Archive of documentation for

MATH 123, Abstract Mathematics 1

Bilkent University, Fall 2017, Laurence Barker

version: 28 January 2018

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Course specification

MATH 123, Abstract Mathematics 1, Fall 2017

Laurence Barker, Bilkent University. Version: 28 January 2018.

Instructor: Laurence Barker, Office SAZ 129,
e-mail: barker at fen dot bilkent dot edu dot tr.

Assistant: Melih Üçer
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Main course text: R. P. Grimaldi, "Discrete and Combinatorial Mathematics", 5th Edition, Pearson, 2004, (New International Edition, 2013).

Some notes will be supplied, on my webpage, for some of the syllabus material.

Classes: Mondays 09:30 - 10:30, Wednesdays 10:40 - 12:30. All classes in room

Office Hours: Mondays 08:40 - 09:30 (in my office SA-129). If you are having difficulty with the course, then you must come to see me. One major cause of difficulty is having done insufficient work earlier in the semester, then finding that one cannot understand anything much. That is normal enough, and I will probably refrain from reciting desiderata whose time has passed.

Syllabus: The format below is, *Week number; Monday date; Subtopics and textbook section numbers.*

1: 18 Sep: Recreational games and puzzles. Illustrations of proofs in that context. Argument by contradiction. Use of graph diagrams.

2: 25 Sep: Sets and finite sets. Setwise definition of a graph and a finite graph. Counting edges using vertex degrees. Proof of equivalence of two characterizations of a tree.

3: 2 Oct: Euler Path Theorem (proved by contradiction by considering minimal counter-example). Method for finding Euler paths. Adaptations of the theorem to multigraphs and directed graphs.

4: 9 Oct: Euler's Characteristic Theorem for planar graphs (proved by minumum counter-example). Necessary conditions for planarity. Techniques for proving non-planarity.

5: 16 Oct: Mathematical Induction. Toy illustrations such as sums of consecutive squares. Genuine illustrations such as Euler Path Theorem, Euler's Characteristic Formula.

6: 23 Oct: Proof of Five-Colour Map Theorem. Midterm 1.

7: 30 Oct: Correspondences and Hall's Marriage Theorem.

8: 6 Nov: Bijections, injections, surjections. Enumeration of bijections and injections.

9: 13 Nov: Binomial Theorem. Enumeraing arrangements of plain balls in coloured boxes.

10: 20 Nov: Equivalence relations and equivalence classes. Inclusion-Exclusion Formula and Stirling number formula for counting surjections or equivalence relations.

11: 27 Nov: Posets. Isomorphism of graphs and posets as a formal equivalence relation (on classes, not sets).

12: 4 Dec: Second order homogenous recurrence relations. Midterm 2.

13: 11 Dec: Euclidian algorithm. Fundamental Theorem of Arithmetic. Inclusion-exclusion and Euler's function.

14: 18 Dec: Review.

The last class for this course is on Wednesday 20 December.

Assessment:

- Quizzes, Homework and Participation 20%.
- Midterm I, 20%, 26 October.
- Midterm II, 25%, 6 December.
- Final, 35%, 5 January.

75% attendance is compulsory. Attendance will be assessed through quiz returns.

Class Announcements: All students, including any absentees from a class, will be deemed responsible for awareness of class announcements.

Homeworks and Quizzes

MATH 123, *Abstract Mathematics 1*, Fall 2017

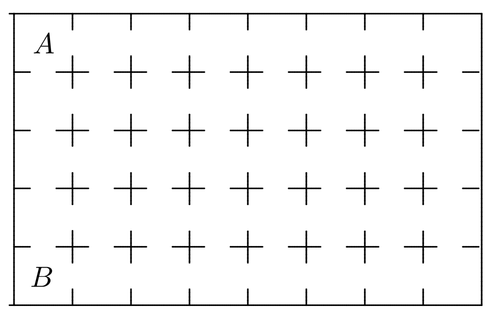
Laurence Barker, Mathematics Department, Bilkent University,
version: 28 January 2018.

Office Hours: Mondays, 08:40 – 09:30, Office Room Fen A 129.

Homeworks

Homework 1, due Wednesday, 27 September.

HW.1.1: A prisoner wanders around in a dungeon consisting of 40 cells arranged in an 8 by 5 grid. She is free to pass from cell to cell via the doorways indicated in the diagram. Can she make a tour of the dungeon, visiting each cell exactly once, starting in the top-left cell labelled A, finishing in the bottom-left cell labelled B?



