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MATH 110, Discrete Mathematics,

Bilkent University, Fall 2014, Laurence Barker

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MATH 110, Discrete Mathematics, Fall 2013

Handout 1: Course specification

Laurence Barker, Mathematics Department, Bilkent University,
version: 28 November 2013.

Course Aims: To introduce some concepts and techniques of discrete and combinatorial mathematics from an applicable point of view.

Course Description: Discrete mathematics is an umbrella name for all areas of applicable mathematics where there is not much structure to work with. Although it is very diverse, certain kinds of technique tend to crop up frequently. We shall be studying three areas, superficially quite separate, but similar in style and technique: the first third will focus largely on graph theory; the middle third on enumerative study of relations; the last third on coding theory.

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Text: R. P. Grimaldi, "Discrete and Combinatorial Mathematics", 5th Ed. (Pearson, 2004). Some other sources may be supplied for some small components of the syllabus material.

Classes: Tuesdays 15:40 - 16:30 FC-Z23C, Fridays, 13:40 - 15:30 FC-Z23C.

Office Hours: Tuesdays, 16:40 - 17:30, SAZ 129.

Please note, Office Hours is not just for the strong students. I will have no sympathy for drowning students who do not come to me for help. Students who are having serious difficulties must come to Office Hours to discuss the mathematics. In fact, I need those students to come, so as to ensure that the classroom material does not lose touch with parts of the audience.

Syllabus:

Week number: Monday date: Subtopics. Section numbers

1: 16 Sept: Illustrations of problems in discrete mathematics. Mathematical induction, 4.1.

2: 23 Sept: Recursive definitions and induction, 4.2. Second order recurrence relations as an application of induction, 10.2

3: 30 Sept: Graphs. trees. Criteria for existence of Euler paths or Euler circuits, proved by mathematical induction, 11.1, 11.2, 11.3.

4: 7 Oct: Euler's characteristic formula for planar graphs, proved by mathematical induction. Degrees of vertices of planar graphs.

5: 14 Oct: Holiday.

6: 21 Oct: The non-planarity of the graphs K_5 and $K_{3,3}$. Non-examinable topic: planar graph colouring.

7: 28 Oct: (Tuesday 29th is Republic Day; no classes.) Review for Midterm 1.

8: 4 Nov: Midterm 1 on 5th November (tested up to week 6 material, not including planar graph colouring). Sets and correspondences. Injective, surjective and bijective functions, 5.1, 5.2, 5.3, 5.6.

9: 11 Nov: Midterm 1 postmortem. Relations, incidence matrices, enumeration of relations, 7.1, 7.2.

10: 18 Nov: Partial orderings, Hasse diagrams, 7.3. Equivalence relations, 7.4.

11: 25 Nov: Stirling numbers of the second kind, 5.3. (Isomorphism of graphs, of partially ordered sets and of other relations, 11.2, non-examinable).

12: 2 Dec: Midterm 2 review. Midterm 2 on 6th December.

13: 9 Dec: Midterm 2 postmortem. Coding theory, Hamming metric, parity-check and generator matrics, 16.5, 16.6, 16.7.

14: 16 Dec: Encoding and decoding linear codes using coset leaders, 16.8.

15: 23 Dec: (Classes on 24, 27, 31 Dec.) [If time permits: Abelian groups, cyclic groups, homomorphisms, 16.1, 16.2. (Lagrange's Theorem, 6.3, non-examinable.)] Review for Final.

Assessment:

- Quizzes, Homework and Participation 15%.
- Midterm I, 25%, Tuesday 5th November.
- Midterm II, 25%, Friday 6th December.
- Final, 35%.

75% attendance is compulsory.

Class Announcements: All students, including any absentees from a class, will be deemed responsible for awareness of class announcements.

MATH 110, Discrete Mathematics, Fall 2013

Handout 2: Homeworks, Quizzes and Practise Exam

Laurence Barker, Mathematics Department, Bilkent University,
version: 30 December 2013.

Office Hours: Tuesdays, 16:40 - 17:30, SAZ 129.

Office Hours would be a good time to ask me for help with the homeworks.

Homework 1

Page 245, Supplementary Exercises for Chapter 4. Questions 4, 6, 12. You may do these questions by mathematical induction or by any other valid method. They are as follows.

4: Show that, given any positive integer n, then 5 divides $n^5 - n$ and 6 divides $n^3 + 5n$.

6: For positive integers n, define

$$s_n = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + ... + \frac{(n-1)}{n!} + \frac{n}{(n+1)!} \ .$$

Calculate s_1, s_2, s_3, s_4, s_5, s_6. Guess a general formula for s_n. Prove your formula.

12: For all positive integers n, show that 57 divides $7^{n+2} + 8^{2n+1}$.

Quiz 1. (24 Sept.)

Prove that, if 32 dominoes pieces are of such a size as to exactly cover the 64 squares of a chess-board, and if 2 opposite corner squares of the chess-board are removed, then the remaining 62 squares of the chess-board cannot be covered by 31 dominoes pieces. (The reason was discussed orally in class. The purpose of the exercise is to write down a proof — a very clear deductive explanation — which can be understood by a reader who was not present during the oral discussion.)

Answer: (As will all my answers here, I can only give illustrations. Always, many other ways of communicating the argument are possible): Each dominoes piece covers one black and one white square. If the mutilated board can be covered, then it has the same number of blacks and whites. The original board does have the same number of blacks and whites, but the two removed squares have the same colour, so the mutilated board does not have the same number of blacks and whites.

Quiz 2

Show that $1^2 + 3^2 + 5^2 + ... + (2n-1)^2 = n(2n-1)(2n+1)/3$ for all positive integers n.

