

Documentation for:  
MATH 110, Discrete Mathematics.  
Spring 2006/07. Laurence Barker, Bilkent University

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MATH 110: DISCRETE MATHEMATICS FINAL

Time allowed: 110 minutes. Please put your name on EVERY sheet of your manuscript. You may take the question-sheet home with you. All the questions carry the same credit. The use of calculators, mobile phones and other electronic devices is prohibited.

LJB, 24 May 2007, Bilkent University.

**1:** Consider a group code with check matrix

$$H = \left[ \begin{array}{cccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{array} \right].$$

- (a) Find the generating matrix. Encode the four message words 0000, 0001, 0011, 0111.
- (b) Write out the part of the decoding table that consists of the message words, the codewords, the coset leaders and the syndromes. (Do not write out the remaining 49 entries of the decoding table.) To do this, you may assume that all the coset leaders have at most one non-zero digit.
- (c) Decode the four received words 0000000, 0000001, 0000011, 0000111.
- (d) For this code, what is the maximum number of errors in transmission such that:
  - (d1) a received word can always be correctly decoded?
  - (d2) an error in a received word can always be detected?

**2:** Let  $n$  be a positive integer. Let  $S$  be the set of binary strings with length  $n$ . Let  $*$  be the function from  $S \times S$  to  $S$  such that  $x_1 \dots x_n * y_1 \dots y_n = z_1 \dots z_n$  where  $z_i = 0$  if and only if  $x_i = y_i$ . Let  $T$  be a subset of  $S$  with the property that if  $x$  and  $y$  belong to  $T$  then  $x * y$  belongs to  $T$ . Consider the relation  $\sim$  on  $S$  such that  $x \sim y$  if and only if  $x * y$  belongs to  $T$ .

- (a) Show that  $\sim$  is an equivalence relation.
- (b) Suppose that  $|T| = 2^m$  for some natural number  $m$ . How many equivalence classes does the equivalence relation  $\sim$  have? Justify your answer.

**3:** (a) State and prove a result about the minimum degree of a vertex of a planar graph. (You may assume Euler's formula for planar graphs.)

(b) On desert planet, all the land is divided up into countries (each country being one continuous region.) Show that, with six colours, it is (mathematically) possible to paint the sand so that each country has sand of just one colour, and there is a change of colour at every border.

**4:** Let  $x \geq y$  be positive integers. Give very brief justifications for your answers to the following questions; standard results may be assumed without proof.

- (a) How many ways are there of putting  $x$  indistinguishable objects into  $y$  distinguishable boxes such that each box has at least one object?
- (b) How many ways are there of putting  $x$  distinguishable objects into  $y$  indistinguishable boxes such that each box has at least one object?
- (c) How many ways are there of putting 12 indistinguishable objects into 5 indistinguishable boxes such that each box has at least one object?

# MATH 110: DISCRETE MATHEMATICS

## MIDTERM 1

LJB, 6 March 2007, Bilkent University.

Time allowed: 110 minutes.

Please put your name on EVERY sheet of your manuscript.

You may take the question-sheet home with you.

All the questions carry the same weight of credit.

The use of calculators, mobile phones and similar electronic devices is prohibited.

**1:** How many ways are there of distributing 500 (indistinguishable) chocolates and 300 (distinguishable) boys among 100 (distinguishable) girls in such a way that each girl receives at least one chocolate?

**2:** Let  $a, b, c$  be complex numbers with  $a \neq 0$ . Suppose that the quadratic equation  $at^2 + bt + c = 0$  has a repeated solution  $t = \alpha$ . Show that the general solution to the recurrence relation  $ax_{n+2} + bx_{n+1} + cx_n$  is  $x_n = (A + nB)\alpha^n$  where  $A$  and  $B$  are constants. (Hint: make use of the fact that  $\alpha = -b/2a$ .)

**3: (a)** Find the real numbers  $e$  and  $f$  such that the recurrence relation  $x_{n+2} + ex_{n+1} + fx_n = 0$  has a solution  $x_n = 123456789(\sqrt{2} + 1)^n - 987654321(\sqrt{2} - 1)^n$ .