HW.1.2: There are n closed doors in a row. A cat hides behind one of the doors. Whenever you open a door, if the cat is not there, then you must close the door, whereupon the cat will move and hide behind a door that is adjacent to its present door. To be sure of finding the cat, how many times do you need to open a door?

Homework 2, due Wednesday, 18 October.

HW.2.1: Consider the graph of a cube. To obtain a graph with an Euler circuit, what is the minimum number of edges that have to be removed?

HW.2.2: (From the textbook, question 2 page 589.) What is the maximum possible number of vertices for a connected graph with 17 edges such that each vertex has degree at least 3?

HW.2.3: For which positive integers n is it possible for there to be a party, with n people, such that each person shakes hands with exactly half of the other people? (Thus, each person is to shake hands with $(n-1)/2$ people.)

HW.2.4: Read the definition of a *directed multigraph* in the textbook. Let G be a finite directed multigraph. For each vertex x, let $d_{\text{in}}(x)$ be the number of edges to x, and let $d_{\text{out}}(x)$ be the number of edges from x. State and prove a version of Euler's Path Theorem for directed multigraphs.

Homework 3, due 09:40 on Monday 4 December.

Reminder: to say that mathematical objects $\mathcal{O}_1$, $\mathcal{O}_2$, ... are **mutually distinct** is to mean that no two of them are the same; if $\mathcal{O}_i = \mathcal{O}_j$ then $i = j$.
Answer questions as simply as possible. For instance, 6 is simpler than $\binom{5}{2} - \binom{5}{1} + \binom{5}{0}$.

HW.3.1: (Recall, when we say that mathematical objects $\mathcal{O}_1$, $\mathcal{O}_2$, ... are **mutually distinct**, we mean that no two of them are the same; if $\mathcal{O}_i = \mathcal{O}_j$ then $i = j$.) Let n be a positive integer and A_1, ..., A_n finite sets. Prove that there exist mutually distinct elements $a_1 \in A_1$, ..., $a_n \in A_n$ if and only if, for all $1 \leq m \leq n$ and $1 \leq i_1 < ... < i_m \leq n$, we have $|A_{i_1} \cup ... \cup A_{i_r}| \geq r$. (Hint: Use Philip Hall's Marriage Theorem.)

HW.3.2: How many distinguishable ways are there of putting 12 objects into 5 numbered boxes if:
(a) the objects are distinguishable from each other?
(b) the objects are indistinguishable from each other?
(Just calculate the answers using standard methods. There is no need to explain the theory behind the methods.)

HW.3.3: Consider the pairs (m, n) of integers m and n such that $m \geq n \geq 1$. For which of those pairs does the inequality $S(m, n) < m!/2$ hold? (Hint: This is an exercise in mathematical induction.)

Homework 4, due Wednesday, 20 December.

HW.4.1: Let A and B be sets. Let X be the set of functions $B \leftarrow A$. Let $\equiv$ be the relation on X such that, given $f, g \in X$, then $f \equiv g$ provided there exist bijections $\alpha : A \leftarrow A$ and $\beta : B \leftarrow B$ satisfying $\beta_\circ f = g_\circ \alpha$.
(a) Show that $\equiv$ is an equivalence relation.
(b) How many equivalence classes does $\equiv$ have?

HW.4.2: How many isomorphism classes of poset of height 2 and width 3 are there?

HW.4.3: Recall, for a positive integer n, the Euler function $\phi(n)$ is defined to be the number of integers z in the range $1 \leq z \leq n$ such that $\gcd(z, n) = 1$.
(a) Directly from the Fundamental Theorem of Arithmetic (without using the formula for $\phi(n)$ proved in class) show that, given positive integers m and n with $\gcd(m, n) = 1$, then $\phi(mn) = \phi(m)\phi(n)$.
(b) Hence give another proof of the formula proved in class, $\phi(n) = n \prod_p (1 - 1/p)$ where p runs over the prime divisors of n.

HW.4.4: For positive integers m and n with $m \geq 2n$, let $\sigma(m, n)$ be the number of ways of putting m coloured balls into n plain boxes such that each box has at least 2 balls. Check that your formula is correct by calculating $\sigma(5, 2)$ directly.

Questionnaire, please hand in by Monday 18th December, midnight.

Please do not write your name. To preserve anonymity, you are welcome to post printed or handwritten answers under my door.

Of course, you do not need to answer all the questions. We are interested in observations that will help us improve our introduction to mathematical abstraction for Bilkent students.

The questions are for future planning, and are purely about the design of the course. My style of teaching is irrelevant, since staff will be rotated anyway.

