreting z as a vertex of H, then z still has even degree $d_H(z) = d_G(z) + 2$. Let n be the number of vertices of H. Then $n = d_H(x) = d_H(y)$. So, if n is even, then H has an Euler circuit, otherwise H has an Eulr path from x to y.

Solution 2.4: We construct a connected graph H such that every vertex of G is a vertex of H and, for every edge ϵ of G, there are $w(\epsilon)$ vertices ϵ_1, ..., $\epsilon_{w(\epsilon)}$ of H. The edges of H are the pairs $\{x, \epsilon_i\}$ where x is an end-point of ϵ. Plainly, G has a weighted Euler circuit if and only if H has an Euler circuit in the ordinary sense. Thanks to Euler's Path Theorem, we deduce the following theorem:

Theorem: Given a conected graph G with weighted edges as above, then G has a weighted Euler circuit if and only if $\mathrm{wd}(x)$ is even for all vertices x.

Comment: The latest theorem can also be proved by adapting the usual proof of the Euler Path Theorem.

Section 1, Quiz 3: *Monday, 19 October.* Let G be a connected planar graph with 12 vertices such that every vertex has degree 5. Find the number of faces.

Solution: Recall that $n - e + f = 2$, where n, e, f are the numbers of vertices, edges and faces, respectively. So $f = 2 - n + e$. By the Handshaking Lemma, $e = 12.5/2 = 30$. Therefore $f = 2 - 12 + 30 = 20$.

Section 2, Quiz 3: *Tuesday, 20 October.* Let G be a connected planar graph with 20 vertices such that every vertex has degree 3. Find the number of faces.

Solution: Arguing as in Section 1 Quiz 3, $e = 20.3/2 = 30$ and $f = 2 - 20 + 30 = 12$.

Comment: Examples of graphs as in Section 1 Quiz 3 and Section 2 Quiz 3 are the icosahedron and the dodecahedron. They are mutual duals, which is why n, e, f for the former coincide with, respectively, f, e, n for the latter.

Section 2, Quiz 4: *Monday, 27 October.* Let G be a connected planar graph such that every cycle has length at least 4. Show that some vertex x has degree $d(x) \leq 3$.

Solution: We may assume that G is not a tree, because otherwise the assertion is trivial. Then $e \leq c(n-2)/(c-2)$ where e is the number of edges, n is the number of vertices and $c = 4$. Hence $e \leq 2n - 4$. So the sum of the degrees of the vertices is $2e$, which is less than or equal to $4n - 8$. Therefore, the average degree is less than 4. The required conclusion follows. □

Homework 3 Due Thursday, 5 November.

3.1: Do Question 2.4 (which has now been discussed in class).

3.2: We say that a graph G is **properly toroidal** provided it can be drawn on a torus in such a way that no two edges intersect and every face is a curved polygon without holes. You may assume that, for such G, we have

$$n - e + f = 0$$

where n, e, f are the numbers of vertices, edges and faces, respectively. Give an example of a properly toroidal graph such that every vertex has degree 6.

3.3: Show that every properly toroidal graph has a vertex x such that $d(x) \leq 6$.

3.4: Let c and d be integers with $c \geq 3 \leq d$. Let G be a planar graph such that every face is a regular polygon with c edges and every vertex has degree d.
(a) Let e be the number of edges. Show that

$$\frac{1}{d} + \frac{1}{c} = \frac{1}{2} + \frac{1}{e} \, .$$

(b) Deduce that the possible values for the pair (c, d) are $(5, 3)$, $(4, 3)$, $(3, 3)$, $(3, 4)$, $(3, 5)$.
(c) For each of those five possible values of (c, d), find e and the number of vertices n and the number of faces f.
(d) Look up the Wikipedia article on **Platonic solid**. Briefly, use part (b) to explain why there are only five Platonic solids: the tetrahedron, the cube, the octahedron, the dodecahedron and the icosahedron. (Strictly speaking, it is not obvious that the dodecahedron and icosahedron exist, but you may assume that they do exist.)

Unmarked homework: Twelve past paper questions recommended for Midterm revision

These questions are to be found in the new version of the file discretepastpapers.pdf on my homepage.

There are many other graph theory, induction and recurrence relation questions in that file. There are not so many suitable questions on binomial coefficients and functions, because the Midterm cuts the enumerative combinatorics part of the course in half. Most of the enumerative combinatorics questions involve concepts that are not on the Midterm syllabus.

I suggest you show me some of your solutions during Office Hours. I will not be able to mark them (accurate marking takes time, and I cannot do it quickly while people are waiting), but I can make comments if I notice an important mistake.