Answer: We shall show, by induction on n, that $A_n = B_n$, where A_n is the left-hand expression and B_n is the right-hand expression. We have $A_1 = 1 = B_1$, so the assertion holds in the case $n = 1$. Now suppose that $n \geq 2$ and that $A_{n-1} = B_{n-1}$. We have

$$B_n - B_{n-1} = n(2n-1)(2n+1)/3 = (n-1)(2n-3)(2n-1)/3$$

$$= \big(n(2n+1) - (n-1)(2n-3)\big)(2n-1)/3 = \big(2n^2 + n - (2n^2 - 5n + 3)\big)(2n-1)/3$$
$$= (6n-3)(2n-1)/3 = (2n-1)^2 = A_n - A_{n-1} \, .$$

Hence, using the assumption $A_{n-1} = B_{n-1}$, we obtain $A_n = B_n$, as required. □

♣ ♡ ♢ ♠

In class, we discussed a more sophisticated application of mathematical induction, as follows. (We also discussed how to do some preliminary work on rough paper to figure out the crucial definitions and calculations that appear in the argument.)

Theorem: *Let a, b, c be real numbers with $a \neq 0$. For each natural number n, let x_n be a real number and suppose that $ax_{n+2} + bx_{n+1} + cx_n = 0$ for all n. Let α and β be complex numbers such that $at^2 + bt + c = a(t-\alpha)(t-\beta)$.*
(1) *If $\alpha \neq \beta$ then there exist real numbers A and B such that $x_n = A\alpha^n + B\beta^n$ for all n.*
(2) *If $\alpha = \beta \neq 0$ then there exist real numbers A and C such that $x_n = (A + nC)\alpha^n$ for all n.*

Proof of part (1): Let

$$A = \frac{\beta x_0 - x_1}{\beta - \alpha} \, , \qquad B = \frac{-\alpha x_0 + x_1}{\beta - \alpha} \, .$$

We shall show, by induction on n, that $x_n = A\alpha^n + B\beta^n$ for all natural numbers n. This is clear for $n = 0$ and $n = 1$. Now suppose that the formula holds for x_n and x_{n+1}. We shall deduce the formula for x_{n+2}. Now

$$ax_{n+2} + b(A\alpha^{n+1} + B\beta^{n+1} + c(A\alpha^n + B\beta^n) = ax_{n+2} + bx_{n+1} + cx_n = 0$$
$$= A\alpha^n(a\alpha^2 + b\alpha + c) + B\beta^n(a\beta^2 + b\beta + c)$$
$$= a(A\alpha^{n+2} + B\beta^{n+2}) + b(A\alpha^{n+1} + B\beta^{n+1}) + c(A\alpha^n + B\beta^n) \, .$$

Cancelling summands and then dividing by a, we obtain $x_{n+2} = A\alpha^{n+2} + B\beta^{n+2}$. The proof of part (1) is complete.

The proof of part (2) is part of Homework 2, below.

Homework 2

1: Give a proof of part (2) of the above theorem. (Warning: this is a hard question. A cunning observation is needed in the manipulations. My expectation is that only a few people will be able to do it.)

2: Let n be a positive integer and let C_n be the graph whose vertices are the binary strings with length n, two vertices being adjacent provided they differ in a single digit. (This graph was discussed in class.) For which values of n does C_n have an Euler path?

Practice Midterm 1

Note 1: Your script will be marked if handed in by Friday 11 October. The course credit available for this exercise is zero.

Note 2: This is just to give an idea of the exam. It is to be handed in at the beginning of class in Week 4. Question 4 is on material not covered before the deadline. The exam will take place in Week 7.

1: 15% Show that $\sum_{i=1}^{n} n(n+1) = n(n+1)(n+2)/3$ for all $n \geq 1$.

2: 30% (a) Let $y_n = n^3$ for all $n \in \mathbb{N}$. Show that $y_{n+2} - 4y_{n+1} + 4y_n = n^3 - 6n^2 + 4$.

(b) Let z_0, z_1, z_2, ... be a sequence such that $z_{n+2} - 4z_{n+1} + 4z_n = 0$ for all $n \in \mathbb{N}$. What can you say about expressing z_n in terms of n? (No proof required. Just apply a standard result.)

(c) Suppose that $x_{n+2} - 4x_{n+1} + 4x_n = n^3 - 6n^2 + 4$ for all $n \in \mathbb{N}$. Also suppose that $x_0 = 2$ and $x_1 = 7$. Using parts (a) and (b), give a formula for x_n. (Hint: consider $x_n - y_n$.)

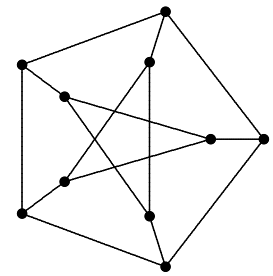
3: 30% (a) State (without proof) Euler's criterion for a connected graph to have (1) an Euler trail, (2) an Euler circuit.

(b) Let G be a connected graph with at least one edge and such that the vertices of G all have the same degree and G has an Euler trail but no Euler circuit. How many vertices does G have?

(c) Let G be a graph with 24 vertices and 96 edges such that all the vertices have the same degree. Does this graph have an Euler circuit?

4: 25% *Warning: this is on material not covered before the deadline for scripts to be marked.* Starting from Euler's characteristic equation $n - e + f = 2$, prove that every planar graph has a vertex with degree 5 or less. Give an example of a planar graph where every vertex has degree 5. (Hint: consider the Platonic solids.)

Homework 3



1: Show that the Peterson graph, depicted, is non-planar.

2: Do Question 4 in the Practice Midterm.

Quiz 3

How many graphs are there with vertex set $\{1, 2, 3\}$?

Homework 4 (set 15th, due 29th November)

1: On a set with size 6:
(a) How many symmetric relations are there?
(b) How many graphs are there?

2: On a set with size $n \geq 2$:
(a) How many equivalence relations are there with exactly 2 equivalence classes?
(b) How many equivalence relations are there with exactly $n - 1$ equivalence classes?
(c) How many with exactly n equivalence classes?