Some of the staff with considerable interest in your answers are not fluent in Turkish, including me. So please answer in English (unless you really do need to use Turkish to explain what you mean).

Question A: The course you have taken has been an experiment. The aim has been to minimize the number of mathematics students who never become skillful at proof. The method has been to focus heavily on proof instead of concepts. Do you think the department should continue with this new approach?

Question B: The course has taken much material from MATH 210 *Finite and Discrete Mathematics.* Would you like the material of MATH 210 to be always taught in MATH 123, allowing MATH 210 to be replaced by something else? (This may be difficult for you to answer at present, but we would still like to know your views.)

Question C: How would you like the design and syllabus of MATH 124 Abstract Mathematics 2 to be different from the design and syllabus of the MATH 123 course you have taken?

Question D: Do you have any other comments about the MATH 123 and MATH 124 part of the Bilkent mathematics undergraduate program?

Quizzes

Q1: For a positive integer n, let K_n denote the complete graph with order n.
(1) How many edges does K_n have?
(2) For which n does K_n have an Euler path?

Q2: Let T be a finite tree with a vertex of degree at least 3. Show that T has at least 3 leaves.

Q3: Which of the following are true? (Give a proof or a counter-example.)
(a) Any function $f : X \to Z$ can be written as $f = g_\circ h$ where g is surjective and h is injective?
(b) The same as part (a), except with g injective and h surjective?

Q4: How many integer solutions $x_i \geq 0$ are there to $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 4$ are there?

Q5: Let $\sqsubseteq$ be a reflexive transitive relation on a set X. Define a relation $\equiv$ on X such that, given $x, y \in X$, then $x \equiv y$ if and only if $x \sqsubseteq y$ and $y \sqsubseteq x$. Show that $\equiv$ is an equivalence relation.

Quiz Solutions

SolQ1: Part (1). Each vertex has degree $n - 1$. So, by the Handshaking Lemma, the number of edges is $n(n-1)/2$.

Part (2). There is an Euler path if and only if $n = 2$ or n is odd. Indeed, the number of vertices of odd degree is 2 if and only if $n = 1$, while the number of such vertices is 0 if and only if n is odd.

Comment: Binomial coefficients will we discussed later in the course. One alternative to part (a) is as follows: The set of vertices has $\binom{n}{2} = n(n-1)/2$ subsets with size 2. In other words, the number of edges is $n(n-1)/2$.

SolQ2: Let e be the number of edges of T. Then T has $e + 1$ vertices. Enumerating the vertices as x_0, x_1, ..., x_e such that $d(x_0) \geq d(x_1) \geq ... \geq d(x_e)$, then $d(x_0) \geq 3$. Since x_0 has 3 neighbours, $e \geq 3$. We have $2e = d(x_0) + d(x_1) + ... + d(x_e)$, so $2e - 3 \geq d(x_0) + ... + d(x_e)$. But each $d(x_i) \geq 1$. Therefore, $d(x_{e-2}) = d(x_{e-1}) = d(x_e) = 1$.

SolQ3: Part (a). True. Let Z' be a set disjoint from X such that there exists a bijection $g' : Z' \to Z$. Put $Y = X \cup Z'$. Let $h : X \to Y$ be such that $h(x) = x$ for all $x \in X$. Let $g : Y \to Z$ be such that $g(x) = f(x)$ and $g(z') = g'(z')$ for $z' \in Z$.

Part (b). True. Let $Y = f(X)$. Let g and h be the restrictions of id_Z and f, respectively.

SolQ4: The answer is $\binom{6+4-1}{4} = \binom{9}{4} = \frac{9.8.7.6}{4.3.2} = 126$.

SolQ5: Routine.

MATH 123: Abstract Mathematics 1. Midterm 1

LJB, 26 October 2017, Bilkent University.

Time allowed: 110 minutes. Please put your name on EVERY sheet of your manuscript. The use of telephones, calculators or other electronic devices is prohibited. The use of red pens or very faint pencils is prohibited too. You may take the question sheet home.

Remember to justify your answers, except in any cases where your answers are obvious.

1: 15 marks. Let G be a finite connected graph with 16 edges such that every vertex has degree 4.
(a) Find the number of vertices of G.
(b) Now suppose that G is planar. Find the number of faces of G.