Page 1, (Midterm 1, MATH 210, Spring 2015), questions 1, 2, 3, 4, 5, 6.

Page 4, (Midterm 2, MATH 210, Spring 2015), questions 1, 2, 5.

Page 25, (Final, MATH 132, Spring 2014), question 2.

Page 32, (Midterm 2, MATH 110, Fall 2013), question 2.

Page 36, (Makeup 2, MATH 110, Fall 2013), question 2.

Section 1, Quiz 4: *Thursday, 5 November.* [In class, we let a_n be the number of ways of cutting an $n \times 2$ rectangle into 1×2 rectangles. We observed that $a_1 = 1$ and $a_2 = 2$ and $a_{n+2} = a_{n+1} + a_n$.]

(a) Complete the proof that $a_n = \dfrac{1}{\sqrt{5}}\left(\left(\dfrac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\dfrac{1-\sqrt{5}}{2}\right)^{n+1}\right)$.

(b) Let $L_{n+2} - L_{n+1} - L_n = 0$ with $L_0 = 2$ and $L_1 = 1$. Find a formula for L_n.

Solution: Part (a). The roots to $X^2 - X - 1$ are $\phi = (1+\sqrt{5})/2$ and $\psi = (1-\sqrt{5})/2$. By a standard formula, $a_n = A\phi^n + B\psi^n$ for some A and B. The initial conditions are satisfied when $A = 1/\sqrt{5}$ and $B = -1/\sqrt{5}$. The required formula for a_n is now established.

Part (b). The auxilliary quadratic equation is the same as in part (a), so $L_n = P\phi^n + Q\psi^n$. The initial conditions are satisfied when $P = Q = 1$. So $L_n = \left(\dfrac{1+\sqrt{5}}{2}\right)^{n+1} + \left(\dfrac{1-\sqrt{5}}{2}\right)^{n+1}$.

Comment: Incidentally, $a_n = F_{n+1}$, a shift of the Fibonacci sequence. The number L_n is caled the n**-th Lucas number**.

Section 2, Quiz 5: *Thursday, 5 November.* [In class, we let b_n be the number of ways of cutting an $n \times 2$ rectangle into 1×2 and 2×2 rectangles. rectangles. We observed that $b_1 = 1$ and $b_2 = 3$ and $b_3 = 5$ and $b_n = b_{n-1} + 2b_{n-2}$ when $n \geq 3$. Give a formula for b_n.

Solution: The quadratic $X^2 - X - 2$ has roots 2 and -1, so $b_n = A2^n + B(-1)^n$ for some A and B. Putting $n = 1$ and $n = 2$, we solve for A and B, obtaining $A = 2/3$ and $B = -1/3$. So $b_n = (2^{n+1} + (-1)^n)/3$.

Section 1, Quiz 5: *Thursday, 26 November.* How many reflexive transitive relations are there on the set $\{1, 2\}$?

Solution: The diagonal entries of the incidence matrics are all 1, by reflexivity. All 4 of the 2×2 zero-one matrices with this property satisfy the transitivity axiom. So the answer is 4.

Section 2, Quiz 6: *Thursday, 26 November.* How many reflexive symmetric transitive relations are there on the set $\{1, 2, 3\}$?

Solution: There are 5 such relations. Their incidence matrices are as shown.

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \quad \begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}, \quad \begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}, \quad \begin{pmatrix}1&1&1\\0&1&0\\1&1&1\end{pmatrix}, \quad \begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}.$$

Comment: This quiz was set prior to any lecture discussion of equivalence relations. If the notion of an equivalence relation has been studied, then the question can be done much more easily: there is 1 equivalence relation on $\{1, 2, 3\}$ with 3 equivalence classes; 3 equivalence relations with 2 equivalence classes; 1 equivalence relation with 3 equivalence classes. So the answer is $1 + 3 + 1 = 5$.

Section 1, Quiz 6: *Thursday, 3 December.* Consider the following theorem and incomplete proof. Finish the proof.

Theorem: Let S be a set whose elements are sets. Let $\equiv$ be the relation on S such that, given $A, B \in S$, we have $A \equiv B$ provided there exists a bijection $A \to B$. Then $\equiv$ is an equivalence relation.

Proof: Let $A, B, C \in S$. Reflexivity: the identity function $\mathrm{id}_A : A \to A$ is a bijection, so $A \equiv A$. Symmetry: Suppose that $A \equiv B$. Let $f : A \to B$ be a bijection $A \to B$. Then $f^{-1} : B \to A$ is a bijection. So $B \equiv A$. Transitivity: Suppose that $A \cong B$ and $B \cong C$. Let $f : A \to B$ and $g : B \to C$ be bijections. Then ...