3: Let $|S| = n$ and $|T| = n + 1$ with $n \geq 1$.
(a) How many injections $S \to T$ are there?
(a) How many surjections $T \to S$ are there?

4: Let $m \leq n$. Let $s(m, n)$ be the number of relations on $\{1, ..., n\}$ with exactly m equivalence classes. Let $t(m, n)$ be the number of surjections $\{1, ..., n\} \to \{1, ..., m\}$. Find and prove a formula relating $s(m, n)$ and $t(m, n)$.

Quiz 4

Prove that $\binom{n}{m} + \binom{n}{m+1} = \binom{n+1}{m+1}$ for all integers m and n with $0 \leq m < n$.

Quiz 5

Let S be a set. Let E be the set of equivalence relations on S. Let $\cong$ be the relation on E whereby, given $\equiv_1$ and $\equiv_2$ in E, then $\equiv_1 \cong \equiv_2$ provided there exists a bijection $f : S \to S$ such that, for all $x, y \in S$, we have $x \equiv_1 y$ if and only if $f(x) \equiv_2 f(y)$.

Homework 5 (set 17th, due 24th November)

This homework is taken from a Fall 2012 exam paper.

1: Consider the coding scheme with encoding function $\mathbb{Z}_2^3 \to \mathbb{Z}_2^6$ given by generating matrix

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} .$$

(a) Write down all 8 message words and their corresponding codewords.
(b) How many errors of transmission can be detected? How many errors of transmission can be corrected? Briefly justify your answers.

2: Consider the coding scheme in Question 1.
(a) Write down the parity-check matrix H.
(b) Find the syndromes for each of the words

$$000001\,, \quad 000010\,, \quad 000100\,, \quad 001000\,, \quad 010000\,, \quad 100000\,, \quad 000110\,, \quad 000101\,.$$

(c) Using part (b), without calculating entries of the decoding table, explain why there must exist a decoding table with coset leaders

$$000000\,, \quad 000001\,, \quad 000010\,, \quad 000100\,, \quad 010000\,, \quad 100000\,, \quad 000110\,, \quad 000101\,.$$

(Warning: there are no marks for doing this by calculating entries of the decoding table.)
(d) Using the syndromes in part (b), decode the message $000111, 111000, 111101$. (Warning: there are no marks, here, for doing this using the decoding table.)

3: Consider, once again, the coding scheme in Questions 1 and 2. Now write out the decoding table with the coset leaders specified in part (c) of Question 2. Use it to check your decoding of the message in part (d) of Question 2.

4: Let n be a positive integer. Let S be the set of linear codes in $\mathbb{Z}_2^n$. We define a relation $\equiv$ on S such that, given two linear codes C and C' in S, then $C \equiv C'$ if and only if there exists a bijection $f : C \to C'$ with the property that $d(c_1, c_2) = d(f(c_1), f(c_2))$ for all $c_1, c_2 \in C$. Show that $\equiv$ is an equivalence relation.

Quiz 6

1: Let $f : B \to A$ and $g : C \to B$ be functions.
(a) Suppose that $f \circ g$ is injective. Must f be injective? Must g be injective? (Give a proof or counter-example.)
(b) Suppose that $f \circ g$ is surjective. Must f be surjective? Must g be surjective?

MATH 110: DISCRETE MATHEMATICS. Midterm 1

Time allowed: 110 minutes. Please put your name on EVERY sheet of your manuscript. The use of telephones and other electronic devices is prohibited. The use of very faint pencils is prohibited too. You may take the question-sheet home.

LJB, 5 November 2013, Bilkent University.

1: 30% Let x_0, x_1, x_2, ... be a sequence of real numbers such that $x_{n+2} + x_{n+1} + x_n = 0$ for all natural numbers n. Suppose that $x_0 = 2$ and $x_1 = -1$.

(a) Using mathematical induction, show that $x_n = \begin{cases} 2 & \text{if } n \text{ is divisible by 3,} \\ -1 & \text{if } n \text{ is not divisible by 3.} \end{cases}$

(b) By solving the equation $t^2+t+1 = 0$ and using a standard formula for recurrence relations, give another proof of the conclusion in (a). (Hint: after finding the solutions α and β to the quadratic equation, calculate α^3 and β^3.)

2: 20% Let G be a connected graph (with finitely many vertices). Suppose that each edge of G is coloured either red or blue. Let G_R be the graph with the same vertices as G and with the red edges. Let G_B be defined similarly for the blue edges. (Thus, all three graphs G and G_R and G_B have the same vertices, and every edge of G is either an edge of G_R or else an edge of G_B.) Taking care to be clear about which well-known theorems you are using, show that:

(a) If G_R and G_B both have Euler circuits, then G has an Euler circuit.

(b) If G_R and G both have Euler circuits and G_B is connected, then G_B has an Euler circuit.

3: 25% **(a)** State and prove a formula relating the number of vertices n, the number of edges e and the number of faces f of a connected planar graph.

(b) Let G and G_R and G_B be as in the Question 2. Now suppose also that G is planar and that G_R and G_B are connected. Let n be the number of vertices. Let f and f_R and f_B be the number of faces of G and G_B and G_R, respectively. Show that $f = n + f_R + f_B - 2$.

4: 25% Let G be a planar graph such that every vertex has degree 3. Suppose that G can be drawn such that every face has exactly 5 edges. Show that G has exactly 20 vertices.

Example solutions to Midterm 1, Fall 2013

There are no “model solutions” to exam questions in mathematics. Often, there is a variety of good arguments, each of which can be succinctly expressed in many different ways.

Solution 1: Part (a). The required formula for x_n plainly holds when $n = 0$ or $n = 1$. Assuming, inductively, that the formula holds for x_n and x_{n+1}, then it plainly holds for x_{n+2}.