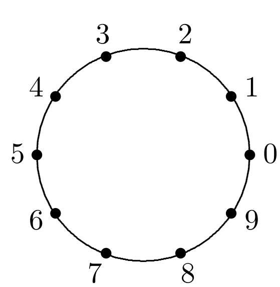
2: 20 marks. The graph K_7 has 7 vertices and 21 edges.
(a) Draw a (clear) diagram of K_7.
(b) Number the vertices of K_7 from 1 to 7 and specify an Euler path for K_7 by listing the vertices in order. (Do not list labled edges in order, because that would be too hard to read).

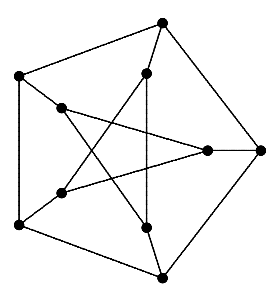
3: 20 marks. Let x_0, x_1, ... be an infinite sequence of integers such that

$$x_{n+3} \leq 2x_{n+2} + 13x_{n+1} + 10x_n$$

for all integers $n \geq 0$. Suppose that $x_0 = 0$ and $x_1 = 4$ and $x_2 = 24$. Show that $x_n < 5^n$ for all integers n with $n \geq 0$. (If you use argument by contradication or argument by mathematical induction, use complete sentences to be clear about the distinctions between definitions, hypotheses, deductions.)

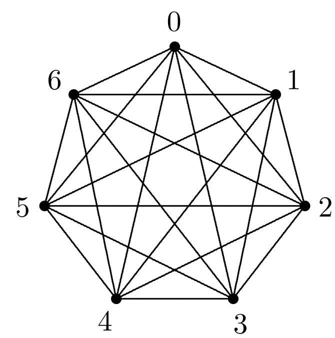
4: 25 marks. The Peterson graph is the graph with 10 vertices and 15 edges shown on the right. (Hint: part (a) might be useful for part (b) or (c).)
(a) Let G be a graph with vertices numbered from 1 to 10. Suppose G has 15 edges, 10 of the edges shown in the diagram on the left, the other 5 edges connecting each point to either its opposite point or else one of the neighbours of the opposite point. (Thus, the vertex 0 is adjacent to one of the three vertices 4 or 5 or 6. The vertex 1 is adjacent to one of the three vertices 5 or 6 or 7. And so on.) Show that G has a cycle with length 4.
(b) Can edges (but no vertices) be added to the Peterson graph to produce a graph with an Euler circuit? If so, what is the minimum number of edges that must be added?
(c) Can edges (but no vertices) be removed from the Peterson graph to produce a graph with an Euler circuit? If so, what is the minimum number of edges that must be removed?





5: 20 marks. Show that the Peterson graph is non-planar.

<u>Solutions to Midterm 1</u>



1: Part (a). By the Handshaking Lemma, the number of vertices is $2.16/4 = 8$.

Part (b). Let f denote the number of faces. Euler's Characteristic Formula tells us that $8 - 16 + f = 2$. So $f = 10$.

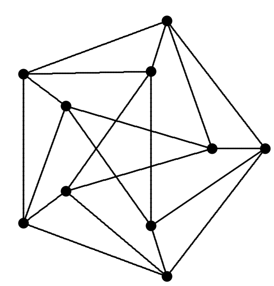
2: Omitting commas, one Euler path is:

0 1 2 3 4 5 6 0 2 4 6 1 3 5 0 3 6 2 5 1 4 0

3: We argue by induction. The required inequality for x_n is clear when $n \leq 2$. Now suppose, inductively, that the assertion holds for x_n and x_{n+1} and x_{n+2}. Then

$$x_{n+3} < 2.5^{n+2} + 13.5^{n+1} + 10.5^n = (2.25 + 13.5 + 10)5^n = 125.5^n = 5^{n+3} .$$

4: Part (a). If every vertex is adjacent to its opposite, then $0, 5, 6, 1, 0$ is a cycle with length 4. So we may assume that some vertex is not adjacent to its opposite. By symmetry, we may assume that there is an edge 04. Then there is an edge $5x$ where $x = 1$ or $x = 9$. Then $0, 4, 5, x, 0$ is a cycle with length 4.



Part (b). Yes, the minumum number of edges to be added is 5. Indeed, adding 5 edges as depicted, the Euler Path Theorem implies that the new graph has an Euler circuit. If fewer edges are added, then some vertices remain of odd degree and, by the same theorem, there cannot be an Euler circuit.

Part(c). No. For a contradiction, suppose otherwise. Then, by the Euler path theorem, 5 edges must be removed, leaving a connected graph where every vertex has degree 2. The new graph must consist of a single circuit with length 10. Since the Peterson graph has no cycle with length less than 5, it satisfies the hypothesis in part (a). The conclusion of part (a) yields a contradiction.