**(b)** Give an example of two non-real complex numbers  $\alpha$  and  $\beta$  and four real numbers  $g, h, A, B$  such that the recurrence relation  $x_{n+1} + gx_{n+1} + hx_n = 0$  has solution  $x_n = A\alpha^n + B\beta^n$  and each  $x_n$  is a real number.

**4:** Using the Euclidian Algorithm, find the value of  $\gcd(1634621, 1655041)$ .

**5:** Below is an algorithm for calculating  $A^n$  where  $A$  is a  $30 \times 30$  matrix and  $n$  is a positive integer represented as a binary string  $n_r n_{r-1} \dots n_1 n_0$  where  $r$  is a positive integer and  $n_r = 1$ . Thus each  $n_j \in \{0, 1\}$  and  $n = \sum_{j=0}^r n_j 2^j = 2^r + \sum_{j=0}^{r-1} n_j 2^j$ . What is the complexity of the algorithm (a) in terms of  $r$ , (b) in terms of  $n$ ?

- Input  $A$  and  $r$  and  $n_0, \dots, n_{r-1}$
- Let  $B := A$  and  $C := I_{30}$  (the identity matrix)
- For  $j = 0$  to  $r - 1$  do
  - If  $n_j = 1$  let  $C := BC$
  - Let  $B := B^2$
- Next  $j$
- Let  $C := BC$
- Output  $C$

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LJB, 27 April 2007, Bilkent University.

**1: (a)** Evaluate the Stirling number  $S(6, 4)$ .

**(b)** Let  $X$  and  $Y$  be sets with sizes  $|X| = 6$  and  $|Y| = 4$ . How many surjections from  $X$  to  $Y$  are there?

**(c)** Consider the equivalence relations  $\equiv$  on  $X$  such that  $\equiv$  has at least 4 equivalence classes. How many such equivalence relations are there?

**2:** The following algorithm determines whether or not there are any repetitions in a given list of numbers  $x_1, \dots, x_n$ .