Part (b). There exist real numbers A and B such that $x_n = A\alpha^n + B\beta^n$. Solving the quadratic equation, $\alpha = (-1 + i\sqrt{3})/2$ and $\beta = (-1 - i\sqrt{3})/2$. Note that $\alpha^2 = \beta$ and $\beta^2 = \alpha$ and $\alpha^3 = \beta^3 = 1$. By considering the cases $n = 0$ and $n = 1$, we see that $A = B = 1$. The required formula for x_n follows.

Comment: Some candidates did not make it clear what their inductive assumption was. Above, assuming that the formula holds for x_n and x_{n+1}, we deduced that it holds for x_{n+2}. If one were just to assume the formula for x_n, one would not be able to deduce it for x_{n+1}.

Another comment: But another inductive way of doing part (a) is by showing that $x_{n+3} = x_n$, then noting that the formula follows from the cases $n = 0$ and $n = 1$ and $n = 2$.

Yet another comment: As is usual in the first midterm of this introductory course, some candidates incoherently imitated inductive arguments that were given in class. Remember, mathematical induction is just a convenient technique for explaining things. It is not a ritual incantation.

Comment on part (b): As slicker arithmetic, note that $(t^2 + t + 1)(t - 1) = t^3 - 1$, hence $\alpha = e^{2\pi i/3}$ and $\beta = e^{-2\pi i/3}$.

Solution 2: We apply the theorem asserting that a connected graph has an Euler circuit if and only if every vertex has even degree. For a vertex v, let $d(v)$ and $d_R(v)$ and $d_B(v)$ denote the degrees of v in G and G_B and G_R, respectively. Then $d(v) = d_R(v) + d_B(v)$. Part (a): if each $d_R(v)$ and $d_B(v)$ is even, then $d(v)$ is even. Part (b): if each $d(v)$ and $d_R(v)$ is even, then $d_B(v)$ is even.

Comment: Part (a) can also be done by noting that an Euler circult for G_B followed by an Euler circuit for G_R amounts to an Euler circuit for G.

Solution 3: Part (a). To prove that $n - e + f = 2$, we argue by induction on f. If $f = 1$ then G is a tree and the required conclusion is clear. For $f \geq 2$, we can remove an edge from a circuit, thus obtaining a graph with $n' = n$ vertices, $e' = e - 1$ edges and $f' = f - 1$ faces. Inductively, we may assume that $n' - e' + f' = 2$. Hence $n - e + f = 2$.

Part (b). Let e, e_R, e_B be the number of edges of G, G_R, G_B, respectively. Since all three graphs are planar and connected, $n - e + f = n - e_R + f_R = n - e_B + f_B = 2$. The required equality follows because $e = e_R + e_B$.

Solution 4: We use the formula $n - e + f = 2$. Counting pairs (ϵ, F) where ϵ is an edge of face F, we see that $5f = 2e$. Also, the sum of the degrees of the vertices is $3n = 2e$. Therefore $n(1 - 3/2 + 3/5) = 2$, in other words, $n = 20$.

Comment: As a variant of essentially the same argument, one can use the standard formula $n - 2 = e(c - 2)/c$, where c is the average number of edges per face; in this case, $c = 5$.

MATH 110: DISCRETE MATHEMATICS. Makeup 1

Time allowed: 110 minutes. Please put your name on EVERY sheet of your manuscript. The use of telephones and other electronic devices is prohibited. The use of very faint pencils is prohibited too. You may take the question-sheet home.

LJB, 30 December 2013, Bilkent University.

1: Let a, b, c be complex numbers with $a \neq 0$. Let x_0, x_1, x_2, ... be a sequence of complex numbers such that $ax_{n+2} + bx_{n+1} + cx_n = 0$ for all natural numbers n. Suppose that the quadratic equation $at^2 + bt + c = 0$ has two distinct solutions α and β.

(a) 20% Show that there exist complex numbers A and B such that $x_n = A\alpha^n + B\beta^n$.

(b) 10% Show that there does not exist a sequence of complex numbers x_0, x_1, ... satisfying the equalities $x_1 = x_4 = 1$ and $x_{n+2} - \sqrt{2}x_{n+1} + x_n = 0$.

2: 20% Let T be a tree, let r be a natural number, and suppose that T has at least r vertices with degree greater than 2. Show that T has at least $r + 2$ vertices with degree 1. (If you use any standard results, be clear about which results you are using.)

3: Let m be an integer with $m \geq 3$. Let G_m be the graph with $2m$ vertices v_1^1, v_2^1, ..., v_m^1, v_1^2, v_2^2, ..., v_m^2 such that there is an edge between v_i^a and v_j^b if and only if $i \neq j$.

(a) 10% Explain why G_m has an Euler circuit.

(b) 10% Show that, if any two edges are removed from G_m, then the resulting graph has an Euler path.

(c) 10% Give an example of a graph G such that G has an Euler circuit but, if any two edges are removed from G, then the resulting graph does not have an Euler path.

4: 20% Let m and G_m be as in the previous question. For which values of m is G_m planar? (You may use any well-known results about planar graphs, but you must be clear about which results you are using.)

MATH 110: DISCRETE MATHEMATICS. Midterm 2

Time allowed: 110 minutes. Please put your name on EVERY sheet of your manuscript. The use of telephones and other electronic devices is prohibited. The use of very faint pencils is prohibited too. You may take the question-sheet home.

LJB, 6 December 2013, Bilkent University.

1: 10% Find a positive integer n such that, given any set N with size $|N| = n$, then the number of reflexive relations on N is equal to the number of symmetric relations on N. (You may use any standard results, provided you are clear about which results you are using.)

2: Let m and n be positive integers with $m \leq n$. Let M and N be sets with sizes $|M| = m$ and $|N| = N$.
(a) 10% How many injective functions $M \to N$ are there? (Explain your reasoning.)
(b) 10% Let $f : M \to N$ and $g : N \to M$ be functions such that $g(f(x)) = x$ for all $x \in M$. Show that f is injective and g is surjective.
(c) 10% Let f be an injective function $M \to N$. How many functions $g : N \to M$ are there such that $g(f(x)) = x$ for all $x \in M$?