5: For a contradiction, suppose the graph is planar. Then $e \leq (n-2)c/(c-2)$ where e is the number of edges, n is the number of vertices and c is the length of the shortest cycle. We have $e = 15$ and $n = 10$ and $c = 5$. Hence $15 \leq 8.5/3 = 40/3$. This is a contradiction, as required.

MATH 123: Abstract Mathematics 1. Midterm 2

LJB, 6 December 2017, Bilkent University.

Time allowed: 110 minutes. Please put your name on EVERY sheet of your manuscript. The use of telephones, calculators or other electronic devices is prohibited. The use of red pens or very faint pencils is prohibited too. You may take the question sheet home.

Make your answers as simple, explicit and clear as possible. Remember to justify your answers, except in any cases where your answers are obvious.

1: 10 marks. Let X be a set with $|X| = 12$. Find the number of subsets Y of X such that $|Y| = 6$.

2: 20 marks. Let r be a nonzero complex number such that $r + 1/r$ is an integer. Show that, given any positive integer n, then $r^n + 1/r^n$ is an integer. (Hint: expand $(r + 1/r)^n$ using the Binomial Theorem.)

3: 30 marks. For integers k and n satisfying $1 \le k \le n$, let $a(k, n)$ be the number of ways of putting n distinguishable objects into n distinguishable boxes such that exactly k boxes are non-empty.
(a) Give a formula for $a(k, n)$ in terms of Stirling numbers (of the second kind).
(b) Evaluate the Stirling numbers $S(5, k)$ for each k in the range $1 \le k \le 5$.
(c) Evaluate $a(k, 5)$ for each k in the range $1 \le k \le 5$.
(d) Making direct use of your answer to part (c), evaluate the sum

$$a(1, 5) + a(2, 5) + a(3, 5) + a(4, 5) + a(5, 5) \; .$$

(e) Evaluate the sum in part (d) by another method, without using parts (a), (b), (c) or (d).

4: 25 marks. For integers k and n satisfying $1 \le k \le n$, let $b(k, n)$ be the number of ways of putting n indistinguishable objects into n distinguishable boxes such that exactly k boxes are non-empty.
(a) Give a formula for $b(k, n)$.
(b) Making direct use of your answer to part (a), find a simple formula for the sum

$$b(1, n) + b(2, n) + ... + b(n - 1, n) + b(n, n)$$

(c) Find a formula for the sum in part (b) by another method, without using part (a).

5: 15 marks. Let n be an even positive integer. A **round-robin tornament** with n players is a competition held in $n - 1$ sessions. During each session, each player plays exactly one game. At the end of the tournament, each player has played every other player. (It can be shown that, for all even n, a scheduling for a round-robin tournament can be devised. That theorem, however, does not concern us here.) Suppose that, in some round-robin tournament, each game ends with a win for one of the two players, a loss for the other; we mean to say, no game ends in a draw. Show that, at the end of the tournament, it is possible to produce a list of $n - 1$ mutually distinct players such that, for each k in the range $1 \le k \le n - 1$, the k-th player in the list won the game she played during the k-th session. (Hint: apply Hall's Marriage Theorem.)

Solutions to Midterm 2

1: The answer is $\binom{12}{6} = \frac{12.11.10.9.8.7}{6.5.4.3.2.1} = 12.11.7 = 840 + 84 = 924$.

2: We argue by induction on n. The case $n = 1$ is trivial. Now suppose that $n \geq 2$ and that $r^m + r^{-m}$ for all positive integers $m < n$. We are to deduce that $r^n + r^{-n}$ is an integer. By the Binomial Theorem

$$r^n + r^{-n} = (r + r^{-1})^n - \binom{n}{1}(r^{n-1} + r^{1-n}) - \binom{n}{2}(r^{n-2} + r^{2-n}) - ...$$

the last term on the right-hand side being $-\binom{n}{n/2}$ or $-\binom{n}{(n-1)/2}(r + r^{-1})$ depending on whether n is even or odd, respectively. It is now clear that the inductive assumption yields the required conclusion.

Comment: This is an argument by what is sometimes called "strong induction". We deduced $r^n + 1/r^n \in \mathbb{Z}$ from the inductive assumption that $r^m + 1/r^m \in \mathbb{Z}$ for all m less than n. If you tried to use the Binomial Theorem in the above way but with the inductive assumption that $r^{n-1} + 1/r^{n-1} \in \mathbb{Z}$, then your argument merits partial credit but it is not completely correct. If you neglected to say what your inductive assumption is then, for your first or second year undergraduate reader, that is no better than specifying the inductive assumption incorrectly.

