Documentation for:

MATH 110, Discrete Mathematics.

Spring 2009. Laurence Barker, Bilkent University

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MATH 110: DISCRETE MATHEMATICS. Midterm I, Spring 09

Time allowed: 110 minutes. Please put your name on EVERY sheet of your manuscript. Do take the question-sheet home with you and, if possible, remember to bring it to the next class. The use of telephones and other electronic devices is prohibited.

LJB, 12 March 2009, Bilkent University.

1: 15% For which integers $n \geq 1$ is K_n a planar graph? (Recall that K_n is the graph with n vertices and $n(n-1)/2$ edges. You may use standard results about the number of vertices and the number of edges of a planar graph.)

2: (a) 5% State Euler's Criterion for a connected graph to have an Euler trail. (You do not need to give a proof of this.)

(b) 15% Give a similar criterion for a connected graph to have a walk that uses each edge an odd number of times. Justify your answer using part (a).

3: 20% Let a, b, c be complex numbers with $a \neq 0$. Suppose that the quadratic equation $at^2 + bt + c = 0$ has a repeated non-zero solution α. Let $x_1, x_2, ...$ be an infinite sequence of complex numbers such that

$$ax_{n+2} + bx_{n+1} + cx_n = 0 \, .$$

Show that there exist complex numbers A and B such that

$$x_n = (A + nB)\alpha^n \, .$$

(Warning: It is not enough just to show that the formula is a solution to the recurrence relation. You must use mathematical induction to prove that every solution has that form.)

4: 20% Let $x_1, x_2, ...$ be an infinite sequence of complex numbers such that $x_1 = x_2 = 0$ and

$$x_{n+2} - 2ix_{n+1} - x_n = 1 - in \, .$$

Find a formula for x_n. (Hint: First try $x_n = (n-1)/2$; but note that, unfortunately, this does not satisfy the initial conditions.)

5: Let G be a connected planar graph such that every vertex has the same degree. Suppose that G can be drawn on the plane in such a way that every face has four edges.

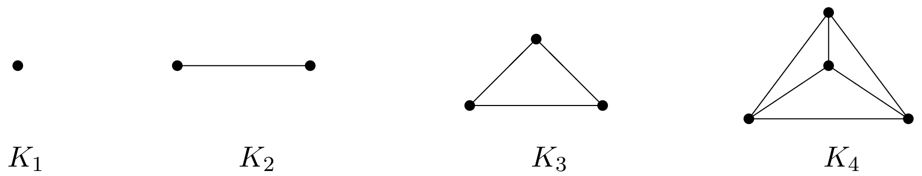
(a) 5% Show that $2f = e$.

(b) 15% Show that $n = 4$ or $n = 8$, where n is the number of vertices.

(c) 5% Give an example of such a graph with $n = 4$ and an example with $n = 8$.

Solutions to Midterm 1.

1: The graph K_n is planar for $n \leq 4$. Indeed, the graphs K_1, K_2, K_3, K_4 can be drawn as shown in the diagram.



Recall that, given a planar graph with n vertices and e edges, with $e \geq 3$, then $e \leq 3n - 6$. For K_5, we have $n = 5$ and $e = 10$. If K_5 is planar then $10 \leq 3.5 - 6 = 9$, a fallacy. So K_5 is non-planar. Finally, for $n \geq 6$, the graph K_n has K_5 as a subgraph, so K_n cannot be planar.

2: (a) A connected graph has an Euler trail if and only if the number of vertices with odd degree is 0 or 2.
(b) Let us call such a walk an **odd walk**. A connected graph has an odd walk if and only if the number of vertices with odd degree is 0 or 2. *Proof:* First suppose there are precisely 0 or 2 vertices with odd degree. Then there is an Euler trail by part (a). Any Euler trail is an odd walk. Conversely, suppose that there is an odd walk from vertex s to vertex t. For any other vertex x, the number of times we enter x is equal to the number of times we exit x. So $d(x)$ is even. If $s = t$ then, again, the number of times we enter s is equal to the number of times we exit s, hence $d(s)$ is even and the number of vertices with odd degree is even. On the other hand, if $s \neq t$, then we enter s one less time than we exit s, and we enter t one more time that we exit t. Hence $d(s)$ and $d(t)$ are odd and the number of vertices with odd degree is 2.