Solution: ... $g \circ f : A \to C$ is a bijection, so $A \cong C$. $\square$

Comment 1: Many people wrote half a page or even a whole page of waffle. Even if the right idea is buried in there, excessive writing makes it difficult for the reader to extract the idea. Try to just give the reader the information she needs, and then stop.

Comment 2: Let me repeat some remarks made in class. We observed that, when every element of S is finite, there is a quicker argument: $\equiv$ is an equivalence relation because, in fact, $\equiv$ is the equivalence relation associated with the function $S \to \mathbb{N}$ that sends each A to its size $|A|$. However, without a notion of size for infinite sets, we cannot apply that idea for arbitrary S. Actually, in the 1870s, Cantor turned the idea around, in effect **defining** the cardinality $|A|$ of a set A to be the equivalence class $[A]$. We mention that there are some logical quibbles here. Replacing S with the collection of all sets is dodgy because that collection is not a set. However, there are some set-theoretic techniques for dealing with this trouble in a satisfactory way.

Homework 4 Due Thursday, 17 December.

4.1: Let X_0, X_1, ..., X_n be sets and let f_1, ..., f_n be functions such that $f_i : X_{i-1} \to X_i$. Let $f = f_n \circ f_{n-1} \circ ... \circ f_1 \circ f_1$.
(a) Show that, if each f_i is injective, then f is injective.
bf (b) Show that, if each f_i is surjective, then f is surjective.

4.2: Let $T = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. We define a relation $\equiv$ on $T \times T$ such that $(a, b) \equiv (c, d)$ if and only if $ad = bc$.
(a) Show that $\equiv$ is an equivalence relation.
(b) How many equivalence classes does $\equiv$ have? (Hint: For each equivalence class E, consider the smallest a and b such that $(a, b) \in E$.)

4.3: This is essentially Exercise 5.3.8 in Grimaldi. A mathematician has 9 papers to referee and 5 research students. In how many ways can she assign papers to the students such that every student referees at least 1 paper?

4.4: This is Exercise 16.7.12 in Grimaldi. Let n and k be integers with $n > 0 \leq k$. Let M be a code with the maximum possible number of codewords subject to the condition that the distance between any two distinct codewords is at least $2k + 1$. Prove:

Gilbert lower bound: $2^n / \sum_{i=0}^{2k} \binom{n}{i} \leq |M|$.

Hamming upper bound: $|M| \leq 2^n / \sum_{i=0}^{k} \binom{n}{i}$.

Miniproject Essay. Due Thursday, 24 December (last day of classes).

For the rules and some guidelines, see the *Assessment* section of intro132fall15.pdf.

I cannot list 139 essay topics. You will have to choose a topic yourself. To give you an indication of some possibilities, the following are the titles of some 5-minute presentations in a discrete mathematics course with fewer students.

Economic Applications of Discrete Mathematics: Game Theory.
Convolution Coding and Viterbi Decoding.
Dijkstra's Shortest Path Algorithm.
Recurrence Relations.
Magic Squares.
Karnaugh Maps.
Map Colouring.
Path-Finding Algorithms.
Boolean Algebra.
Contrast Adjustment of Images by Histogram Smoothing.
Prisoner's Dilemma and Nash Equilibrium.
The Ford-Fulkerson Algorithm.
The Papoulis-Gerchberg Algorithm.
The Bellman-Ford Algorithm for Single-Source Shortest Paths.
Reed Solumon Codes,
Gambler's Ruin as a Markov Chain Process.
Some Mathematical Puzzles.
Hidden Markov Models and Viterbi Decoding.

Section 1, Quiz 7: *Thursday, 10 December.* State and prove a formula for $S(n+1,n)$ where n is a positive integer.

Solution: The number of ways of putting $n+1$ coloured balls into n plain boxes is

$$S(n+1,n) = \binom{n+1}{2} = n(n+1)/2$$

because the arrangement is determined by the choice of the two balls that are to share a box.

Section 2, Quiz 7: *Thursday, 10 December.* How many ways are there of putting 7 coloured balls into 4 numbered containers such that the blue ball goes into container number 2 and no box is left empty?

Solution: There are $3!S(6,3)$ arrangements with only the blue ball in container 2. There are $4!S(6,4)$ arrangements with at least one other ball in that container. So the answer is

$$4!S(6,4) + 3!S(6,3) = 24.65 + 6.90 = 2100\ .$$

MATH 132: DISCRETE AND COMBINATORIAL MATHEMATICS.

