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MATH 110, Discrete Mathematics

Bilkent University, Fall 2015, Laurence Barker

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MATH 110, Discrete Mathematics, Fall 2014

Course specification

Laurence Barker, Mathematics Department, Bilkent University,
version: 10 November 2014.

Course Aims: To introduce some concepts and techniques of discrete and combinatorial mathematics from an applicable point of view. To convey a practical sense of mathematical proof as very clear deductive explanation.

Course Description: Discrete mathematics is an umbrella name for all areas of applicable mathematics where there is not much structure to work with. Although it is very diverse, certain kinds of technique tend to crop up frequently. We shall be studying three areas, superficially quite separate, but similar in style and technique: the first third will focus largely on graph theory; the middle third on enumerative study of relations; the last third on coding theory.

Course Requirements: Most mathematics cannot be learned just by listening. To take in the concepts and techniques, you have to study the course notes and do some exercises yourself.

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Course Text: R. P. Grimaldi, "Discrete and Combinatorial Mathematics", 5th Ed. (Pearson, 2004).

Classes: Tuesdays 15:40 - 16:30 FC-Z23D, Fridays, 13:40 - 15:30 FC-Z23D.

Office Hours: Tuesdays, 16:40 - 17:30, SA-129.

Please note, Office Hours is not just for the strong students. I will have no sympathy for drowning students who do not come to me for help. Students who are having serious difficulties must come to Office Hours to discuss the mathematics. In fact, I need those students to come, so as to ensure that the classroom material does not lose touch with parts of the audience.

Syllabus:

Week number: Monday date: Subtopics. Section numbers

1: 15 Sept: Illustrations of problems in discrete mathematics. Sets and statements. Mathematical induction, 4.1.

2: 22 Sept: Recursive definitions and induction, 4.2. Second order recurrence relations, 10.2

3: 29 Sept: Graphs. trees. 11.1, 11.2.

4: 6 Oct: Criteria for existence of Euler paths or Euler circults, 11.3

5: 13 Oct: Euler's characteristic formula for planar graphs, proved by mathematical induction. Degrees of vertices of planar graphs, 11.4

6: 20 Oct: The non-planarity of the graphs K_5 and $K_{3,3}$, 11.4

7: 27 Oct: Review for Midterm 1. Midterm 1 on 31 October.

8: 3 Nov: Binomial coefficients, 1.2, 1.3, 1.4

9: 10 Nov: Sets and correspondences. Injective, surjective and bijective functions, 5.1, 5.2, 5.3, 5.6.

10: 17 Nov: Relations, incidence matrices, enumeration of relations, 7.1, 7.2. Equivalence relations, 7.4.

11: 24 Nov: Stirling numbers of the second kind, 5.3, 7.4

12: 1 Dec: Coding theory, Hamming metric, parity-check and generator matrics, 16.5, 16.6, 16.7.

13: 8 Dec: Encoding and decoding linear codes using coset leaders, 16.8.

14: 15 Dec: Presentations.

15: 22 Dec: Presentations.

16: 29 Dec: Review for Final.

Assessment:

On Friday 7th November, because of the very high midterm scores, a vote was taken to select one of the three assessment options: (a) Midterm 2 and no presentation, as originally planned; (b) Midterm 2 and presentations; (c) presentations and no Midterm 2. Option (c) was selected by an overwhelming majority.

Because the Midterm 1 course credit has changed from 25% to 30%, the two students who scored less than 60% in Midterm 1 may, if they request, sit a Midterm Retake at the end of the semester. If they score more than 55% in the Retake, their Midterm mark will be evaluated as 55% (because it would be unfair if they were to catch up with students who do not have the Retake option).

- Quizzes, Homework and Participation 15%.
- Midterm, 30%, Friday 31 October.
- Presentations, 15%, Last two full weeks of semester (see syllabus).
- Final, 40%.

The 20-minute presentations will be graded mainly for clear communication to the audience. Do not try to impress the audience with the depth of your knowledge! Instead, try to teach the audience something interesting. The use of electronic machinery is optional.

75% attendance is compulsory.

Class Announcements: All students, including any absentees from a class, will be deemed responsible for awareness of class announcements.

MATH 110, Discrete Mathematics, Fall 2014
Homeworks, Quizzes, Presentations

Laurence Barker, Mathematics Department, Bilkent University,
version: 6 January 2015.

Office Hours: Tuesdays, 16:40 - 17:30, SAZ 129.

Office Hours would be a good time to ask me for help with the homeworks.