3: We argue by induction. Let $A = x_0$ and $B = x_1/\alpha - A$. Base step: we have $x_0 = A = (A + 0.B)\alpha^0$ and $x_1 = (A + B)\alpha = (A + 1.B)\alpha^1$. So the assertion holds for $n = 0$ and $n = 1$. Induction Step: Fix n and suppose that $x_n = (A + nB)\alpha^n$ and $x_{n+1} = (A + (n + 1)B)\alpha^{n+1}$. Then

$$a(A + (n + 2)B)\alpha^{n+2} + bx_{n+1} + cx_n$$
$$= a(A + (n + 2)B)\alpha^{n+2} + b(A + (n + 1)B)\alpha^{n+1} + c(A + nB)\alpha^n$$
$$= (A + nB)\alpha^n(a\alpha^2 + b\alpha + c) + B\alpha^{n+1}(2a\alpha + b) = 0 = ax_{n+2} + bx_{n+1} + cx_n$$

because $a\alpha^2 + b\alpha + c = 0$ and $\alpha = -b/2a$. Canceling, we obtain $x_{n+2} = (A + (n + 2)B)\alpha^n$, as required.

4: Letting $y_n = (n-1)/2$, it is easy to see that $y_{n+2}-2iy_{n+1}-y_n = 1-in$. Putting $x_n = y_n+z_n$, then z_n satisfies the homogeneous recurrence relation $z_{n+2} - 2iz_{n+1} - z_n = 0$. The auxiliary quadratic equation $t^2 - 2it - t = 0$ has a repeated solution $t = i$, so $z_n = (A + nB)i^n$ for some A and B. Thus $x_n = (n - 1)/2 + (A + nB)i^n$.

We now solve for A and B using the initial conditions. We have $0 = x_1 = (A + B)i$ and $0 = x_2 = 1/2 + (A + 2B)i^2 = 1/2 - A - 2B$. Therefore $B = 1/2 = -A$. In conclusion,

$$x_n = (n - 1)/2 + (n/2 - 1/2)i^n = (n - 1)(1 + i^n)/2\,.$$

5: (a) Consider the pairs (ϵ, F) where ϵ is an edge on a face F. Since each face has 4 edges, the number of such pairs is $4f$. But each edge belongs to 2 faces, so the number of such pairs is $2e$. Therefore $2f = e$.

(b) Since G is a connected planar graph, $n - e + f = 2$. By part (a), $2n - e = 4$. Letting d be the degree of the vertices, then $2e = dn$, hence $4n - dn = 8$, in other words, $n = 8/(4 - d)$. But n and d are integers with $n \geq 0 \leq d$, so $d \in \{0, 2, 3\}$. But, if $d = 0$ then the graph has no edges, yet $n = 2$. This is impossible, because a graph with 2 vertices and no edges is not connected. So $d = 2$ or $d = 3$, hence $n = 4$ or $n = 8$.
(c) For $n = 4$, the vertices and edges of a square comprise such a graph. For $n = 8$, the vertices and edges of a cube comprise such a graph. (I omit the diagrams, which would take a long while to produce using my LaTeX word-processing program.)

Comments on common mistakes in Midterm 1

1: One person disposed of the case $n \geq 5$ using Kurotowsky's Theorem (not on the syllabus). But one ought to avoid using deep theorems to prove trivial little results, because that gives a false impression of difficulty. Furthermore, sometimes the trivial little result is used in the proof of the deep theorem. I have not seen a proof of Kurotowsky's theorem, but I have no doubt that the non-planarity of K_5 is a necessary lemma. I did, however, award full marks in this case.

Others argued as follows:

> *For K_n, we have $e = n(n-1)/2$. If K_n is planar, then $e \leq 3n - 6$, hence $(n-3)(n-4) \leq 0$, so $n = 3$ or $n = 4$. Therefore K_n is planar if and only if $n = 3$ or $n = 4$.*

As a way of dealing with the case $n \geq 5$, this is very elegant. To complete the argument, one just has to deal with the case $n \leq 4$, and it would be enough to draw the four diagrams as in my solution above. However, as the argument stands, there are two mistakes.