3: Let n be any positive integer and let N be a set with size $|N| = n$.
(a) 10% Explain why, for any integer m in the range $0 \leq m \leq n$, the number of subsets of size m in N is equal to the binomial coefficient $\binom{n}{m} = \dfrac{n!}{m!(n-m)!}$.
(b) 10% Let $n_1, n_2, ..., n_k$ be natural numbers such that $n = n_1 + n_2 + ... + n_k$. Show that there are exactly $n!/n_1!n_2! \dots n_k!$ ways of choosing subsets N_1, N_2, ..., N_k such that each $|N_i| = n_i$ and $N_1 \cup N_2 \cup ... \cup N_k = N$.
(c) 10% Give a counter-example to the following statement: *There are exactly* $n!/k!\, n_1! \dots n_k!$ *equivalence relations* $\equiv$ *on* N *such that* $\equiv$ *has exactly* k *equivalence classes and the equivalence classes have sizes* n_1, n_2, ..., n_k.

4: Let a be any positive integer, let $n = 3a$ and let N be a set with size $|N| = n$.
(a) 10% How many equivalence relations $\equiv$ on N are there such that all the equivalence classes of $\equiv$ have size a?
(b) 10% Let d be the Hamming metric on the set $\mathbb{Z}_2^n$ (the set of binary strings with length n). How many triples (x, y, z) of elements $x, y, z \in \mathbb{Z}_2^n$ are there such that $d(x, y) = d(y, z) = 2a$?
(c) 10% How many of the triples (x, y, z) satisfy $d(x, y) = d(y, z) = d(x, z) = 2a$?

Example solutions to Midterm 2, Fall 2013

Reminder: Of course, there are no "model solutions" to mathematical questions. Often, a conclusion can be justified in many different ways. Always, an argument can be expressed in many different styles.

Solution 1: On N, the number of reflexive relations is $2^{n(n-1)}$. The number of symmetric relations is $2^{n(n+1)/2}$. Those two numbers are equal when $n = 3$.

Comment: The formulas for the numbers of relations can be obtained by considering incidence matrices. The equality $2^{n(n-1)} = 2^{n(n+1)/2}$ holds if and only if $n \in \{0, 3\}$. So $n = 3$ is the unique positive integer solution. None of these observations is required, though.

Solution 2: Part (a). Enumerate $M = \{x_1, ..., x_n\}$. To choose an injection f, there are n choices for $f(x_1)$, then $n-1$ choices for $f(x_2)$, and so on. Finally, there are $n-m+1$ choices for $f(x_m)$. So there are $n(n-1)...(n-m+1) = n!/(n-m)!$ injections $M \to N$.

Part (b). Given $x_1, x_2 \in M$ such that $f(x_1) = f(x_2)$, then $x_1 = g(f(x_1)) = g(f(x_2) = x_2$. So f is injective. Each element $x \in M$ is the image of $f(x)$ under g, so g is surjective.

Part (c). To choose such g, the values at elements of the image of f are already determined but, for each of the $n-m$ elements y of M that are not in the image, there are m choices for $g(y)$. So the number of choices for g is m^{n-m}.

Comment 1: The phrasing for part (c) requires wordpower. A clumsier style, but easier to compose, involves enumerations, as follows. Enumerate $M = \{x_1, ..., x_m\}$ and $N = \{y_1, ..., y_n\}$ such that $f(x_i) = y_i$. To choose g, we must have $g(y_i) = x_i$ when $1 \le i \le m$, but there are m choices for $g(y_i)$ when $m+1 \le i \le n$. So there are m^{n-m} choices for g.

Comment 2: Parts (b) and (c) were not done well in the exam, undoubtedly because we spent little time on such problems in lectures. Remarkably, several candidates attempted to apply the theory of Stirling numbers of the second kind (unsuccessfully, because that theory is not applicable to this problem). A study of those numbers later in the course and, generally, more practise with combinatorics of abstract functions would seem to be approporiate.

Solution 3: Part (a). To choose, in order, m mutually distinct elements x_1, ..., x_m of N, there are n choices for x_1, then $n-1$ choices for x_2, and so on, finally $n-m+1$ choices for x_m. So there are $n(n-1)...(n-m+1) = n!/(n-m)!$ choices altogether. By the same argument, for each subset M of N with size m, there are $m!$ was of ordering the elements of M. So there are $n!/m!(n-m)!$ choices of M.

Part (b). By part (a), there are $\binom{n}{n_1}$ choices for N_1, then $\binom{m_1}{n_2}$ choices for N_2, generally, $\binom{m_{j-1}}{n_j}$ choices for N_k, where $m_j = n - n_1 - ... - n_j$. We have

$$\binom{n}{n_1}\binom{m_1}{n_2}...\binom{m_{k-2}}{n_{k-1}}\binom{m_{k-1}}{n_k} = \frac{n!}{n_1!m_1!} \cdot \frac{m_1!}{n_2!m_2!} ... \frac{m_{k-2}!}{n_{k-1}!m_{k-1}!} \cdot \frac{m_{k-1}!}{n_k!m_k!} = \frac{n!}{n_1!...n_k!} \ .$$

Part (c). Put $k = 2$ and $n_1 = 1$ and $n_2 = 2$. Then the number of equivalence classes is 3, whereas $n!/k!n_1!...n_k! = 3!/2!1!2! = 3/2$.

Comment 1: An alternative to part (b) would be to argue by induction on k. The case $k = 1$ is trivial, and the case $k = 2$ is part (a). Now suppose that $k \geq 3$ and that the required conclusion holds for all n in the case where k is replaced by $k - 1$. Then the number of choices for N_1 is $n!/n_1!(n - n_1)!$, the number of choices for N_2, ..., N_k is $(n - n_1)!/n_2!...n_k!$, hence the number of choices for N_1, ..., N_k is

$$\frac{n!}{n_1!(n-n_1)!} \cdot \frac{(n-n_1)!}{n_2!...n_k!} = \frac{n!}{n_1!...n_k!} \ .$$

This part of the question was intended as an exercise in mathematical induction. I am surprised no-one did it that way. Presumably, some candidates must have noticed that induction can be used, but decided, correctly, that a more direct argument would be just as clear.