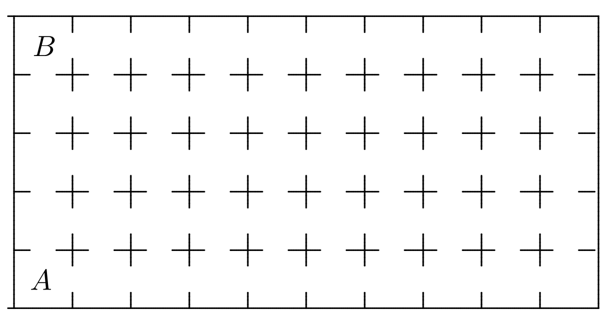
The course textbook is R. P. Grimaldi, "Discrete and Combinatorial Mathematics" 5th edition (Pearson, 2004). Many of the questions are taken from there. The solutions are to be discussed in class.

Homework 1 due Friday 26th September

The purpose of these two questions is to practise the art of clear deductive explanation. Just try to make your argument easy to understand.

Question 1.1: Prove that, if 32 dominoes pieces are of such a size as to exactly cover the 64 squares of a chess-board, and if 2 opposite corner squares of the chess-board are removed, then the remaining 62 squares of the chess-board cannot be covered by 31 dominoes pieces. (Hint: consider the colours of the squares.)

Question 1.2: A prisoner is trapped in a dungeon consisting of 50 cells arranged in a 10 by 5 grid. She is free to pass from cell to cell via the doorways indicated in the diagram. Can she make a tour of the dungeon, visiting each cell exactly once, starting in the bottom-left cell labelled A, finishing in the top-left cell labelled B? (Hint: consider the number of up moves, the number of down moves, the number of left moves, the number of right moves.)



Solution 1.1: Each dominoes piece covers one black and one white square. If the mutilated board can be covered, then it has the same number of blacks and whites. The original board does have the same number of blacks and whites, but the two removed squares have the same colour, so the mutilated board does not have the same number of blacks and whites.

Solution 1.2: No, such a tour is impossible. For a contradiction, suppose such a tour exists. Let U, D, L, R be the number of moves, from one cell to the next one, in the upwards, downwards, leftwards, rightwards directions, respectively. Then $U - D = 4$. So U and D and either both odd or else both even. It follows that $U+D$ is even. We also have $L=R$, so $L+R$ is even. But the total number of moves made in the tour is $U+D+L+R=49$, which is odd. This is a contradiction, as required. □

Alternative Solution (supplied by class): Colour the squares alternately black and white, as a chess-board. If such a tour is possible, then the number of moves is 49, which is odd. But the starting and finishing squares have the same colour. This is a contradiction. So such a tour does not exist.

Quiz 1: *Tuesday, 23 September.* Show that $1^3+2^3+...+n^3 = n^2(n+1)^2/4$ for all positive integers n.

Solution 1: Let $A_n = 1^3+...+n^3$ and $B_n = n^2(n+1)^2/4$. We shall show, by induction, that $A_n = B_n$ for all integers $n \geq 1$. Plainly, $A_1 = 1 = B_n$. Suppose that $n \geq 2$ and $A_{n-1} = B_{n-1}$. We have

$$B_n - B_{n-1} = \big((n+1)^2 - (n-1)^2\big)n^2/4 = n^3 = A_n - A_{n-1} \ .$$

Cancelling, we deduce that $A_n = B_n$. □

Quiz 2: *Friday, 10 October.* How many isomorphism classes of graphs with order 3 are there?

Solution 2: There are 4. There is one with 0 edges, one with 1 edge, one with 2 edges, one with 3 edges. [Diagrams given in class.] □

Quiz 3: *Tuesday, 14 October.* **(a)** What is the minimum number of edges that can be added to the depicted graph to produce a graph with an Euler circuit? **(b)** Add such edges and find an Euler circuit.

[The graph was depicted in class. The answer to part (a) was 4. The easiest way of finding an Euler circuit was to apply the method described in the proof of the Euler Path Theorem.]

Homework 2 due Friday 17th October

2.1: Show that $\sum_{i=1}^{n} 1/i(i+1) = n/(n+1)$.

2.2: Show that the Fibonacci numbers F_0, F_1, ... satisfy $F_{n+1} < (7/4)^n$ for $n \geq 1$.

2.3: Solve $x_{n+2} - 6x_{n+1} + 9x_n = 0$ with $x_0 = 1$ and $x_1 = 6$.