One of the mistakes lies in the fact that the inequality $e \leq 3n - 6$ is valid only when $e \geq 3$. (Recall that, in the proof of that inequality, presented in class, we made use of the fact that every face has at least three edges.) So the above argument applies only when $n \geq 3$. We must deal separately with the cases $n = 1$ and $n = 2$ (and, indeed, the graph K_n is planar in those two cases).

The other mistake — made by many people — lies in the fact that the criterion works only one way: Assuming that $e \geq 3$, then planar implies $e \leq 3n - 6$, but $e \leq 3n - 6$ does not imply planar. The argument shows that K_3 and K_4 satisfy $e \leq 3n - 6$, but it does not show that K_3 and K_4 are planar. To confirm that K_4 is planar, I think one ought to draw an appropriate diagram. To confirm that K_1, K_2, K_3 are planar, one might as well again draw diagrams. (Or, alternatively, I think the reader would understand if one were just to point out that K_1, K_2, K_3 are obviously planar. But this is still a situation where, at the very least, one ought to alert the reader to the fact that something is obvious, just to reassure the reader that the matter has been considered.)

I awarded 11/15 for the above solution with the wrong answer $n \in \{3, 4\}$. I awarded 13/15 for the above argument combined with the right answer $n \leq 4$ yet without an explanation as to why K_n is planar for $n \leq 4$. Here are two interesting variants of of the argument.

One person gave the above argument and then tried to deal with the case $n \leq 4$ as follows: "For $n = 1, e = 0, f = 1$, have $n - e + f = 1 - 0 + 1 = 2$, planar. For $n = 2, e = 1, f = 1$, have $n - e + f = 2 - 1 + 1 = 2$, planar. For $n = 3, e = 3, f = 2$, have $n - e + f = 3 - 3 + 2 = 2$, planar. For $n = 4, e = 6, f = 4$, have $n - e + f = 4 - 6 + 4 = 2$, planar." But the number of faces f is defined only for planar graphs. No diagrams were drawn, but perhaps the writer mentally visualized them and counted the numbers of faces for K_1, K_2, K_3, K_4. If so, then the diagrams would already confirm that those four graphs are planar! Or perhaps the writer just solved for f using the equation $n - e + f = 2$. If so, then the calculations imply nothing. Mark: 13/15.

Another candidate wrote the following, with no words at all:

$$3n - 6 \geq e,\ n \geq 1,\ 3n - 6 \geq n(n-1)/2,\ 6n - 12 \geq n^2 - n,\ 0 \geq n^2 - 7n + 12,$$
$$0 \geq (n-3)(n-4),\ n = 3,\ n = 4,\ K_3,\ K_4.$$

What does any of that mean? Well, sure enough, if we already know the argument, then it is easy to guess what the writer is thinking. But the whole point of a proof is to explain something to someone who does not already know it. Does the inequality $3n - 6 \geq e$ hold for all n, or just for some n, or is it an assumption? (Actually, it holds when K_n is planar and $e \geq 3$.) What does the writer mean by ending with the symbols K_3, K_4? Does he or she mean to say that K_3 and K_4 are planar graphs, or that they are non-planar graphs, or that they are rabbits? (Actually, the calculations have merely shown that K_3 and K_4 satisfy the inequality $3n - 6 \geq e$.) Mark awarded: 5/15. *You must use words to explain what is going on.*

2: Language point: we may say that a graph *is* connected or that it *has* an Euler trail.

(a) Several people answered with something like: *A connected graph has an Euler trail if and only if exactly* 2 *of the vertices have odd degree. The graph has an Euler circuit if and only if all of the vertices have even degree. An Euler circuit is also an Euler trail.* This does not make sense. If all the vertices have even degree, is there an Euler trail or not? Answer: yes, there is an Euler trail.

Another daft answer: "For a connected graph to have an Euler trail, we must have 2 vertices with odd degree and each edge is visited only once." This is a deranged mixture of Euler's *criterion* for the existence of an Euler trail (a condition concerning a graph) and the *definition* of an Euler trail (a condition concerning a trail). A corrected version: *For a connected graph to have an Euler trail, we must have exactly* 0 *or* 2 *vertices with odd degree.* Tip: do not add extra spurious information in the hope of picking up floating marks. That would just make things more difficult for a reader, so the examiner is likely to subtract marks. (In this case, I subtracted one mark for omitting the 0 vertices of odd degree, and I subtracted another mark for the confusing extra information.)