Midterm, Fall 2015, Bilkent, LJB, 18 November 2015

Time allowed: two hours.
Please put your name and section number on *every* sheet of your manuscript.
Submit one sheet of paper for each question that you attempt (maximum five sheets).
No sheet may have solutions for two separate questions. (Misplaced parts will not be marked).
You may use extra sheets of paper for rough work (which does not need to be handed in).
Do not place sheets of written work in such a way that other candidates can see it.
All use of electronic devices is strictly prohibited. Please remember to silence your phone.
Please DO NOT hand in the question sheet! The examiner already knows the questions.

Question 1: 10% The game of simplified nim begins with two piles of matches. Each pile has the same number of matches. Player A moves first, then Player B, then A, then B, and so on. On each move, the player must remove some or all of the matches from one of the piles, and may not remove any matches from the other pile. The winner is the player who removes the last match. Using argument by induction, show that, if Player B plays correctly, then Player B will win.

Question 2: 20% Let x_0, x_1, ..., be an infinite sequence such that $x_0 = 1$ and $x_1 = -2$ and $x_{n+2} + 2x_{n+1} + x_n = 0$ for all $n \geq 0$. Find a formula for x_n.

Question 3: 30% Let G_1 and G_2 be conected graphs with no shared vertices. Let x_1 and x_2 be vertices of G_1 and G_2 respectively. Let G be the graph consisting of the vertices and edges of G_1 and G_2 together with the edge $x_1 x_2$.
(a) Is it true that, if G_1 and G_2 both have Euler circuits, then G must have an Euler path but no Euler circuit? (Give a proof or a counter-example.)
(b) Is it true that, if G has an Euler path but no Euler circuit, then G_1 and G_2 must both have Euler circuits? (Give a proof or a counter-example.)
(c)[1] Suppose that G_1 and G_2 both have 7 vertices and 21 edges. Find an Euler path for G, describing your Euler path by listing the vertices in order. (Diagrams with tiny handwriting will not be marked.)

Question 4: 20% Let m and n be positive integers with $m \geq n$. How many ways are there of putting m plain balls into n coloured boxes such that no box is left empty? (We understand plain to mean indistinguishable and coloured to mean distinguishable.)

Question 5: 20% Let G be a connected planar graph with n vertices and e edges.
(a) State, without proof, a relationship between n and e. Use it to deduce that some vertex of G has degree less than 6.
(b) Show that, for any planar diagram of G, some face shares an edge with less than 6 other faces.

[1] In the original exam paper, this question had a typo.

Solutions to MATH 132 Midterm Fall 2015

1: We must prove that, for all positive integers n, the position with n matches in both piles is a losing position. We shall argue by induction on n. The case $n = 1$ is trivial. Now suppose that $n \geq 2$ and that the assertion holds for all positive integers less than n. Letting m be the number of matches removed by the first player, then then other player should remove m matches from the other pile, thus obtaining the position with $n - m$ matches in both piles, which is a losing position by the inductive assumption.

Comment: This is an argument by so-called "strong induction". The inductive assumption is "$n \geq 2$ and ... the assertion holds for all positive integers less than n". Note that the argument does not work with the "weak induction" assumption "$n \geq 2$ and the assertion holds with n replaced by $n - 1$". If you failed to state any inductive assumption, then your argument is incomplete.

There were 5 marks for finding the winning strategy.

Some candidates did not understand the meaning of the term "inductive assumption". The term is the name of the assumption at the beginning of the "inductive step". When doing the "inductive step", you start with the inductive assumption, then you make deductions from that until you arrive at the next term of the sequence of statements that you are trying to prove.

Below, after this section on solutions and coments, four outstanding induction arguments are quoted from exam scripts.

2: The quadratic equation $X^2 + 2X + 1 = 0$ has unique solution $X = -1$. Therefore $x_n = (C + nD)(-1)^n$ for some C and D. We have $1 = x_0 = C$ and $-2 = (C + D)(-1)^1$, hence $D = 1$ and $x_n = (1 + n)(-1)^n$.

Comment: This is a routine question. Recall, to obtain grade C, you should be able to do this kind of question, and you should also be able to partially deal with some more difficult problem questions and some easy theoretical questions.

Almost all the marks are for getting the method right and presenting it clearly. For mistakenly finding repeated solution $X = 1$, then making another mistake of calculation to arrive at $C = 1 = -D$, concluding that $x_n = 1 - n$, the mark would be 16 out of 20.

3: Part (a). Yes. Suppose that G_1 and G_2 have Euler circuits C_1 and C_2. We may assume that C_1 starts and finishes at x_1 and that C_2 starts and finishes at x_2. Then the G has an Euler path consisting of C_1 followed by the edge x_1x_2 followed by C_2.

Part (b). No. One counter-example is the case where G_1 has 2 vertices and G_2 has 1 vertex.