2.4: Solve $x_{n+2} + 2x_{n+1} + 2x_n = 0$ with $x_0 = 2$ and $x_1 = -2$.

Solution 2.1: Let $S_n = \sum_i 1/i(i+1)$. We shall show, by induction, that $S_n = n/(n+1)$. First note that $S_1 = 1/2 = 1/(1+1)$. Now suppose that $n \geq 2$ and that $S_{n-1} = (n-1)/n$. We have $S_n - S_{n-1} = 1/n(n+1) = n^2/n(n+1) - (n-1)(n+1)/n(n+1) = n/(n+1) - (n-1)/n$. Cancelling, we deduce that $S_n = n/(n+1)$. □

Alternative Solution (supplied by class): Since $1/i(i+1) = 1/i - 1/(i+1)$, the sum is

$$(1-1/2)+(1/2-1/3)+(1/3-1/4)+...+(1/n-1/(n+1)) = 1-1/(n+1) = n/(n+1) \ . \qquad \square$$

Solution 2.2: We argue by induction. The inequality is clear when $n = 1$ or $n = 2$. Suppose that $n \geq 3$ and $F_{n-1} < (7/4)^{n-2}$ and $F_n < (7/4)^{n-1}$. Then

$$F_{n+1} = F_n + F_{n-1} < (7/4)^{n-1} + (7/4)^{n-2} = (7/4)^{n-2}(7/4 + 1)$$
$$= (7/4)^{n-2}11/4 < (7/4)^{n-2}49/16 = (7/4)^n \ . \qquad \square$$

Solution 2.3: The equation $t^2 - 6t + 9 = 0$ has unique solution $t = 3$. So $x_n = (C + nD)3^n$ for some C and D. We have $C = x_0 = 1$ and $3(C + D) = x_1 = 6$. Hence, $D = 1$. Therefore $x_n = (1 + n)3^n$.

Solution 2.4: The equation $t^2 + 2t + 2$ has solutions $t = -1 + i$ and $t = -1 - i$. So $x_n = A(-1 + i)^n + B(-1 + i)^n$ for some A and B. We have $A + B = x_0 = 2$ and $A(-1 + i) + B(-1 - i) = x_1 = -2$. Solving, we find that $A = B = 1$. So $x_n = (-1 + i)^n + (-1 - i)^n$.

Quiz 4: *Friday, 17 October.* Repeat of Homework 2.2 (set because many people misunderstood internet accounts of the Fibonacci sequence; when the sequence is expressed as 1, 1, 2, 3, 5, ... the implicit numbering is F_1, F_2, F_3, F_4, ...)

Homework 3 due Friday 28th November

3.1: Book question 1.4.12: Find the number of integer solutions to

$$x_1 + x_2 + x_3 + x_4 + x_5 < 40$$

where (a) each $x_i \geq 0$, (b) each $x_i \geq -3$.

3.2: Book question 1.4.28: For integers n and r where $1 \leq n$ and $0 \leq r \leq n/2$, let $b_{n,r}$ be the number of binary strings z with length n such that 01 occurs exactly r times in z. For example, when $n = 4$ and $r = 1$ the possible strings are

$$0001, 1001, 1101, 0010, 0011, 1010, 1011, 0100, 0110, 0111 \ .$$

(a) For $n \geq 4$, give a formula for $b(n, 2)$.
(b) For $n \geq 6$, give a formula for $b(n, 3)$.
(c) For $n \geq 1$, give a conceptual proof of the formula

$$2^n = \binom{n+1}{1} + \binom{n+1}{3} + \ldots + \begin{cases} \binom{n+1}{n} & \text{if } n \text{ is odd,} \\ \binom{n+1}{n+1} & \text{if } n \text{ is even.} \end{cases}$$

3.3: Book question 5.2.20: If $A = \{1, 2, 3, 4, 5\}$ and B is a set such that there are 6720 injective functions $A \to B$, then what is $|B|$?

3.4: Book question 5.3.10: We have 7 differently coloured balls and 4 numbered boxes.
(a) How many ways are there of placing the balls in the boxes such that no box is left empty?
(b) One of the balls is blue. How many ways are there of placing the balls such that the blue ball is in box number 2 and no container is left empty?

(c) Now suppose that we erase the numbering of the boxes so that we can no longer distinguish between them. How many ways are there of placing the balls, allowing the possibility that some boxes can be left empty?