Comment 2: It is not hard to show that the formula appearing in part (c) holds if and only if $n_1 = n_2 = ... = n_k$. So the smallest counter-example is the one specified above.

Solution 4: Part (a). As a special case of the formula stated in part (a) of Question 3, the number of ways of choosing the three equivalence classes, N_1, N_2, N_3, in some order, is $(3a)!/(a!)^3$. For each of the equivalence relations $\equiv$, there are $3! = 6$ ways of ordering the three equivalence classes. So there are $(3a)!/3!(a!)^3$ equivalence relations $\equiv$ as specified.

Part (b). We write $x = x_1...x_n$ where each $x_i \in \{0, 1\}$. Recall, $d(x, y) = |\{i : x_i \neq y_i\}|$. There are 2^n choices for y. For each y, the number of choices for x is $\binom{n}{2a}$, because that is the number of ways of choosing the $2a$ indices i for which $x_i \neq y_i$. Similarly, the number of choices for z is $\binom{n}{2a}$. So the number of choices for (x, y, z) is

$$2^n \binom{n}{2a}^2 = 2^{3a}((3a)!/a!(2a)!)^2 \ .$$

Part (c). There are 2^n choices for x. For each x, choosing y and z amounts to choosing the three sets $\{i : x_i \neq y_i = z_i\}$ and $\{i : x_i \neq y_i \neq z_i\}$ and $\{i : x_i = y_i \neq z_i\}$. Each of those three sets has size a, they are mutually disjoint and their union is the whole set of indices i. The argument in part (a) shows that there are $(3a)!/(a!)^3$ ways of choosing those three sets (in order, of course). So the number of choices for (x, y, z) is now

$$2^n \binom{n}{2a} \binom{2a}{a} = 2^{3a} \frac{(3a)!}{(a!)^3} \ .$$

Comment: Part (c) is exceptionally difficult for a first course in combinatorial methods. I would not have been at all surprised if no-one had been able to do it under exam conditions. Congratulations are due to the four candidates who did it successfully. The question is a special case of the following problem: Given any positive integer n then, in the metric space $\mathbb{Z}_2^n$, how many triangles are there with given edge-lengths? For entertainment, I record the following theorem, which supplies a complete answer to that question.

Theorem: *Let n be a positive integer and a, b, c natural numbers. Consider the triples (x, y, z) where $x, y, z \in \mathbb{Z}_2^n$ and $d(x, y) = c$ and $d(x, z) = b$ and $d(y, z) = a$. There exists such a triple if and only if the following conditions hold: each of a, b, c is less than or equal to the sum of the other two; $a + b + c$ is even; $(a + b + c)/2 \leq n$. In that case, the number of such triples is $2^n n!/\alpha!\beta!\gamma!(n - \alpha - \beta - \gamma)!$ where $2\alpha = b + c - a$ and $2\beta = a + c - b$ and $2\gamma = a + b - c$.*

Proof: First suppose that such (x, y, z) exists. Consider the sets

$$A = \{i : y_i \neq z_i\} , \qquad B = \{i : x_i \neq z_i\} , \qquad C = \{i : x_i \neq y_i\} .$$

Since the three elements $x_i, y_i, z_i \in \{0, 1\}$ cannot be mutually distinct, any element of A or B or C must belong to exactly two of those three sets. In other words, $A \cup B \cup C$ is the disjoint union of $B \cap C$ and $A \cap C$ and $A \cap B$. Let $\alpha = |B \cap C|$ and $\beta = |A \cap C|$ and $\gamma = |A \cap B|$. Then

$$a = |A| = \beta + \gamma , \qquad b = |B| = \alpha + \gamma , \qquad c = |C| = \alpha + \beta .$$

Noting that $a + b + c = 2(\alpha + \beta + \gamma)$, we see that a, b, c satisfy the specified existence criterion.

Conversely, suppose that the existence criterion is satisfied. Let $\alpha = (b + c - a)/2$ and $\beta = (a + c - b)/2$ and $\gamma = (a + b - c)/2$, all of which are integers because $a + b + c$ is even. None of a, b, c is greater than the sum of the other two, so α, β, γ are non-negative. Plainly, $\alpha + \beta + \gamma \leq n$. Choosing x, y, z is the same as choosing x, A, B, C. The required conclusion follows because there are 2^n choices for x and, by part (b) of Question 3, there are $n!/\alpha!\beta!\gamma!(n - \alpha - \beta - \gamma)!$ choices for A, B, C. $\square$

MATH 110: DISCRETE MATHEMATICS. Makeup 2

Time allowed: 110 minutes. Please put your name on EVERY sheet of your manuscript. The use of telephones and other electronic devices is prohibited. The use of very faint pencils is prohibited too. You may take the question-sheet home.

LJB, 30 December 2013, Bilkent University.

1: 10% A relation $<$ on a set X is said to be **antisymmetric** provided there do not exist elements $x, y \in X$ such that $x < y$ and $y < x$. Suppose that X is a finite set with size n. How many antisymmetric relations on X are there?

2: (a) 10% Give an example of a function $g : Y \to X$ and two functions $f_1, f_2 : X \to Y$ such that $f_1 \circ g = f_2 \circ g = \mathrm{id}_Y$ and $f_1 \neq f_2$. (Recall, the identity function on Y is the function $\mathrm{id}_Y : Y \to Y$ such that $\mathrm{id}_Y(y) = y$ for all $y \in Y$. Thus, the equations $f_1 \circ g = f_2 \circ g = \mathrm{id}_Y$ mean that $f_1(g(y)) = f_2(g(y))$ for all $y \in Y$.)
(b) 10% Give an example of a function $g : Y \to X$ and two functions $h_1, h_2 : X \to Y$ such that $g \circ h_1 = g \circ h_2 = \mathrm{id}_X$ and $h_1 \neq h_2$.
(c) 10% Show that, given a function $g : Y \to X$ and two functions $f, h : X \to Y$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ h = \mathrm{id}_X$ then g is bijective and $f = h = g^{-1}$.