Part (c)[2]. Let a_0, ..., a_6 be the vertices of G_1. Let b_0, ..., b_6 be the vertices of G_2. Since G_1 has $7.6/2 = 21$ edges, G_1 is a copy of K_7. Renumbering if necessary, we may assume that $x_1 = a_0$. Similarly, G_2 is a copy of K_7 and we may assume that $x_2 = b_0$. One Euler path of G is

$$a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_0, a_2, a_4, a_6, a_1, a_3, a_5, a_0, a_3, a_6, a_2, a_5, a_1, a_4, a_0,$$

$$b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_0, b_2, b_4, b_6, b_1, b_3, b_5, b_0, b_3, b_6, b_2, b_5, b_1, b_4, b_0.$$

[2]Because of the typo in the original exam paper, no marks were awarded for this part.

Alternative for part (a): If G_1 and G_2 have Euler circuits, then all their vertices have even degree, hence x_1 and y_1 are the only two vertices of G with odd degree. Since G is connected, it follows that G has an Euler path from x_1 to x_2.

Comment: Many candidates lost 1 mark for failing to state the answer to (a), another mark for failing to state the answer to (b). The answer to (a) is "Yes" or "It is true". The answer to (b) is "No" or "It is false". You should state answers or conclusions at the beginning or at the end of your explanation. Your reader does not wish to expend energy trying to deduce your answer from a long paragraph of discussion.

To prove a universal positive assertion, say, "All ravens are black", you must present a general proof that, given any raven, then it is black. To refute a universal positive assertion, you need only present a counter-example, a non-black raven.

Thus, part (a) requires a general proof about any two graphs G_1 and G_2 that have Euler circuits. But part (b) just requires you to specify one counter-example. In part (b), it is not helpful to give a story about why some particular line of reasoning would not work. The discovery of a flaw in an argument does not imply that the conclusion of the argument is false. Can you see the huge logical flaw in the following argument? "Ravens haunt the darkness of the night. However, this does not imply that all ravens are black. Therefore, some raven is non-black."

4: By placing one ball in each box and then considering the possible arrangements of the other balls, we see that the answer is the number of ways of placing $m - n$ balls in n boxes. By a standard formula, this number is $\binom{(m-n)+(n-1)}{n-1} = \binom{m-1}{n-1}$.

Comment: Of course, the answer can also be expressed as $\binom{m-1}{m-n}$.

Many candidates gave, as answer, the memorized formula $\binom{n+m-1}{n-1}$. But this is the standard answer to the similar question where there is no requirement that each box has at least one ball. One cannot do this kind of question if one remembers only the standard formula. It is better to forget the formula and to understand the idea behind it.

5: Part (a). Recall, $e \leq 1$ or $e \leq 3n - 6$. Let x_1, ..., x_n be the vertices. Trivially, if $e \leq 1$ then $d(x_1) < 6$. Otherwise, $d(x_1) + ... + d(x_n) = 2e \leq 6n - 12$. Since the average of the degrees $d(x_i)$ is less than 6, at least one of the $d(x_i)$ is less than 6.

Part (b). Let $\widehat{G}$ be a planar diagram representing G. Let F be the graph such that the vertices of F are the faces of $\widehat{G}$, two vertices of F being adjacent provided, as faces of $\widehat{G}$, they share at least one edge. By regarding G as a map of countries, reinterpreting the vertices of F as capital cities and reinterpreting the edges of F as border-crossing roads between capital cities, we see that F is connected and planar. The required conclusion now follows by applying part (a) to F.

Comment: Of 139 registered students, only 2 were able to do part (b). We saw the key idea in lectures, when we were discussing duality of Platonic solids.

Four responses to Question 1: how not to apply mathematical induction

The four candidates whose solutions appear below need not care about it. The best mathematicians can make awful howlers, especially when under pressure to think quickly. I presume that, instead of writing nothing, these four candidates were trying to pick up a mark or two by random luck. Nevertheless, imitation is no substitute for understanding.

Conceptual style:

Proof by induction. Base case: Player B plays correctly and Player A plays poorly and Player B wins. Inductive step: we assume that Player B plays correctly on every move and wins the game and Player A plays poorly, and this assumption is our inductive hypothesis. $\square$

Conceptual style:

First move m_1 is played correctly by player B. Let's assume any move m_k for all k is played correctly by player B. If player B plays move m_{k+1} correctly, player B wins the game. For each correct move if the next move is correct all moves are correct. $\square$

