Documentation for:

MATH 110, Discrete Mathematics.

Spring 2007/08. Laurence Barker, Bilkent University

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MATH 110: DISCRETE MATHEMATICS. Midterm I, Spring 08

Time allowed: 110 minutes. Please put your name on EVERY sheet of your manuscript. Do take the question-sheet home with you (and maybe bring it to the next class). The use of calculators, mobile phones and other electronic devices is prohibited.

LJB, 20 March 2008, Bilkent University.

1: 15% Show that $\sum_{i=1}^{n} n(n+1) = n(n+1)(n+2)/3$ for all $n \geq 1$.

2: 30% (a) State (without proof) Euler's criterion for a connected graph to have (1) an Euler trail, (2) an Euler circuit.

(b) Let G be a connected graph with at least one edge and such that the vertices of G all have the same degree and G has an Euler trail but no Euler circuit. How many vertices does G have?

(c) A four-dimensional geometric figure called the **24-cell** has 24 vertices, 96 edges, 96 two-dimensional faces and 24 three-dimensional faces. The vertices and edges form a connected graph such that all the vertices have the same degree. Does this graph have an Euler circuit?

3: 35% (a) Let $y_n = n^3$ for $n \geq 0$. Show that $y_{n+2} - 4y_{n+1} + 4y_n = n^3 - 6n^2 + 4$.

(b) Find the general solution to the recurrence relation $z_{n+2} - 4z_{n+1} + 4z_n = 0$.

(c) Using parts (a) and (b), solve the recurrence relation $x_{n+2} - 4x_{n+1} + 4x_n = n^3 - 6n^2 + 4$ with initial conditions $x_0 = 2$ and $x_1 = 7$.

4: 20% We define a **multigraph** in the same way that we define a graph, except that we no longer require there to be at most one edge between any two given points. We say that a multigraph is **planar** provided it can be drawn on the plane with no intersecting edges. For example, the Seven Bridges of Königsberg Problem can be expressed in terms of a planar multigraph with 4 vertices, 7 edges and 5 faces. Let G be a connected multigraph with n vertices, e edges and f faces. Prove that $n - e + f = 2$ by adapting the well-known inductive proof of this equation for ordinary graphs. You may assume the preliminary lemma which asserts that $n = e + 1$ for a tree.

MATH 110: DISCRETE MATHEMATICS. Makeup I, Spring 08

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LJB, 20 March 2008, Bilkent University.

1: 20% Guess a formula for $\sum_{i=1}^{n} n(n+1)(n+2)$ and then prove it by induction.

2: 30% State and prove Euler's criterion for a connected graph to have an Euler circuit.

3: 30% Solve the recurrence relation relation $x_{n+2} - 5x_{n+1} + 6x_n = 2n^2 - 6n - 1$ with $x_0 = x_1 = 0$.

4: 20% Show that the graph $K_{3,3}$ is non-planar. (This is the graph with 6 vertices x_1, x_2, x_3, y_1, y_2, y_3 and 9 edges having the form $x_i y_j$.)

MATH 110: DISCRETE MATHEMATICS. Midterm II, Spring 2008

Time allowed: 110 minutes. Please put your name on EVERY sheet of your manuscript. You may take the question-sheet home with you. The use of calculators, mobile phones or similar electronic devices is prohibited throughout the exam.

LJB, 1 May 2008, Bilkent.

1. 50%: Consider a group code with check matrix $H = \left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$.

(a) 5% Write down the generating matrix G.
(b) 5% Encode all eight of the message words.
(c) 5% Show that the following seven received words all have different syndromes.

000000 , 000001 , 000010 , 000100 , 001000 , 010000 , 100000 .