(b) Many people responded with one or two paragraphs of amorphous discussion, but without giving an answer. *A connected graph has such a walk if and only if ... what?* I do not award many marks to people who just make correct relevant statements without arriving at any conclusion. So be brave: commit yourself to a definite answer, and then try to prove it.

3: Rather disappointing. From the scripts, anyone with scant experience of undergraduate teaching might surmise that many of the students have no grasp at all of the reasoning behind induction arguments! Actually, my guess is that most of the people in the class do understand the reasoning in the context of simple exercises, but many get confused in more complicated situations. I shall try to remember to include at least one induction argument in Midterm II. Many scripts offered solutions with the following form:

Base step: $a(A + 2B)\alpha^2 + b(A + B)\alpha + cA = ... = 0$. *Induction step:*
$a(A + (n+2)B)\alpha^{n+2} + b(A + (n+1)B)\alpha^{n+1} + C(A + nB)\alpha^n = ... = 0.$

Even when the manipulations were entirely correct (including the tricky part where one makes use of the equation $\alpha = -b/2a$), I awarded no marks for offerings which had that form. (I do not think I am being harsh, because I issued a warning about this matter in the question.) But what is wrong with that attempted solution?

Well, first of all, just prefixing a calculation with "Base step: blah blah. Induction Step" does not magically turn the calculation into an argument by induction. Sure enough, the manipulation $a(A+2B)\alpha^2+b(A+B)\alpha+cA = ... = 0$ does actually prove something. It proves that if $x_n = (A+nB)\alpha^n$ then $ax_{n+2}+bx_{n+1}+cx_n = 0$. But the question did not ask for a proof of that fact. It asked for a proof that if $ax_{n+2}+bx_{n+1}+cx_n = 0$ then $x_n = (A+nB)\alpha^n$ for some A and B.

The crux of the matter is to find the right inductive assumption. In the induction step, one shows that if we *assume* the formula for x_n and x_{n+1}, then we can *deduce* the formula for x_{n+2}. (The assumption does need to be specified clearly, because some people claimed, wrongly, that if we assume the formula just for x_{n-1} then we can deduce the formula for x_n.) As soon as the assumption in the induction step has been specified, the reader does not really need any more help with that step; any idiot can then carry out the manipulations. Indeed, for a more advanced reader, one could deal with the induction step just by saying: *Assuming the stated formula for x_n and x_{n+1}, then a straightforward manipulation using the equality $\alpha = -b/2a$ yields the stated formula for x_{n+2}.* In the base step, too, it is necessary to give the reader some help. We have to explain that some particular values of A and B are to be selected, namely $A = x_0$ and $B = x_1/\alpha - A$, whereupon the stated formula holds for x_0 and x_1.

This ought to have been an easy question, since it was bookwork. I already presented the argument in class, and also in the internet notes discrete2.pdf.

5 (a): Many people offered arguments such as the following one: "**(a)** $\bar{c} = 4$ and $\bar{c}f = 2e$ so $4f = 2e$, so $2f = e$." I admit that the requirements of the question might not have been sufficiently clear. (It is difficult to phrase bookwork questions for first-years in such a way that they know what can be assumed and what has to be proved.) The equations $2f = e$ and $\bar{c}f = 2e$ both occurred in the middle of arguments presented in class. But neither of them are standard lemmas. The question demanded a *proof* that $2f = e$, not just a quotation of the more general formula $\bar{c}f = 2e$. My scheme was to generously award 2/5 for the above, and 3/5 to those who explained that $\bar{c}$ is the number of edges per face (or, more generally, the average number of edges per face). I awarded 5/5 to anyone who mentioned the key idea of the proof, which is to note that each edge is associated with two faces. (Only two people gave a really lucid counting-by-pairs argument).

Olympiad results. Or: How come we survived and the Neanderthals went extinct?