Analytically formal style:

$$a_n = \underbrace{1+2+3+\ldots+k}_{\text{matches}} = k(k+1)/2$$

$$S_i = 1+2+3+\ldots+n = n(n+1)/2$$

$$P(k+1) = (k+1)(k+2)/2$$

1) $1 \stackrel{?}{=} 1$ ✓

2) $1+2+3+\ldots+k+(k+1) = \dfrac{k(k+1)}{2} + (k+1)$

$$= k^2+k+2k+2 = \frac{k^2+3k+2}{2} = \frac{(k+2)(k+1)}{2} \;\checkmark \quad \square$$

Logically formal style:

$$S_n - S_{n-1} = T_n - T_{n-1}$$

$A \rightarrow a$ times n (piles) $\longrightarrow S_n$

$$na - (na - a) = an - a - (an - 2a)$$

$B \rightarrow a$ times $n-1$ (piles) $\longrightarrow$ let's say T_n $\square$

MATH 132: DISCRETE AND COMBINATORIAL MATHEMATICS.

Midterm Makeup, Fall 2015, Bilkent, LJB, 21 December 2015

Time allowed: two hours.
Please put your name and section number on *every* sheet of your manuscript.
Submit one sheet of paper for each question that you attempt (maximum five sheets).
No sheet may have solutions for two separate questions. (Misplaced parts will not be marked).
You may use extra sheets of paper for rough work (which does not need to be handed in).
Do not place sheets of written work in such a way that other candidates can see it.
All use of electronic devices is strictly prohibited. Please remember to silence your phone.
Please DO NOT hand in the question sheet! The examiner already knows the questions.

Question 1: 20% Recall that the Fibonacci numbers F_0, F_1, ... are defined by the condition that $F_0 = 0$ and $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$. Show that F_{4m} is divisible by 3 for all positive integers m.

Question 2: 20% Let x_0, x_1, ... be an infinite sequence such that $x_0 = x_1 = 1$ and $x_{n+2} - 2x_{n+1} + 2x_n = 0$ for all $n \geq 0$. Consider an integer $m \geq 0$. Give a formula for x_{4m} that does not involve complex numbers.

Question 3: 20% We have 8 red balls, which are indistinguishable from each other. We also have 15 blue balls, which are indistinguishable from each other. How many ways are there of putting the balls into 4 numbered boxes such that, for each box, the number of blue balls is greater than the number of red balls?

Question 4: 20% A **cycle**, recall, is a circuit that has no repeated vertices except that the last vertex is the same as the first vertex. Let G be a graph. Show that every vertex of G has even degree if and only if the edges of G can be coloured in such a way that, for each colour, the edges with that colour form a cycle.

Question 5: 20% Let G be a planar graph with at least 3 edges. Let n and e be the number of vertices and the number of edges, respectively. Show that $e = 3n - 6$ if and only if it is impossible to add a new edge such that the new graph is planar.

Solutions to MATH 132 Midterm Makeup, Fall 2015

1: We shall prove, by induction, that $F_{4m} \equiv 0$ and $F_{4m+1} \equiv 1$ modulo 3. The case $m = 0$ is given by the initial conditions. Now suppose that $m \geq 1$ and that $F_{4m-4} \equiv 0$ and $F_{4m-3} \equiv 1$. Then $F_{4m-2} \equiv 1$, hence $F_{4m-1} \equiv 2$, yielding the required conclusion. □

2: The quadratic equation $X^2 - 2X + 2 = 0$ has solutions $1 + i$ and $1 - i$. So $x_n = A(1+i)^n + B(1-i)^n$ for some A and B. The initial conditions yield $A = B = 1/2$. Therefore

$$x_{4m} = ((1+i)^{4m} + (1-i)^{4m})/2 = (-4)^m .$$

3: Put one blue ball in each box, then glue each red ball to one of the remaining blue balls. The number of ways of putting the 8 glued pairs of balls into the boxes times the number of ways of putting the last 3 blue balls into the boxes is

$$\binom{8+4-1}{8}\binom{3+4-1}{3} = \binom{11}{8}\binom{6}{3}$$

$$= \frac{11\,.\,10\,.\,9\,.\,6\,.\,5\,.\,4}{3\,.\,2\,.\,3\,.\,2} = 11\,.\,5\,.\,3\,.\,5\,.\,4 = 11\,.\,3\,.\,100 = 3300 .$$

4: Suppose that every vertex of G has even degree. Arguing by induction on the number of edges e, we shall prove that the edges of G can be coloured as specified. The case $e = 0$ is trivial. Now suppose that $e \neq 0$ and that the assertion holds in all cases with fewer edges. Any forest with at least one edge has at least one vertex with degree 1. So G is not a forest, in other words, G has a cycle of positive length. Colour the edges of that cycle red. Removing the red edges, we obtain a graph H satisfying the required conditions and with fewer edges. By the inductive assumption, we can colour the edges of H in the required way, without using red. Half of the required conclusion is now established. The converse half is obvious. □

5: Suppose it is impossible to add a further edge. Then G must be connected, every face must be a triangle, and every edge must have two distinct faces on either side. The number of pairs (ϵ, F), where ϵ is an edge of face F, is $2e = 3f$. But $n - e + f = 2$, hence $e = 3n - 6$.