3: Let M and N be finite sets with sizes $m = |M|$ and $n = |N|$. Let X be the set of all functions $M \to N$.
(a) 10% Let $\equiv$ be the relation on X such that, given functions $f, g : M \to N$, then $f \equiv g$ if and only if there exist bijections $u : M \to M$ and $v : N \to N$ satisfying $f = v \circ g \circ u$. Show that $\equiv$ is an equivalence relation.
(b) 10% When $n = 2$, how many equivalence classes does $\equiv$ have?
(c) 10% When $m = 5$ and $n = 3$, how many equivalence classes does $\equiv$ have?

4: Let n be a positive integer. A **ternary string with length** n is a string $x = x_1 x_2 ... x_n$ where each $x_j \in \{0, 1, 2\}$. The set of ternary strings of length n is written as $\mathbb{Z}_3^n$. The **Hamming metric** on $\mathbb{Z}_3^n$ is defined to be the function d such that $d(x, y) = |\{i : x_i \neq y_i\}|$. In other words, the distance between x and y is the number of places i such that the i-th digit of x is different from the i-th digit of y. A **ternary code** of length n is a non-empty subset of $\mathbb{Z}_3^n$. When working with such a code C, the elements of C are called the **codewords**.
(a) 10% Given $x \in \mathbb{Z}_3^n$ and given an integer r such that $0 \leq r \leq n$, how many elements $y \in \mathbb{Z}_3^n$ are there such that $d(x, y) = r$?
(b) 10% Let k be an integer such that $2k + 1 \leq n$ and let C be a code that has as many codewords as possible, subject to the condition that the minimum distance between any two distinct codewords is $2k + 1$. Show that

$$\frac{3^n}{\sum_{r=0}^{2k} 2^r \binom{n}{r}} \;\leq\; |C| \;\leq\; \frac{3^n}{\sum_{r=0}^{k} 2^r \binom{n}{r}} \,.$$

(c) 10% For a coding scheme with code C (used in digital communication based on ternary logic), how many errors of transmission can be detected? How many errors of transmission can be corrected? (Briefly justify your answers.)

MATH 110: DISCRETE MATHEMATICS. Final, Fall 13

Time allowed: two hours. Please put your name on EVERY sheet of your manuscript. The use of telephones and other electronic devices is prohibited. The use of very faint pencils is prohibited too. You may take the question-sheet home.

LJB, 3 January 2014, Bilkent University.

1: 40% Consider the encoding function with generating matrix $G = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$.

(a) 6% Write down the codewords for each of the 8 message words.
(b) 6% Find the parity-check matrix H, and find the syndromes for each of the 8 strings

$$000000\ ,\quad 100000\ ,\quad 010000\ ,\quad 001000\ ,\quad 000100\ ,\quad 000010\ ,\quad 000001\ ,\quad 000011\ .$$

(c) 5% Explain why the 8 strings in part (b) can be used as the coset leaders. (Do not forget to explain why the minimal weight condition for selecting coset leaders is satisfied.)
(d) 6% Taking the 8 strings in part (b) as the coset leaders, use the method of calculating syndromes to decode the received words 111111, 011111, 001001.
(e) 6% Write down only the following part of the decoding table: the top line of consisting of the message words, the next line consisting of the codewords, and the last line (which begins with the coset leader 000011).
(f) 5% Instead of 000011, which other strings could be used as the last coset leader?
(g) 6% For this code, how many single-digit errors of transmission can be detected? How many single-digit errors of transmission can be corrected?

2: 30% For positive integers m and n with $m \geq n$, we define the Stirling number $S(m, n)$ be the number of equivalence relations on $\{1, 2, ..., m\}$ that have n equivalence classes. All your arguments in this question must be deduced from that definition. (You may not use any general formulas without proof.)
(a) 10% Show that $S(m + 1, n) = S(m, n - 1) + nS(m, n)$ for all integers $m \geq n \geq 1$.
(b) 5% Briefly explain why $S(m, 1) = 1 = S(m, m)$ for all integers $m \geq 1$.
(c) 5% Using mathematical induction and parts (a) and (b), show that $S(n+1, n) = n(n+1)/2$ for all integers $n \geq 1$.
(d) 5% Directly from the above definition of $S(m, n)$, without using mathematical induction, give another proof that $S(n + 1, n) = n(n + 1)/2$.
(e) 5% Using parts (a) and (b), evaluate $S(7, 4)$.

3: 30% We define a **preorder** on a set X to be a relation $\preceq$ on X such that $\preceq$ is reflexive and transitive.
(a) 10% Let P be the set of preorders on X. We define a relation $\cong$ on P such that, given elements $\preceq_1$ and $\preceq_2$ of P, then $\preceq_1 \cong \preceq_2$ provided there exists a bijection $f : X \to X$ with the property that, for all $x, y \in X$, we have $x \preceq_1 y$ if and only if $f(x) \preceq_2 f(y)$. Show that $\cong$ is an equivalence relation.
(b) 10% Now suppose that $|X| = 3$. How many equivalence classes of preorders on X are there? (Hint: You may find it helpful to describe the equivalence classes by drawing suitable directed graphs. If you do so, you must explain how your diagrams are to be interpreted.)
(c) 10% Still assuming that $|X| = 3$, find the size of each equivalence class. Use that to show that the number of preorders on X is 29. (Little credit will be given for laboriously listing all 29 preorders. That could be done using a computer. The aim of the question is to count the preorders in a systematic and understandable way.)