Bronze Medal: **5:** "**(a)** $c = 4$ and $cf = 2e$ so $4f = 2e$, so $2f = e$. **(b)** If G is planar then $n - e + f = 2$. For $n = 4$ we have $4 - 2f + f = 2$, so $f = 2$ and $e = 4$ so $4 - 4 + 2 = 2$. So $n = 4$, $e = 4$, $f = 2$ satisfy the condition. For $n = 8$ we have $8 - 2f + f = 2$ so $f = 6$ and $e = 12$ so $8 - 12 + 6 = 2$. So $n = 6$, $e = 12$, $f = 6$ satisfy the condition." (For readability, I have edited the English and I have inserted some words.)

Part (a), here, is tolerable; see a comment above. For part (b), the whole aim is to show that n cannot be anything other than 4 or 8. It is no use assuming the conclusion, then solving some equations, and then noting — how amazing! — that the solutions still satisfy the equations even after the solutions have been written down on paper.

Silver medal: **5:** "$n - e + f = 2$. Suppose that our graph is a square. In our square, every vertex has the same degree, $\deg(a) = \deg(b) = \deg(c) = \deg(d) = 2$. Our faces f_1 and f_2 have four edges. **(a)** $f = 2$, $e = 4$, so $2f = e$, $2.2 = 4$, $\surd$. **(b)** $n = 4$, $n - e + f = 2$, $f = 6$, $e = 12$."

(The script has a diagram of a square with the vertices labeled a, b, c, d and the inside and outside faces labeled f_1 and f_2. No doubt the final two equalities arise from some fragment of insight into the case $n = 8$.)

The particular does not imply the general, not even when the particular has been described in loving detail. My pet rabbit is a mammal. My pet rabbit is very tame, and it eats carrots from out of my hand without ever nibbling my fingers. Therefore all mammals are tame. In an earlier epoch, the name for people who reasoned this way was: sabre-tooth tiger food.

Gold Medal: **3:** "In this question, let us assume that the given question is our initial condition, $x_n = (A + nB)\alpha^n$."

MATH 110: DISCRETE MATHEMATICS. Midterm II, Spring 09

Time allowed: 110 minutes. Please put your name on EVERY sheet of your manuscript. Do take the question-sheet home with you and, if possible, remember to bring it to the next class. The use of telephones and other electronic devices is prohibited.

LJB, 16 April 2009, Bilkent University.

1: 15% (a) 10% Let $\equiv_X$ be an equivalence relation on a set X, let $\equiv_Y$ be an equivalence relation on a set Y, and let $\equiv$ be the relation on $X \times Y$ such that, given $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, then $(x_1, y_1) \equiv (x_2, y_2)$ if and only if $x_1 \equiv_X x_2$ and $y_1 \equiv_Y y_2$. Show that $\equiv$ is an equivalence relation.
(b) 5% Let ξ be the number of equivalence classes of $\equiv_X$ and let η be the number of equivalence classes of $\equiv_Y$. (We are assuming, of course, that $\equiv_X$ and $\equiv_Y$ do both have finitely many equivalence classes). Show that $\xi\eta$ is the number of equivalence classes of $\equiv$.

2: 20% (a) 10% Let n and m be positive integers. Show that the number of ways of putting n indistinguishable balls into m distinguishable boxes is $\binom{n+m-1}{n}$. (Hint: One proof involves counting certain binary strings.)
(b) 3% Briefly explain why part (a) implies that $\binom{n+m-1}{n}$ is the number of integer solutions $x_i \geq 0$ to the equation $x_1 + ... + x_m = n$.
(c) 7% Find the number of integer solutions to $0 \leq y_1 \leq y_2 \leq y_3 \leq y_4 \leq y_5 \leq 6$?