Conversely, suppose that $e = 3n - 6$. Let G_1, ..., G_k be the connected components of G. Let n_i and e_i, respectively, be the number of vertices and edges of G_i. If G is a forest, then $e < n$, hence $e \geq 3(e+1) - 6 = 3e - 3$, contradicting the condition that $e \geq 3$. So G is not a forest and we may assume that G_1 is not a tree, hence $e_1 \leq 3n_1 - 6$. For $i \neq 1$, we have $e_i < n_i$ when G_i is a tree, $e_i \leq 3n_i - 6$ otherwise. The equality $e = 3n - 6$ now implies that $k = 1$, in other words, G is connected.

Let f be the number of faces of G, and let c be the average length of the circuit around the border of a face. A counting argument as above, counting (ϵ, F) twice when F is on both sides of ϵ, yields $2e = cf$. Again using $n - e + f = 2$, we obtain $n - 2 = e(1 - 2/c)$, hence $3n - 6 = e = (n-2)c/(c-2)$. But $c \geq 3$ and the function $c \mapsto c/(c-2)$ is strictly monotonically decreasing, hence $c = 3$, in other words, every face is a triangle.

MATH 132: DISCRETE AND COMBINATORIAL MATHEMATICS

Final, Fall 2015, Bilkent, LJB

2 January 2016

Time allowed: two hours.
Please put your name and section number on *every* sheet of your manuscript.
Submit one sheet of paper for each question that you attempt (maximum five sheets).
No sheet may have solutions for two separate questions. (Misplaced parts will not be marked).
You may use extra sheets of paper for rough work (which does not need to be handed in).
Do not place sheets of written work in such a way that other candidates can see it.
All use of electronic devices is strictly prohibited. Please remember to silence your phone.
Please do not hand in the question sheet! The examiner already knows the questions.

Question 1: 40% Consider the coding scheme such that the message words and received words are binary strings and the generating matrix is $G = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$.

(a) Write down the 8 message words and their corresponding codewords.
(b) Without writing down the decoding table, explain why the decoding function cannot use all of the weight 1 binary strings as coset leaders. (Recall, the *coset leaders* are the received words that appear in the column of the decoding table underneath the message word with weight zero.)
(c) Making use of syndromes, and without writing down the decoding table, show that the binary strings

000000 , 000001 , 000010 , 000100 , 001000 , 010000 , 000011 , 000110

can be used as the coset leaders.
(d) Without writing down the decoding table, decode the words 000000, 111000, 000101.
(e) How many errors of transmission can be detected? How many errors of transmission can be corrected? Briefly justify your answers.

Question 2: 20% How many ways are there of putting 9 distinguishable balls into 6 indistinguishable boxes with no box left empty and then putting the boxes into 3 indistinguishable bags with no bag left empty?

Question 3: 20% Given a partial ordering $\leq$ on a set X, the *Hasse graph* of $\leq$ is the graph such that X is the set of vertices and any Hasse diagram of $\leq$ is also a diagram of the graph.
(a) Up to isomorphism, how many trees with 4 vertices are there?
(b) Up to isomorphism, how many partial orderings $\leq$ on a set of size 4 are there such that the Hasse graph of $\leq$ is a tree?

Question 4: 20% Let X be a set and let f be a function $X \to X$. For integers $n \geq 0$, we define f^n to be the function $X \to X$ such that $f^0 = \mathrm{id}_X$ (the identity function on X) and $f^{n+1} = f \circ f^n$ (thus $f^1(x) = f(x)$ and $f_2(x) = f(f(x))$ and $f^3(x) = f(f(f(x)))$ and so on). Suppose that, for all $x \in X$, there exists an integer $n_x \geq 1$ such that $f^{n_x}(x) = x$.
(a) Give an example such that $f^n \neq \mathrm{id}_X$ for all integers $n \geq 1$.
(b) Show that if $|X| = 2016$, then $f^n = \mathrm{id}_X$ for some integer $n \geq 1$.

Solutions to MATH 132 Final, Fall 2015

1: Part (a). The codewords for each message word are as shown:

mess. words	000	001	010	011	100	101	110	111
codewords	000000	001101	010111	011010	100111	101010	110000	111101

Part (b). The parity-check matrix is $H = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$.