3: Part (a). To choose an arrangement as specified, there are $\binom{n}{k}$ choices for which boxes are to be non-empty. Then there are $k!S(n,k)$ choices of surjection from the set of balls to the set of non-empty boxes. Therefore

$$a(k,n) = \binom{n}{k} k!\, S(n,k) = \frac{n!\, S(k,n)}{(n-k)!} .$$

Part (b). Using the recurrence relation $S(m,n) = S(m-1,n-1) + nS(m-1,n)$ and the initial conditions $S(m,1) = 1 = S(m,m)$, we obtain the following table.

$S(m,n)$	1	2	3	4	5	n
1	1					
2	1	1				
3	1	3	1			
4	1	7	6	1		
5	1	15	25	10	1	
m						

Part (c). Using (a) and (b), we obtain the table on the left, hence the table on the right.

k	1	2	3	4	5
$S(k,5)$	1	15	25	10	1
$5!/(5-k)!$	5	20	60	120	120

k	1	2	3	4	5
$a(k,5)$	5	300	1500	1200	120

Part (d). We have $a(1,5) + ... + a(5,5) = 5 + 300 + 1500 + 1200 + 120 = 3125$.

Part (e). The sum $a(1,5)+...+a(5,5)$ is equal to the number of ways of putting 5 coloured objects into 5 coloured boxes. This is the number of functions $\{1,2,3,4,5\} \to \{1,2,3,4,5\}$, which is $5^5 = 25.125 = 25.120 + 125 = 3000 + 125 = 3125$.

4: Part (a). To choose an arrangement as specified, we first choose the set of non-empty boxes, then place one ball in each of those boxes, then choose how many further balls to place in each of those boxes. For the first choice, there are $\binom{n}{k}$ possibilities. For the second choice, $n-k$ balls are to be placed in k boxes, and the number of possibilities is $\binom{n-k+k-1}{n-k} = \binom{n-1}{n-k}$. Therefore, $b(k,n) = \binom{n}{k}\binom{n-1}{n-k} = \binom{n}{k}\binom{n-1}{k-1}$.

Part (b). We have $b(1,n)+...+b(n,n) = \sum_{k=1}^{n} \binom{n}{k}\binom{n-1}{n-k} = \binom{2n-1}{n}$. Indeed, the second equality holds because, to choose n elements of the set $\{1,...,2n-1\}$, we can choose k in the range $1 \leq k \leq n$, then choose k elements from $\{1,...,n\}$ and $n-k$ elements from $\{n+1,...,2n-1\}$.

Part (c). The formula in part (b) holds because $b(1,n)+...+b(n,n)$ is the number of ways of putting n plain balls into n coloured boxes.

5: Let X be a set of days, Y the set of players. Let $\mathfrak{C}$ be the correspondence from X to Y such that, given $x \in X$ and $y \in Y$, then $x\mathfrak{C}y$ provided player y is a winner on day x. By Hall's Marriage Theorem, it suffices to show that, given $U \subseteq X$, then $|U| \leq |\mathfrak{C}(U)|$. We may assume U is non-empty. Consider the integer $m = n/2$. There are m winners each day. So $m \leq |\mathfrak{C}(U)|$ and we may assume that $|U| = n-k$ where $0 \leq k \leq m$. If there exists a player y who lost on every day in U, then $n-k$ players beat y on a day in U. Therefore, there exist at most k players who lost on every day in U. Hence $n-k \leq |\mathfrak{C}(U)|$, as required. □

MATH 123: Abstract Mathematics 1. Final

LJB, 5 January 2018, Bilkent University.

Time allowed: 2 hours. Please put your name on EVERY sheet of your manuscript. The use of telephones, calculators or other electronic devices is prohibited. The use of red pens or very faint pencils is prohibited too. You may take the question sheet home.

Make your answers as simple, explicit and clear as possible. Remember to justify your answers, except in any cases where your answers are obvious.

1: 20 marks. Using the Euclidian algorithm, find integers a and b such that

$$\gcd(119, 399) = 119a + 399b \ .$$

2: 20 marks. Let p be a prime number, x an integer. Using induction and the Binomial Theorem, prove Fermat's Little Theorem: $x^p \equiv_p x$. (We mean, $x^p \equiv x$ modulo p.)