3: 50% In this question, do not write out the decoding table. No marks will be awarded for arguments that involve writing out the decoding table.
(a) 7% Let $G = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & a_{1,1} & a_{1,2} & a_{1,3} \\ 0 & 1 & 0 & a_{2,1} & a_{2,2} & a_{2,3} \\ 0 & 0 & 1 & a_{3,1} & a_{3,2} & a_{3,3} \end{array}\right]$ be the generating matrix for an encoding function $\mathbb{Z}_2^3 \to \mathbb{Z}_2^6$. Write down the corresponding parity-check matrix H. Show that we can take four of the coset leaders to be the strings 000000, 000001, 000010, 000100.
(b) 7% From now on, let $\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Encode all the message words.
(c) 7% Show that we can take five of the coset leaders to be the strings 000000, 000001, 000010, 000100, 100000.
(d) 7% Taking five of the coset leaders to be as in part (c), show that 010000 and 001000 cannot appear as coset leaders.
(e) 7% Show that we can take the other three coset leaders to be 000011, 000110, 000101. (Your argument should make use of part (d).)
(f) 8% Taking the coset leaders to be as in parts (c) and (e), find their syndromes. Then decode the received words 101111, 111101, 111000, 000111 by finding their corresponding syndromes, coset leaders and codewords.
(g) 7% What is the rate of this code? What are its error detection and correction properties?

4: 15% Prove, by mathematical induction, that

$$\sum_{i=0}^{n} i(i+1)(i+2) = \frac{n(n+1)(n+2)(n+3)}{4}$$

for all positive integers n. (Do not prove this by quoting standard formulas for similar sums.)

MATH 110: DISCRETE MATHEMATICS. Final, Spring 09

Time allowed: two hours. Please put your name on EVERY sheet of your manuscript. The use of telephones and other electronic devices is prohibited. The use of very faint pencils is prohibited too. You may take the question-sheet home.

LJB, 30 May 2009, Bilkent University.

1: 36% (Recall, the Stirling number $S(m,n)$ is defined to be the number of equivalence relations $\equiv$ on the set $\mathbb{N}_m = \{1, ..., m\}$ such that $\equiv$ has n equivalence classes.)
(a) 6% Directly from the above definition, show that $S(m,1) = S(m,m) = 1$ and, for all $m \geq n \geq 1$, we have $S(m+1, n+1) = S(m,n) + (n+1)S(m, n+1)$.
(b) 6% Hence write out a table showing the values of $S(m,n)$ for all m and n in the range $1 \leq m \leq 8$ and $1 \leq n \leq 5$.
(c) 6% Express the number of surjective functions $\mathbb{N}_m \to \mathbb{N}_n$ in terms of $S(m,n)$. Briefly justify your answer.
(d) 6% Consider integers $m \geq n \geq r \geq 1$. Suppose that m (distinguishable) people are placed in n indistinguishable bags such that each bag contains at least one person. Then the bags are given to r (distinguishable) ogres[1] in such a way that each ogre receives at least one bag. Give a formula for the number of ways of doing this. (The formula is to be in terms of Stirling numbers.)
(e) 6% Now suppose we no longer demand that each ogre receives at least one bag. How may ways are there of doing it now?
(f) 6% Using part (b), evaluate your formulas in parts (d) and (e) in the case where there are 3 ogres, 5 bags and 8 people. (Explicit numerical answers are required.)

2: 24% (a) 9% Write down the Euclidian algorithm for finding the greatest common divisor of two positive integers a and b.
(b) 6% Prove that the algorithm terminates (comes to an end after only finitely many steps).
(c) 9% Hence show that there exist integers x and y such that a and b are both divisible by $xa + yb$.

3: 15% (a) 5% Up to isomorphism, how many connected graphs with three vertices are there?
(b) 5% We say that a poset is **connected** provided its Hasse diagram, regarded as a graph, is connected. Up to isomorphism, how may connected posets with three elements are there?
(c) 5% Let us say that two posets are **equivalent** provided their Hasse diagrams, regarded as graphs, are isomorphic to each other. How many equivalence classes of connected posets with 3 elements are there?

4: 25% (a) 10% For integers $1 \leq m \leq n$, consider the linear encoding functions $\mathbb{Z}_2^m \to \mathbb{Z}_2^n$ such that no two message words correspond to the same codeword. How many such encoding functions are there?
(b) 5% For such encoding functions, how many different codes are there? (Recall that the code is just the set of codewords.)
(c) 10% What are the answers to the previous two parts if we no longer demand that the encoding functions are linear? (But we still demand that no two message words correspond to the same codeword.)

[1] An ogre is a monster such as Shrek.

MATH 110: DISCRETE MATHEMATICS.