The weight 1 words 100000 and 010000 cannot both be used as coset leaders because they have the same syndrome, 111.

Part (c). The syndromes of the given received words are as shown.

rec. words	000000	000001	000010	000100	001000	010000	000011	000110
syndromes	000	001	010	100	101	111	011	110

By part (b), the 8 proposed coset leaders satisfy the minimum weight condition. Moreover, the 8 corresponding syndromes are mutually distinct, as required.

Part (d). The decodings are 000 and 110 and 001, respectively. The calculations are shown in the next table.

r	Hr^T	s	c	w
000000	000	000000	000000	000
111000	101	001000	110000	110
000101	101	001000	001101	001

Part (e). The minimum distance between codewords is 2, so one error can be detected and no errors can be corrected.

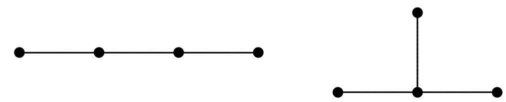
2: There are $S(9,6)$ ways of putting the balls in the boxes. That having been done, the boxes are now distinguishable by means of the balls they contain. So there are $S(6,3)$ ways of putting the boxes in the bags.

Using the recurrence relation $S(m+1,n) = S(m,n-1) + nS(m,n)$ with initial conditions $S(1,m) = S(m,m) = 1$, we obtain the following table of Stirling numbers of the second kind. (Unnecessary entries have been omitted from the table.) The number of arrangements of balls in boxes in bags is $S(9,6)S(6,3) = 2646\,.\,90 = 264600 - 26460 = 238140$.

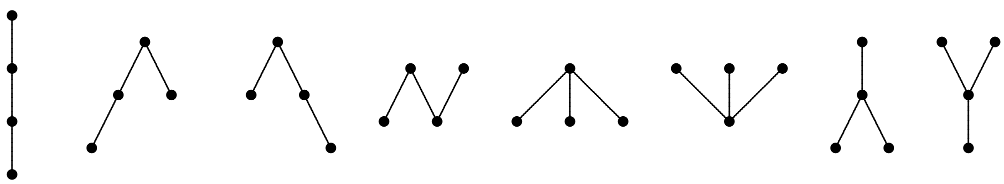
$S(m,n)$		n 1	2	3	4	5	6
m	1	1					
	2	1	1				
	3	1	3	1			
	4	1	7	6	1		
	5		15	25	10	1	
	6			90	65	15	1
	7				350	140	21
	8					1050	266
	9						2646

Comment: For full marks, it is necessary to explain why the answer is $S(9,6)S(6,3)$ and then to calculate it. In the present context (an exam where use of electronic devices is not allowed), the alternating sum formula for $S(m,n)$ is not very practical. Of course, in "real life" applications too, the appropriate choice of method of calculation may depend on the available computing power.

3: Part (a). There are 2 such trees, as depicted.



Part (b). Up to isomorphism, there are 4 partial orderings whose Hasse diagram describes a tree of the first kind. There are 4 partial orderings for the tree of the other kind. So there are 8 partial orderings satisfying the conditions specified in the question. They have Hasse diagrams as shown.



4: Part (a). Let X be the set of positive integers $\{1,2,3,...\}$. We define f such that, given $x \in X$, and writing $q = r(r+1)/2 = 1+2+...+r$ where r is the smallest positive integer satisfying $x \leq q$, then

$$f(x) = \begin{cases} q-r+1 & \text{if } x = q, \\ x+1 & \text{otherwise.} \end{cases}$$

We have $f^r(x) = x$. In fact, for a natural number m, we have $f^m(x) = x$ if and only if m is divisible by r. Given any integer $n \geq 1$ then, choosing any r such that $r > n$, then putting $x = q$, we have $f^n(x) = q - r + n < x$, hence $f^n(x) \neq \mathrm{id}_X$.

Part (b). We shall show, more generally, that the conclusion holds whenever the set X is finite. It is easy to see that each function f^n is bijective. Since X is finite, there are only finitely many bijections $X \to X$. Therefore, $f^a = f^b$ for some integers $a > b \geq 0$. Putting $n = a - b$, then $f^n = (f^b)^{-1} \circ f^a = (f^a)^{-1} \circ f^a = \mathrm{id}_X$.

Comment: The problem is equivalent to the following: At a dinner party, each guest is seated in a chair and each chair seats a guest. Let f be a permutation of the chairs. Whenever the band finishes a song, the guests move in such a way that the guest in chair C moves to chair $f(C)$. Is it possible that every guest eventually returns to her original chair, yet the the original seating arrangement is never repeated? The above solution shows that it is possible when there are infinitely many guests, but it is not possible when there are only finitely many guests.