Solution 3.1: Part (a). This is the number of natural number solutions to $x_1 + ... + x_6 = 39$, namely, $\binom{44}{5}$. Part (b). Put $y_i = x_i + 3$. The answer is the number of natural number solutions to $y_1 + ... + y_5 < 55$, which is the number of natural number solutions to $y_1 + ... + y_6 = 54$, namely, $\binom{59}{5}$.

Solution 3.2: A binary string with length n and exactly r occurrences of the substring 01 begins with x_1 ones, then x_2 zeroes, then 01, then x_3 ones, x_4 zeroes, and so on, finally 01, then x_{2r+1} ones, then x_{2r+2} zeroes, where x_1, ..., x_{2r+2} is a sequence of natural numbers satisfying $x_1 + ... + x_{2r+2} = n - 2r$. In other words, $b_{n,r}$ is the number of ways of placing $n - 2r$ balls in $2r + 2$ boxes. By a standard formula,

$$b_{n,r} = \binom{(n-2r) + (2r+2) - 1}{2r+1} = \binom{n+1}{2r+1} .$$

In particular, $b(n, 2) = \binom{n+1}{5}$ for $n \geq 4$. Also, $b(n, 3) = \binom{n+1}{7}$ for $n \geq 6$. The number of binary strings of length n is

$$2^n = \sum_{0 \leq r \leq n/2} b_{n,r} = \sum_{0 \leq r \leq n/2} \binom{n+1}{2r+1}$$

where the notation indicates that r runs over the integers in the range $0 \leq r \leq n/2$.

Solution 3.3: Write $|B| = n$. We have $n(n-1)(n-2)(n-3)(n-4) = 6270$. By direct calculation, $n = 8$.

Solution 3.4: For part (a), $4!S(7, 4) = 24.350 = 8400$. For part (b), $8400/4 = 2100$.

Quiz 5: *Friday, 14 November.* How many integer solutions are there to $x_1+x_2+x_3+x_4 = 15$ with $x_i \geq i$?

Solution 5: Putting $y_i = x_i - i$, we are to count the natural number solutions to $y_1 + y_2 + y_3 + y_4 = 5$. The answer is the number of ways of placing 5 plain balls into 4 coloured boxes, which is $\binom{5+4-1}{4-1} = 8.7.6/3.2 = 56.$ □

Quiz 6: *Tuesday, 2 December.* Let n be an integer with $n \geq 2$. Let E be the set of binary strings with length n that have even weight. Let F be the set of binary strings of length n that have odd weight.

(a) Specify mutually inverse bijections $f : E \to F$ and $g : F \to E$.

(b) Hence show that $\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + ... = 2^{n-1}$.

Solution 6: Part (a). Let f and g be the functions that change only the last digit of a binary string. Plainly, f and g are mutually inverse bijections. Part (b). Since the number of elements of $\mathbb{Z}_2^n$ with weight r is $\binom{n}{r}$, the sum of binomial coefficients is $|F| = |\mathbb{Z}_2^n|/2 = 2^{n-1}$.

Homework 4 due Friday 19th December

4.1: Question 3 of Midterm 2 on page 8 of the file arch110spr09.pdf on my homepage.

Quiz 7: *Tuesday, 9th December.* Consider the coding scheme with generating matrix

$$G = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} .$$

Encode the words 0000, 1111, 1010.

Solution 7: The encodings are 0000000, 1111111, 0101010, respectively.

Presentations

Yunus Can Aybaş, Economic Applications of Discrete Mathematics: Game Theory.

Osman Musa Aydın, Convolution Coding and Viterbi Decoding.

Erdem Bıyık, Dijkstra's Shortest Path Algorithm.

Alperen Elihoş, Recurrence Relations.

Muhammed Erşat Emek, Magic Squares.

Mehmet Erhan Erköseoğlu, Karnaugh Maps.

Ayşe Özge Göktürk, Map Colouring.

Necati Çağatay Gürsoy, Path-Finding Algorithms.

Burak Kakillioğlu, Boolean Algebra.

İdil Kanpolat, Contrast Adjustment of Images by Histogram Smoothing.

Berksu Kısmat, Prisoner's Dilemma and Nash Equilibrium.

Çağla Kıvılcım Koru, The Ford-Fulkerson Algorithm.

Elvan Kuzucu, The Papoulis-Gerchberg Algorithm.

Ahmed Furkan Özkalay, Hamilton Paths.

Meltem Duygu Şafak, The Bellman-Ford Algorithm for Single-Source Shortest Paths.