Example solutions to Final, MATH 110, Fall 2013

Reminder: Of course, there are no "model solutions" in mathematics. Mathematics thrives on diversity of methods and styles.

1: Part (a). Each message word w has codeword wG. The message words 000, 001, 010, 011, 100, 101, 110, 111 have corresponding codewords 000000, 001101, 010111, 011010, 100110, 101011, 110001, 111100.

Part (b), $H = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$, syndromes 000, 110, 111, 101, 100, 010, 001, 011.

Part (d). Decodings are 111, 010, 001. The calculation is expressed in the following table, where r denotes the received word, s the syndrome, v the coset leader, $c = r + v$ the codeword, w the message word.

r	s	v	c	w
111111	011	000011	111100	111
011111	101	001000	010111	010
001001	100	000100	001101	001

Part (e). The specified part of the decoding table is as follows.

message words	000	001	010	011	100	101	110	111
codewords	000000	001101	010111	011010	100110	101011	110001	111100
last line	000011	001110	010100	011001	100101	101000	110010	111111

Part (f). The only other possible strings for the last coset leader are 010100 and 101000. Indeed, all the other strings in the last line have greater weight.

Part (g). The minimal weight of a nonzero codeword is 3, so 2 errors are detectable, 1 error is correctable.

2: Part (a). To choose an equivalence relation on $\{1, ..., m+1\}$ with n equivalence classes, we can either choose an equivalence relation on $\{1, ..., m\}$ with $n-1$ classes, then put $m+1$ in a class of its own, or else we can choose an equivalence relation on $\{1, ..., m\}$ with n classes, then put $m+1$ in one of those classes. In the former case, there are $S(m, n-1)$ choices. In the latter case, bearing in mind that there are n possible classes in which to place $m+1$, there are $nS(m, n)$ choices.

Parts (b). There is only one equivalence relation on $\{1, ..., m\}$ such that all the elements of $\{1, ..., m\}$ are equivalent to each other, and there is only one equivalence relation such that no two distinct elements are equivalent.

Part (c). Let $s(n) = S(n+1, n)$. By parts (a) and (b), $s(1) = 1$ and $s(n) = s(n-1) + n$ for all $n \geq 2$. To prove that $s(n) = n(n+1)/2$, we shall argue by induction on n. The case $n = 1$ is trivial. Now suppose that $n \geq 2$ and that the required formula holds for $s(n-1)$. Thus, $s(n-1) = n(n-1)/2$. We deduce that $s(n) = n(n-1)/2 + n = n(n+1)/2$, as required.

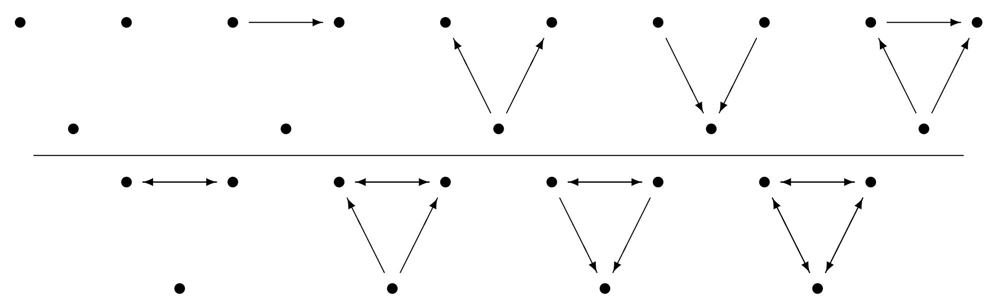
Part (d). The integer $S(n+1, n)$ is the number of equivalence relations such that all the classes have size 1 except for one class of size 2. The number of ways of choosing the class of size 2 is $\binom{n+1}{n} = n(n+1)/2$.

Part (e). By straightforward recursive calculation, $S(7, 4) = 350$.

Comment: A few candidates still had some difficulty with induction. Recall, it is necessary to tell the reader what the inductive assumption is. Otherwise, the reader may not be able to guess what is intended to be implying what. In the above inductive proof in part (c), the inductive assumption was that $n \geq 2$ and that the required formula holds for $s(n-1)$. That is not the only way of organizing the argument. We could, for instance, show that the formula for $s(n)$ implies the formula for $s(n+1)$. One candidate, who did excellently in several other questions, very clearly explained that he or she was showing that the formula for $s(n)$ implies the formula for $s(n-1)$. Of course, that does not work, but it is a correctable slip, and it does earn more credit than arguments which were unclear about what was being assumed and what was being deduced.

3: Part (a). In the notation of the definition of $\cong$, let us call f an **isomorphism** from $\preceq_1$ to $\preceq_2$. Consider elements $\preceq_1, \preceq_2, \preceq_3 \in P$. By considering the identity function on X, we see that $\preceq_1 \cong \preceq_1$. So $\cong$ is reflexive. If $\preceq_1 \cong \preceq_2$, then there is an isomorphism f from $\preceq_1$ to $\preceq_2$, whereupon the inverse f^{-1} is an isomorphism from $\preceq_2$ to $\preceq_1$, hence $\preceq_2 \cong \preceq_1$. We have shown that $\cong$ is symmetric. If $\preceq_1 \cong \preceq_2$ and $\preceq_2 \cong \preceq_3$, then there is an isomorphism f from $\preceq_1$ to $\preceq_2$ and there is an isomorphism g from $\preceq_2$ to $\preceq_2$, whereupon the composite $g \circ f$ is an isomorphism from $\preceq_1$ to $\preceq_3$, hence $\preceq_1 \cong \preceq_3$. We have shown that $\cong$ is transitive.

Part (b). The 9 equivalence classes are indicated in the following diagrams, where an arrow $x \leftarrow y$ indicates that $x \neq y$ and $x \preceq y$.



Part (c). Running through the equivalence classes in the order of the diagrams, the sum of the sizes of the equivalence classes is $1+6+3+3+6+3+3+3+1 = 29$.