(d*) 5% Explain why every decoding table has eight coset leaders and seven of them must be the words in part (c). Explain why there are three possibilities for the eighth coset leader, namely 110000 or 000101 or 001010.
(e) 5% Write down just the first three rows of the decoding table, taking 000000 and 000001 and 000010 as the coset leaders for those three rows. *Do not write down any more rows of the decoding table.*
(f) 5% Using part (e), decode the received word 010111.
(g) 5% *Without writing down any more rows of the decoding table*, take 110000 as the eighth coset leader and decode the received words 011110 and 111111.
(h) 5% *Without writing down any more rows of the decoding table*, now take 000101 as the eighth coset leader, and decode the received words 011110 and 11111.
(i) 5% What is the rate of this code?
(j) 5% What is the maximum number of transmission errors such that the errors can always be detected? What is the maximum number of transmission errors such that the errors can always be corrected?

2. 20%: Let n be a positive integer. Give two different proofs of the equality

$$\sum_{j=0}^{n} \binom{n}{j}^2 = \binom{2n}{n}$$

by the following two methods:
(a) Expand both sides of the equation $(x+y)^n(x+y)^n = (x+y)^{2n}$ using the binomial theorem.
(b) Letting $A = \{1, 2, ..., n\}$ and $B = \{n+1, n+2, ..., 2n\}$, consider the number of ways of choosing subsets $S \subseteq A$ and $T \subseteq B$ with $|S| = j$ and $|T| = n - j$.

3. 20%: Let $X = \{1, 2, ..., 100\}$. Let $\equiv$ be the relation on X such that, given elements x and y of X, then $x \equiv y$ if and only if $x^2 - y^2$ is divisible by 5.
(a) Show that $\equiv$ is an equivalence relation.
(b*) How many equivalence classes does $\equiv$ have?

4. 10% How many injective functions are there from $\{1, 2, ..., 99\}$ to $\{1, 2, ..., 100\}$?

MATH 110: DISCRETE MATHEMATICS. Final, Spring 08

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LJB, 31 May 2008, Bilkent University.

1: 5% Give an example of a poset with 6 elements and draw its Hasse diagram.

2: 25% Let n and e be positive integers and let G_1 and G_2 be graphs with n vertices and e edges.

(a) Show that $e \leq n(n-1)/2$.

(b) Show that, if $e = n(n-1)/2$ or $e = n(n-1)/2 - 1$, then G_1 and G_2 are isomorphic.

(c) Show that, if $n \geq 4$ and $e = n(n-1)/2 - 2$, then G_1 and G_2 are not necessarily isomorphic.

3: 20% Given functions $f_1 : A_1 \to B_1$ and $f_2 : A_2 \to B_2$, let us write $f_1 \equiv f_2$ when there exist bijections $\alpha : A_1 \to A_2$ and $\beta : B_1 \to B_2$ such that $f_2(\alpha(a)) = \beta(f_1(a))$ for all $a \in A_1$.

(a) Show that $\equiv$ is an equivalence relation.

(b) Now confine attention to just those functions $f : A \to B$ such that $|A| = |B| = 3$. How many equivalence classes of such functions are there?

4: 25% Solve the recurrence relation $x_{n+2} - 7x_{n+1} + 10x_n = 4$ with initial conditions $x_0 = 3$ and $x_1 = 2$. (Hint: the attempted solution $x_n = 1$ satisfies the recurrence relation but fails the initial conditions.)

5: 25% Let C and D be group codes in $\mathbb{Z}_2^n$.

(a) Show that the intersection $C \cap D$ is a group code.

(b) Let $C + D$ denote the set whose elements have the form $c + d$ where $c \in C$ and $d \in D$. Show that $C + D$ is a group code.

(c) Given $c_1, c_2 \in C$ and $d_1, d_2 \in D$, show that $c_1 + d_1 = c_2 + d_2$ if and only if there exists an element $e \in C \cap D$ such that $c_2 = c_1 + e$ and $d_2 = d_1 + e$.

(d) Let $\equiv$ be the equivalence relation on $C \times D$ such that $(c_1, d_1) \equiv (c_2, d_2)$ if and only if $c_1 + d_1 = c_2 + d_2$. (This is obviously an equivalence relation, and there is no need to justify the fact.) What can you say about the sizes of the equivalence classes and the number of equivalence classes?

(e) Using part (d), show that $|C + D| = \dfrac{|C| \, . \, |D|}{|C \cap D|}$.