Midterm I Makeup, Spring 09

Time allowed: 110 minutes. Please put your name on EVERY sheet of your manuscript. The use of telephones and other electronic devices is prohibited. You may take the question sheet home.

LJB, 1 June 2009, Bilkent University.

1: 25% (a) 5% State, without proof, Euler's formula for a connected planar graph with n vertices, e edges and f faces.

(b) 15% Starting from the formula in mart (a), state and prove an inequality involving n and e that is satisfied by a planar graph whose faces all have at least four edges. Your inequality must be strong enough for the application in part (c).

(c) 5% Show that the graph $K_{3,3}$ is non-planar. (This is the graph with six vertices x_1, x_2, x_3, y_1, y_2, y_3 and nine edges $x_i y_j$.)

2: 25% (As usual, all graphs are understood to have only finitely many vertices.)

(a) 7% Prove that, for any graph with an Euler circuit, every edge has even degree.

(b) 7% Prove that, for any graph that has an Euler trail which is not an Euler circuit, exactly two vertices have odd degree.

(c) 4% Give a counter-example to the assertion "For any graph such that every vertex has even degree, there is an Euler circuit." State, *without proof*, a corrected version of that assertion.

(d) 7% State and prove an assertion similar to part (c), but for graphs such that exactly two vertices have odd degree.

3: 25% Solve the recurrence relation $x_n = 1 + x_0 + x_1 + ... + x_{n-1}$ with the initial condition $x_0 = 1$. (Hint: work out the first few values of x_n, guess the solution, then prove it by induction. Make sure you are clear about what the inductive assumption is.)

4: 25% (a) 10% The Fibonacci numbers are defined by the recurrence relation $F_{n+2} = F_{n+1} + F_n$ with $F_0 = 0$ and $F_1 = 0$. Find a general formula for F_n.

(b) 15% Solve the recurrence relation $x_{n+2} = x_{n+1} + x_n + 2n(n+1)(n!)$ with the initial condition $x_0 = 0$ and $x_1 = 1$. (Hint: First experiment with the trial solution $x_n = 2(n!)$.)

MATH 110: DISCRETE MATHEMATICS.

Midterm II Makeup, Spring 09

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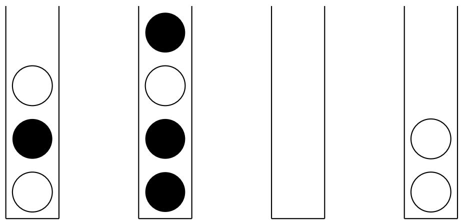
LJB, 1 June 2009, Bilkent University.

1: 15% (a) Let X be a set, and let $E(X)$ be the set of equivalence relations on X. Given $\equiv_1, \equiv_2 \in E(X)$, we write $\equiv_1 \cong_X \equiv_2$ provided there exists an invertible function $f : X \to X$ with the property that, for all $x, x' \in X$, we have $x \equiv_1 x'$ if and only if $f(x) \equiv_2 f(x')$. Show that $\cong_X$ is an equivalence relation on $E(X)$.

2: 20% Show that there are $\dfrac{(n+m+r-1)!}{n!\,m!\,(r-1)!}$ ways of placing n indistinguishable white balls and m indistinguishable black balls in r distinguishable pipes. Note that the ordering of the balls in each pipe is significant; in the diagram, the top ball of the first pipe is white, the top ball of the second pipe is black, there are no balls in the third pipe, and the top ball of the fourth pipe is white.



3: 45% The Hamming 7-4-code has parity check matrix

$$\left[\begin{array}{cccc|ccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right].$$

(a) 10% Write down the generating matrix. For each message word, write down the corresponding codeword.

(b) 15% Show that there is only one way of choosing the coset leaders and, furthermore, the coset leaders are precisely the 8 received words that have weight 0 or 1. For each of the coset leaders, find the corresponding syndrome.

(c) 10% By finding corresponding syndromes — and without writing out the decoding table — decode the words 1111111, 0111111, 0011111, 0001111.

(d) 10% What is the rate of this code? What are its error detection and correction properties?

4: 20% Prove that, for all natural numbers n and r, we have

$$\sum_{i=0}^{n} i(i+1)(i+2)...(i+r) = \frac{n(n+1)(n+2)...(n+r+1)}{r+2}.$$