Mert Tozoğlu, Reed Solumon Codes,

Taner Ulukaya, Gambler's Ruin as a Markov Chain Process.

Melek Gizem Yazıcı, Some Mathematical Puzzles.

Gülşah Yıldız, Hidden Markov Models and Viterbi Decoding.

MATH 110: DISCRETE MATHEMATICS. Midterm

Time allowed: 110 minutes. Please put your name on EVERY sheet of your manuscript. The use of telephones and other electronic devices is prohibited. The use of very faint pencils is prohibited too. You may take the question-sheet home.

Do not forget to justify your answers in terms which could be understood by people who know the background theory but are unable to do the questions themselves.

LJB, 31 October 2014, Bilkent University.

1: 20% Give formulas for the terms of the sequence x_0, x_1, x_2, ... in the following cases.

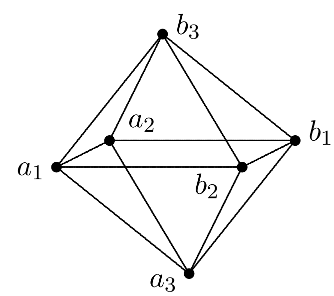
(a) $4x_{n+2} - 4x_{n+1} + x_n = 0$ with $x_0 = x_1 = 1$.

(b) $x_{n+2} + 5x_{n+1} + 6x_n = 0$ with $x_0 = 3$ and $x_1 = -8$.

(c) $x_{n+2} + (2^n + 1)x_{n+1} + (3^n + 1)x_n = 0$ with $x_0 = x_1 = 0$.

2: 20% Let a_0, a_1, a_2, ... be an infinite sequence such that $a_0 = 0$ and $a_n - a_{n-1} = n(n+1)$ for all positive integers n. Show that $a_n = n(n+1)(n+2)/3$.

3: 20% For any positive integer m, there is a graph called the m-**hyperoctahedron** with $2m$ vertices a_1, a_2, ..., a_m, b_1, b_2, ..., b_m. For each integer i in the range $1 \leq i \leq m$, the vertex a_i is adjacent to all the other vertices except b_i. For each i, the vertex b_i is adjacent to all the other vertices except a_i. The case $m = 3$ is as depicted.



(a) Find an Euler circuit for the 3-hyperoctahedron. (Specify the circuit by listing the vertices it visits in order.)

(b) Find an Euler circuit for the 4-hyperoctahedron.

(c) For which values of m does the m-hyperoctahedon have an Euler circuit?

4: 20% For which values of m is the m-hyperoctahedron a planar graph?

5: 20% **(a)** Let T be a tree such that T has exactly 6 vertices and exactly 3 of the vertices have degree 1. Using the Handshaking Lemma, find the number of vertices with degree 2 and the number of vertices with degree 3.

(b) Up to isomorphism, how many trees are there such that there are exactly 6 vertices and exactly 3 of them have degree 1?

Midterm Solutions. version: 6 November 2014

There is no such thing as a "model solution". Often, in mathematics, there are many good ways of justifying a correct conclusion.

1: Part (a). The equation $t^2 - t + 1/4$ has unique solution $t = 1/2$. So $x_n = (C + nD)/2^n$ for some C and D. We have $1 = x_0 = C$ and $1 = x_1 = (C + D)/2$. Hence $D = 1$ and $x_n = (1 + n)/2^n$.

Part (b). The equation $t^2 + 5t + 6 = 0$ has solutions $t = -2$ and $t = -3$. So $x_n = A(-2)^n + B(-3)^n$ for some A and B. Since $3 = x_0 = A + B$ and $-8 = x_1 = -2A - 3B$, we have $A = 1$ and $B = 2$. Therefore $x_n = (-2)^n + 2(-3)^n = (-1)^n(2^n + 2.3^n)$.

Part (c). Obviously, $x_n = 0$ for all n.

Comment: The answer to (c) might not be obvious at first sight but, in my view, when the answer has been stated, it becomes obvious. Some teachers might feel that, at an introductory level, an argument by induction is needed. It may be a matter of opinion. Certainly, though, at a graduate student level, the answer could just be stated and no explanation would be needed. (Note how informative the word "obviously" is in this context. It helpfully informs the reader that we have thought about whether to supply a justification and, after deliberation, we have decided that no explanation is needed.)