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♣ for  $i = 1$  to  $n - 1$ 
  ◇ for  $j = i + 1$  to  $n$ 
    ♠ if  $x_i = x_j$  then print "there is a repetition" and stop
  ◇ next  $j$ 
♣ next  $i$ 
♣ print "there is no repetition" and stop

```

**(a)** Determine the worst-case complexity of the algorithm in terms of  $n$ .

**(b)** Now suppose that the numbers  $x_1, \dots, x_n$  are all 7-digit natural numbers. If  $n \geq 10\,000\,000$  then there must be a repetition, so let us modify the algorithm by beginning it with the line:

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♡ if  $n > 9\,999\,999$  then print "there is a repetition" and stop.

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What is the worst-case complexity now?

**3: (a)** Let  $G$  be a graph with  $e$  edges. State (without proof) a formula for  $e$  in terms of the degrees of the vertices of  $G$

**(b)** Let  $T$  be a tree with  $e$  edges and  $n$  vertices. State (without proof) a formula expressing  $e$  in terms of  $n$ .

**(c)** Using the formulas in (a) and (b), prove that if  $T$  has a vertex with degree at least 3, then  $T$  has at least 3 vertices with degree 1.

**(d)** Let  $m$  be a positive integer. Give an example of a tree with  $m$  vertices and with an Euler path.

**(e)** Up to isomorphism, how many trees are there with  $m$  vertices and with an Euler path?

**4: (a)** State and prove Euler's theorem concerning the number of edges  $n$ , the number of edges  $e$  and the number of faces  $f$  for a connected planar graph.

**(b)** Let  $n$  be an integer with  $n \geq 4$ . Suppose that  $G$  is a connected planar graph with  $n$  vertices and  $3n - 6$  edges. Show that every face has precisely 3 edges.

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LJB, 21 May 2007, Bilkent University.

**1:** Let  $m$  and  $n$  be positive integers.

**(a 5 %)** Define the Stirling number  $S(m, n)$  by a formula involving binomial coefficients.

**(b 5%)** Define the Stirling number  $S(m, n)$  in terms of enumeration of equivalence relations.

**(c 5%)** Let  $A_1, \dots, A_n$  be finite sets. Suppose there exist integers  $k_1, \dots, k_n$  such that

$$k_r = |A_{i_1} \cap \dots \cap A_{i_r}|$$

for all  $1 \leq r \leq n$  and  $1 \leq i_1 < \dots < i_r \leq n$ . Give a formula for  $|A_1 \cup \dots \cup A_n|$  in terms of  $k_1, \dots, k_n$ .

**(d 10%)** Hence show that your two above definitions of  $S(m, n)$  agree with each other.

**2:** Given sequences of real numbers  $\underline{x} = (x_1, \dots, x_n)$  and  $\underline{y} = (y_1, \dots, y_n)$ , their covariance is defined to be

$$\text{cov}(\underline{x}, \underline{y}) = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y}) \quad \text{where} \quad \bar{x} = \frac{1}{n} \sum_{i=1}^n x_i \quad \text{and} \quad \bar{y} = \frac{1}{n} \sum_{i=1}^n y_i .$$

**(a 10 %)** Compose an efficient algorithm to calculate  $\text{cov}(\underline{x}, \underline{y})$  for given sequences  $\underline{x}$  and  $\underline{y}$ . (Hint: first calculate the sums for  $\bar{x}$  and  $\bar{y}$ . Then calculate the sum for  $\text{cov}(\underline{x}, \underline{y})$ .)

**(b 15%)** What are the best-case and worst-case complexities of your algorithm? (Warning: no marks will be awarded if your algorithm has a complexity that is larger than necessary.)

**3:** A directed multigraph is defined to be a quadruple  $(V, E, t, h)$  where  $V$  and  $E$  are sets and where  $t$  and  $h$  are functions  $E \rightarrow V$ . The elements of  $V$  are called vertices. The elements of  $E$  are called edges. Intuitively, we regard  $V$  as a set of points, and each edge  $\epsilon$  is regarded as an arrow from its tail vertex  $t(\epsilon)$  to its head vertex  $h(\epsilon)$ . The degree of a vertex  $v$  is equal to the number of edges from  $v$  plus the number of edges to  $v$ . In symbols

$$d(v) = d_t(v) + d_h(v) .$$

By adapting a standard proof of Euler's well-known theorem about existence of Euler circuits for ordinary graphs, state and prove a version of that theorem for directed multigraphs.

**4:** Let  $G$  be a connected planar graph. Suppose that it is possible to remove  $m$  edges from  $G$  so that the remaining graph is a forest with  $r$  connected components. (A forest is a graph such that every connected component is a tree.) Express the number of faces of  $G$  in terms of  $m$  and  $r$ . Justify your answer, clearly stating any standard results that you use.

## MATH 110: DISCRETE MATHEMATICS

### *Emergency Makeup for Makeup to Midterm 2*

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LJB, 27 April 2007, Bilkent University.

**1:** Using a suitable recurrence relation (which you may assume without proof) write down a table showing the Stirling numbers  $S(m, n)$  for all  $m$  and  $n$  in the range between 1 and 7.

**2:** In an electronic machine with 8-bit bytes, natural numbers are to be stored as records  $(a_0, \dots, a_n, 128)$  where each  $0 \leq a_i \leq 127$  and  $a_n \neq 0$ . (Thus, the leading digit of each byte is 0 except for the end-of-record marker 10000000.) Compose an algorithm to add two natural numbers  $(a_0, \dots, a_n, 128)$  and  $(b_0, \dots, b_n, 128)$ , outputting the result as a record  $(c_0, \dots, c_m, 128)$  where  $c_m \neq 0$ . What is the complexity of your algorithm?

**3:** Let  $G$  be a graph with  $n$  vertices,  $e$  edges and  $c$  components. (A graph with  $c$  components is a disjoint union of  $c$  connected graphs.) Prove that  $G$  is a forest with  $c$  components if and only if  $n = e + c$ . (A forest, recall, is a disjoint union of trees.)

**4:** State Euler's formula for connected planar graphs. Now drop the assumption that the graph is connected, and generalize the formula. Your generalized formula should involve the number of vertices  $n$ , the number of edges  $e$ , the number of faces  $f$  and the number of components  $c$ .