3: 20 marks. Let X be a set with size $|X| = 3$.
(a) How many partial ordering relations on X are there?
(b) What let $\leq_1$ and $\leq_2$ be partial ordering relations on X. What is meant when it is said that $\leq_1$ and $\leq_2$ are *isomorphic?* (Give a definition.)
(c) Consider the isomorphism relation $\cong$ on the set of partial ordering relations on X. Show that $\cong$ is an equivalence relation?
(d) How many isomorphism classes of partial orderings on X are there? (In other words, how many equivalence classes of $\cong$ are there?)
(e) Classify the isomorphism classes of partial orderings on X by drawing the corresponding Hasse diagrams. (In other words, classify the equivalence classes of $\cong$.) Find the number of partial orderings in each isomorphism class. (In other words, find the size of each equivalence class.)

4: 20 marks. How many equivalence relations on $\{1, 2, 3, 4, 5, 6, 7, 8\}$ are there?

5: 20 marks. Let P be a poset with finitely many elements. (We mean, P is a non-empty finite set equipped with a partial ordering relation $\leq$.) Recall, the **height** of P is the maximum h such that there exist elements x_1, ..., x_h of P satisfying $x_1 < ... < x_h$. An **antichain** in P is a subset $\{a_1, ..., a_m\}$ of P such that $a_i \not\leq a_j$ whenever $i \neq j$. Prove Mirsky's Theorem: the height of P is equal to the minimum number k such that P is the union of k antichains

Solutions to Final

1: We have

$$399 = 3.199 + 42\ , \qquad 119 = 2.42 + 35\ , \qquad 42 = 1.35 + 7\ .$$

Therefore, $\gcd(199, 399) = 7$. We have

$$7 = 42 - 35 = 42 - (119 - 2.42) = 3.42 - 119 = 3(399 - 3.199) - 199 = 3.399 - 10.119\ .$$

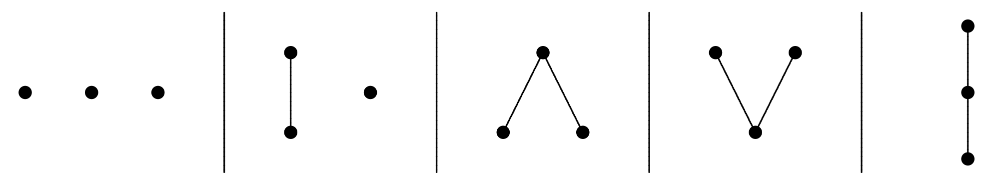
In conclusion, $\gcd(199, 399) = 3.399 - 10.119$.

2: Write $x = qp + r$ where $0 \leq r \leq p - 1$. We argue by induction on r. The case $r = 0$ is trivial. Now suppose $r \geq 1$ and $y^p \equiv_p y$ where $y = x - 1$. Then

$$x^p = (y+1)^p = y^p + 1 + \sum_{s=1}^{p-1} \binom{p}{s} y^s\ .$$

Each term of the sum is divisible by p. So $x^p \equiv y^p + 1 \equiv_p y + 1 \equiv_p x$. $\square$

3: Part (d). The 5 isomorphism classes of partial orderings on X are illustrated by the following Hasse diagrams (separated by vertical lines).



Part (e). The number of partial orderings on X belonging to those isomorphism classes are, respectively: 1, 6, 3, 3, 6.

Part (a). The total number of partial orderings on X is $1 + 6 + 3 + 3 + 6 = 19$.

Part (b). The relations $\leq_1$ and $\leq_2$ are said to be **isomorphic** provided there exists a permutation f of X such that, for all $x, y \in X$, we have $x \leq_1 y$ if and only if $f(x) \leq_2 f(y)$.

Part (c). Let $\leq_1$, $\leq_2$, $\leq_3$ be partial orderings on X. For reflexivity, observe that, by considering the identity function on X, we have $\leq_1 \cong \leq_2$. For symmetry, by considering the inverse of an isomorphism, we see that if $\leq_1 \cong \leq_2$ then $\leq_2 \cong \leq_1$. Finally, for transitivity, suppose $\leq_1 \cong \leq_2$ and $\leq_2 \cong \leq_3$. Let f and g be such that, for all $x, y \in X$, we have$x \leq_1 y$ if and only if $f(x) \leq_2 f(y)$ and $x \leq_2 y$ if and only if $g(x) \leq_3 g(y)$. Consider the composite function $h = g_\circ h$. Then, for all x and y, we have $x \leq_1 y$ if and only if $h(x) \leq_3 h(y)$. Hence $\leq_1 \cong \leq_3$.

4: Using the recurrence relation $S(m, n) = S(m-1, n-1) + nS(m-1, n)$ and the evident initial conditions, we obtain the following table of Stirling numbers of the second kind.