2: Let $b_n = n(n + 1)(n + 2)/3$. We shall show, by induction, that $a_n = b_n$. First note that $a_0 = 0 = b_0$. Now suppose that $n \geq 1$ and that $a_{n-1} = b_{n-1}$. We have

$$b_n - b_{n-1} = (n + 2 - (n - 1))n(n + 1)/3 = n(n + 1) = a_n - a_{n-1} .$$

Cancelling, we deduce that $a_n = b_n$.

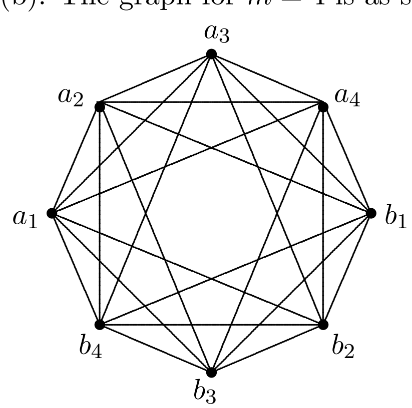
Comment: In this argument, the inductive assumption is that $n \geq 1$ and $a_{n-1} = b_{n-1}$. It is necessary to tell the reader that the conditions $n \geq 1$ and $a_{n-1} = b_{n-1}$ are being *assumed*. Otherwise, it does not make sense to *deduce* that $a_n = b_n$.

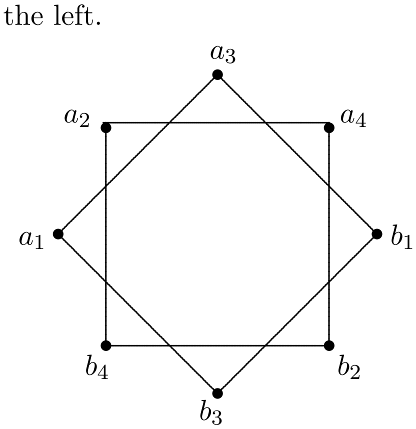
There are other ways of doing it. We could *assume* that $a_n = b_n$ and then, after showing that $b_{n+1} - b_n = a_{n+1} - a_n$, we could *deduce* that $a_{n+1} = b_{n+1}$.

As always, some people inexplicably "deduced" something without telling the reader what *assumptions* were being made. Some people mysteriously wrote $a_n - a_{n-1} = n(n + 1)$ and $a_n = n(n + 1)(n + 2)/3$ and $a_{n-1} = (n - 1)n(n + 1)/3$ near the top of the page and then, bizarrely, deduced something trivial such as $1 = 1$.

3: Part (a). One Euler circuit for $m = 3$ is $a_1, b_2, b_3, a_2, a_3, b_2, b_1, a_3, a_1, b_3, b_1, a_2, a_1$.

Part(b). The graph for $m = 4$ is as shown on the left.





Deleting the edges of the circuit

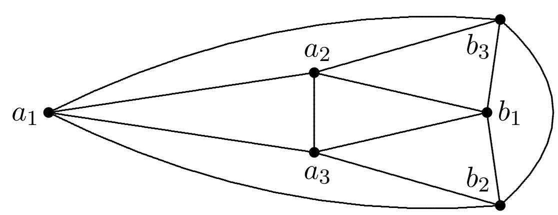
$$a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, a_1, a_4, b_3, a_2, b_1, b_4, a_3, b_2, a_1$$

we obtain the graph shown on the right, whose two components have Euler circuits a_1, a_3, b_1, b_3, a_1 and a_2, a_3, b_2, b_3, a_2. Splicing the two shorter circuits into the larger circuit, we obtain the Euler circuit

$$a_1, a_3, b_1, b_3, a_1, a_2, a_3, b_2, b_3, a_2, a_3, a_4, b_1, b_2, b_3, b_4, a_1, a_4, b_3, a_2, b_1, b_4, a_3, b_2, a_1 \ .$$

Part (c). For the m-hyperoctahedron, the degree of each vertex is $2m - 2$, which is always even. The graph is connected if and only if $m \geq 2$.

4: Let H_m be the m-hyperoctahedron. We shall show that H_m is planar if and only if $m \leq 3$. Since any subgraph of a planar graph is planar, and since H_n is a subgraph of H_m whenever $n \leq m$, it suffices to show that H_3 is planar and H_4 is non-planar. The next diagram shows that H_3 is planar.



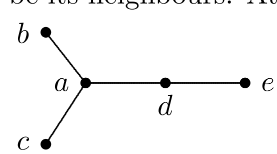
Suppose, for a contradiction, that H_4 is planar. Then $e \leq 3n - 6$ where e is the number of edges of H_4 and n is the number of vertices. But $e = 8.6/2 = 24$ because $n = 8$ and all the vertices have degree 6. We deduce that $24 \leq 3.8 - 6 = 18$, a contradiction, as required.

Comment: The condition $e \leq 3n - 6$ holds if and only if $m \leq 3$. This shows that the graph is non-planar when $m \geq 4$. It does not show that the graph is planar when $m \leq 3$. For $e \geq 2$, the condition $e \leq 3n - 6$ is *necessary* for the graph to be planar, but it is not *sufficient*.

5: Part (a) Number the vertices of T from 1 to 6. Let d_i be the degree of vertex i. We can choose the numbering such that $d_1 \leq d_2 \leq ... \leq d_6$. The given condition on T is that $d_1 = d_2 = d_3 = 1$ and $d_4 \geq 2$. The number of edges of T is 5. By the Handshaking Lemma, $d_1 + d_2 + ... + d_6 = 5.2 = 10$. So $d_4 + d_5 + d_6 = 10 - 3 = 7$. Since $2 \leq d_4 \leq d_5 \leq d_6$, we must have $d_4 = d_5 = 2$ and $d + 6 = 3$. In conclusion, there are exactly 2 vertices with degree 2 and ther is exactly 1 vertex with degree 3.

Part (b). There are exactly 2 isomorphism classes of trees satisfying the specified conditions. They are as depicted.

To see this, let a be the unique vertex with degree 3 and let b, c, d be its neighbours. At least one of b, c, d must be adjacent to another vertex e. Without loss of generality, d is adjacent to e. The last vertex f must be adjacent to b or c or e. If f is adjacent to e, we obtain the first depicted tree, otherwise we obtain the second.



MATH 110: DISCRETE MATHEMATICS. Final, Fall 14/15

Time allowed: two hours. Please put your name on EVERY sheet of your manuscript. The use of telephones and other electronic devices is prohibited. The use of very faint pencils is prohibited too. You may take the question-sheet home.

LJB, 3 January 2015, Bilkent University.

1: 40% Consider the encoding function with generating matrix $G = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$.

(a) Write down the codewords for each of the 8 message words.
(b) Find the parity-check matrix H.
(c) Explain why there are exactly 3 binary strings with weight 2 that can appear as a coset leader. What are those 3 binary strings?
(d) Consider the decoding scheme where one of the coset leaders is the binary strong 100001. Find all the other coset leaders and write down their corresponding syndromes.
(e) For the above decoding scheme, using the method of decoding via syndromes (without writing out the decoding table), decode the received words 111111, 111000, 000111.
(f) For this code, how many single-digit errors of transmission can be detected? How many single-digit errors of transmission can be corrected?
(g) What is the rate of this code?

2: 35% **(a)** State a recurrence relation for the Stirling numbers $S(m, n)$ where m and n are positive integers.
(b) Write out a table giving the value of $S(m, n)$ for all m and n with $m \leq 8$ and $n \leq 4$.
(c) Let X be a set with $|X| = 8$. What is the number of equivalence relations $\equiv$ on X such that $\equiv$ has exactly 5 equivalence classes?
(d) What is the number of equivalence relations $\equiv$ on X such that all the equivalence classes of $\equiv$ have the same size as each other?
(e) For a function $f : X \to X$, the **image** of f, denoted $f(X)$, is the set of values of f. That is to say, $f(X) = \{f(x) : x \in X\}$. How many functions $f : X \to X$ are there such that $|f(X)| = 5$?
(f) How many functions $f : X \to X$ are there such that, for all $x \in X$, there are exactly 2 elements $y \in X$ satisfying $f(y) = f(x)$?

3: 25% Let $\leq$ be a partial ordering relation on a set S. We call $\leq$ a **local ordering** provided, for all $x, y, z \in S$ such that $x \leq y \geq z$ or $x \geq y \leq z$, we have $x \leq z$ or $z \geq x$.
(a) Let $\sim$ be the relation on S such that, given $x, y \in S$, then $x \sim y$ if and only if $x \leq y$ or $y \leq x$. Show that $\leq$ is a local ordering if and only if $\sim$ is an equivalence relation.
(b) How many isomorphism classes of local orderings are there on the set $\{1, 2, 3, 4\}$?
(c) How many local orderings are there on $\{1, 2, 3, 4\}$?

Example solutions to Final, MATH 110, Fall 2013

Reminder: Of course, there are no "model solutions" in mathematics. Mathematics thrives on diversity of methods and styles.