$S(m,n)$	1	2	3	4	5	6	7	8	n
1	1								
2	1	1							
3	1	3	1						
4	1	7	6	1					
5	1	15	25	10	1				
6	1	31	90	65	15	1			
7	1	63	301	350	140	21	1		
8	1	127	966	1701	1050	266	28	1	
m									

Now $S(8,n)$ is the number of equivalence relations on $\{1,...,8\}$ with n equivalence classes. So the total number of equivalence relations on that set is

$$\sum_{n=1}^{8} S(8,n) = 1 + 127 + 966 + 1701 + 1050 + 266 + 28 + 1$$

$$= 128 + 2667 + 1316 + 29 = 2795 + 1345 = 4140 .$$

5: Let h be the height of P. By considering a chain with size h, we see that P cannot be expressed as the union of fewer than h antichains. It remains to show that P can be expressed as the union of h antichains. For each $x \in P$, let $n(x)$ be the largest integer such that there exists a chain $x_1 < ... < x_{n(x)-1} < x_{n(x)} = x$ in P. For $1 \leq n \leq h$, let $A_n = \{x \in P : n(x) = n$. Then A_1, ..., A_h are antichains whose union is P.

MATH 123: Abstract Mathematics 1. Makeup Final

LJB, 19 January 2018, Bilkent University.

Time allowed: 2 hours. Please put your name on EVERY sheet of your manuscript. The use of telephones, calculators or other electronic devices is prohibited. The use of red pens or very faint pencils is prohibited too. You may take the question sheet home.

Make your answers as simple, explicit and clear as possible. Remember to justify your answers, except in any cases where your answers are obvious.

1: 20 marks. Give a precise specification of the Euclidian algorithm for finding the greatest common divisor of two non-zero integers a and b. Prove that the algorithm does return a divisor of a and b. (You do not need to show that the algorithm returns the greatest common divisor. You may not assume the Fundamental Theorem of Arithmetic.)

2: 20 marks. Let $\equiv$ be the relation on the set of integers $\mathbb{Z}$ such that, given integers x and y, then $x \equiv y$ if and only if there exist integers s and t satisfying $x - y = 357s + 429t$.
(a) Show that $\equiv$ is an equivalence relation.
(b) How many equivalence classes does $\equiv$ have?

3: 30 marks. Let X be a set with size $|X| = 12$. Consider the equivalence relations $\equiv$ on X such that exactly 2 of the equivalence classes of $\equiv$ have the same size. How many such equivalence relations $\equiv$ on X are there?

4: 10 marks. Let $\leq$ be a partial ordering on a finite set P. A **chain** in P is a subset $C \subseteq P$ such that, for all $x, y \in C$, we have $x \leq y$ or $y \leq x$. An **antichain** in P is a subset $A \subseteq P$ such that, for all $a, b \in A$, we have $a \not< b$. Let n be the maximum size of an antichain in P. Let m be the minimum positive integer such that P is the union of m chains.
(a) Show that $n \leq m$.
(b) We are to prove that $n = m$. We argue by induction on $|P|$. Let x_0 be a minimum element of P. Let x_1 be a maximum element of P. Let P' be the set obtained from P by deleting x_0 and x_1. Let A be an antichain in P' with maximal size. Why can we assume that $|A| = m$?

5: 20 marks. In the notation of the previous question, let P^+ be the set of all $x \in P$ such that $a \leq x$ for some $a \in A$. Let P^- be the set of all $x \in P$ such that $x \leq a$ for some $a \in A$.
(a) Show that $|P^+| < |P| > |P^-|$.
(b) By applying the inductive hypothesis to P^+ and P^-, complete a proof that $n = m$.

Hints of solutions to Makeup Final

1: Bookwork.

2: The condition is that $x \equiv_3 y$.

3: The relevant partitions of 12 are:

$$10+1+1\,, \quad 8+2+2\,, \quad 8+2+1+1\,, \quad 7+3+1+1\,, \quad 7+2+2+1\,, \quad 6+6\,,$$

$$6+4+1+1\,,\ 6+3+3\,,\ 5+3+2+2\,,\ 5+3+2+1+1\,,\ 4+4+3+1\,,\ 4+3+3+2\,.$$

There are 12 of them. The answer is $12!\,12/2 = 12!\,6$.

4: Easy.

5: Part (a). We have $x_0 \notin P^+$.
Part (b). We have $P^+ \cap P^- = A$ and $P^+ \cup P^- = P$.