1: Part (a). The codewords for 000, 001, 010, 011, 100, 101, 110, 111 are, respectively,

000000, 001101, 010011, 011110, 100110, 101011, 110101, 111000 .

Part (b), $H = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$.

Part (c). The received words r of weight 0 or 1 have mutually distinct syndromes σ, as indicated in the table.

r	000000	000001	000010	000100	001000	010000	100000
σ	000	001	010	100	101	011	110

The last coset leader must have weight at least 2 and syndrome 111. There are exactly 3 ways of obtaining 111 as the sum of 2 columns of H. The 3 codewords associate with those 3 sums are 100001 and 010100 and 001010.

Part (d). Letting r run over the other 7 coset leaders, then the corresponding syndromes σ are as shown in the table in part (c).

Part (e). For $r \in \{111111, 111000, 000111\}$, the corresponding decodings w are as shown, where σ is the syndrome, s the coset leader, $c = r + s$ the codeword associated with w.

r	σ	s	c	w
111111	111	100001	0111110	011
111000	000	000000	111000	111
000111	111	100001	100110	100

Part (f). The minimum weight of a nonzero codeword is 3. So 2 errors can be detected, 1 error corrected.

Part (g). The rate is $3/6 = 1/2$.

2: Part (a), $S(m+1, n) = S(m, n-1) + nS(m, n)$ for $2 \leq n \leq n$. (The initial conditions are $S(m, 1) = S(m, m) = 1$.)

Part (b). The table is as shown.

$S(m,n)$		1	2	3	4	n
m	1	1				
	2	1	1			
	3	1	3	1		
	4	1	7	6	1	
	5	1	15	25	10	
	6	1	31	90	65	
	7	1	63	301	350	
	8	1	127	966	1701	

Part (c). Continuing the table, $S(6, 5)$ and $S(7, 5)$ and $S(8, 5)$ are 15 and 140 and 1050, respectively. The number of equivalence relations as specified is $S(8, 5) = 1050$.

Part (d). When all the equivalence classes have size 1, the number of equivalence relations is 1. To choose an equivalence relation where every class has size 2, select an element x_0 of X, choose a partner for x_0, select an unpartnered element x_1, choose a partner for x_1, select an unpartnered element x_2, choose a partner for x_2, then partner the last two elements of X together. In that case, the number of equivalence relations is $7.5.3 = 105$. When the size is 4, the number of equivalence relations is $\binom{8}{4}/2 = 8.7.6.5/4.3.2.2 = 35$. When the size is 8, there is only 1 equivalence relation. The answer is $1 + 105 + 35 + 1 = 142$.

Part (e). To choose f, there are $\binom{8}{5} = 8.7.6/3.2 = 56$ choices for $f(X)$, then $5!S(8,5)$ choices for f. So the total number of choices for f is $56 \,.\, 5! \,.\, S(8,5) = 56.120.1050 = 7056000$.

Part (f). Let $\equiv_f$ be the equivalence relation on X such that, given $x, y \in X$, then $x \equiv y$ provided $f(x) = f(y)$. As we saw in part (d), there are 105 choices for $\equiv_f$. For each $\equiv_f$, there are $8.7.6.5 = 1680$ choices for f. So the total number of choices for f is $105.1680 = 176400$.

3: Part (a). The relation $\sim$ is reflexive and symmetric. The relation $\leq$ is a local ordering if and only if $\sim$ is transitive.

Part (b). A partial ordering is a local ordering if and only if the Hasse diagram is a disjoint union of graphs having the form •— • —...—•. Each isomorphism class corresponds to a sequence of positive integers $(x_1, ..., x_r)$ where $x_1 > ... > x_r$ and $x_1 + ... + x_r = 4$. The correspondence is such that the isomorphism class corresponds to the sequence such that the connected components of the Hasse diagram have sizes x_1, ..., x_r. Noting that

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1 \,,$$

we see that the number of isomorphism classes is 5.

Part (c). When the Hasse diagram has shape 4, we mean, when the Hasse diagram is •— • — • —•, the number of local orderings is $4! = 24$. When the shape is $3 + 1$, we mean, when the Hasse diagram is •—•—• •, the number of local orderings is, again, 24. When the shape is $2 + 2$, there are 12 possibilities for $\leq$. When the shape is $2 + 1 + 1$, there are, again, 12 possibilities. When the shape is $1 + 1 + 1 + 1$, there is exactly 1 possibility. So the number of local orderings is $24 + 24 + 12 + 12 + 1 = 73$.