Archive of past papers, solutions and homeworks for MATH 110, Discrete Mathematics, Fall 2012, Laurence Barker

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MATH 110, Discrete Mathematics, Fall 2012

Handout 1: Course specification

Laurence Barker, Mathematics Department, Bilkent University,
version: 25 September 2012.

Course Aims: To introduce some concepts and techniques of discrete and combinatorial mathematics from an applicable point of view.

Course Description: We shall focus on two major areas of discrete mathematics: graph theory and coding theory. Many of the principles and techniques that we shall encounter are frequently applicable to other areas of discrete mathematics.

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Text: R. P. Grimaldi, "Discrete and Combinatorial Mathematics", 5th Ed. (Pearson, 2004). Some other sources may be supplied for some small components of the syllabus material.

Classes: Tuesdays 15:40 - 16:30 B-206, Fridays, 13:40 - 15:30 B-206.

Office Hours: Tuesdays, 16:40 - 17:30, SAZ 129.

Syllabus:

Week number: Monday date: Subtopics. Section numbers

1: 17 Sept: Mathematical induction. Some illustrations. Relations for Fibonacci numbers. 4.1, 4.2.

2: 24 Sept: Recurrence relations as an illustration of mathematical induction. A formula for Fibonacci numbers. 10.2.

3: 1 Oct: Introduction to graph theory. Graph isomorphism. Trees and spanning trees. Criteria for existence of Euler paths or Euler circuits, proved by mathematical induction. 11.1, 11.2, 11.3.

4: 8 Oct: Euler's characteristic formula for planar graphs, proved by mathematical induction. The non-planarity of the graphs K_5 and $K_{3,3}$. 11.4

5: 15 Oct: Flow networks. Ford-Fulkerson Algorithm. 13.3.

6: 22 Oct: Sets and relations. Injective, surjective and bijective functions. 5.1, 5.2, 5.3, 5.6.

Bayram followed by Republic Day, 25 - 29 October.

7: 29 Oct: Partial ordering relations. Equivalence relations. 7.3, 7.4.

8: 5 Nov: Review for Midterm 1.

9: 12 Nov: Introduction to coding theory. Linear codes. Hamming codes. 16.1, 16.6.

10: 19 Nov: Abelian groups. Cosets and coset representatives. 16.2, 16.3.

11: 26 Nov: Encoding and decoding linear codes using coset leaders. 16.8.

12: 3 Dec: Review for Midterm 2.

13: 10 Dec: Presentations.

14: 7 Dec: Presentations.

15: 24 Dec: Review for Final.

Assessment:

- Quizzes and Participation 15%.
- Midterm I, 20%, Friday 9 November.
- Midterm II, 20%, Friday 7 December.
- Final, 30%.
- Presentations, 15%.

Attendance is measured by quiz scripts. Some quizzes will pop up without prior announcement. Attendance as measured by presence of quiz scripts will result in an F or XF grade.

Class Announcements: All students, including any absentees from a class, will be deemed responsible for awareness of class announcements.

Advice: Practice, practice, practice. The more extra exercises you do throughout the semester, the easier the exercises will become. No-one can do it by procrastinating lazily and then frantically cramming during the few days prior to an exam.

MATH 110 Discrete Mathematics, Fall 2012

Homeworks, Quizzes and Presentation Projects

Laurence Barker, Mathematics Department, Bilkent University,
version: 28 December 2012.

Homeworks

Homework 1: Exercises 4.1.2 (summations by induction), 4.2.12 (identity for Fibonacci numbers).

Homework 2: Exercises 11.1.4, 11.1.6, 11.1.8 on pages 518 and 519.

Homework 3: Exercises 11.3.22 to 11.3.28.

Homework 4: Exercises 16.8.3, 16.8.4 (decoding using decoding table or syndromes).

Quizzes

Quiz 1: (20 Sept) Evaluate $1 + a + a^2 + ... + a^{n-1} + a^n$ where a is a real number.

Quiz 2: (27 Sept) Consider the Fibonacci numbers F_0, F_1, F_2, ... Recall that $F_0 = 0$ and $F_1 = 1$ and $F_{n+2} - F_{n+1} - F_n = 0$. Show that

$$\sum_{i=1}^{n} F_{i-1}/2 = 1 - F_{n+2}/2^n \ .$$

Quiz 3: (2 Oct) In the graph of a cube (depicted), find the maximum distance between two vertices.

Quiz 4: Let G be a connected planar graph with e edges and n vertices. Suppose that $e \geq 2$.
(a) State (do not prove) an inequality for e in terms of n.
(b) State (do not prove) an equality for e in terms of the degrees of the vertices.
(c) Find an inequality for the average degree of a vertex.
(d) Deduce that any planar graph has a vertex with degree less than or equal to 5.

Quiz 5: Let m and n be positive integers with $m < n$. Let $E : \mathbb{Z}_2^m \to \mathbb{Z}_2^n$ and $D : \mathbb{Z}_2^n \to \mathbb{Z}_2^m$ be functions such that $D \circ E = \mathrm{id}_{\mathbb{Z}_2^m}$.
(a) Is E is injective? Surjective? Bijective?
(b) Is D injective? Surjective? Bijective?

Presentation titles

Mustafa Arda Ahi: Convolution codes.

Bulunt Efe Akmek: Hidden Markov models.

Yaşar Daysel: Stirling numbers.

Canberk İnciler: Boolean algebras.

Koray Kılınç: Vitterby decoding.

Dilara Oğuz: The RSA cryptosystem.

Noyan Cem Sevüktekin: LDPC coding and Tanner algorithms.

Devrim Şahin: Path finding algorithms.

Gizem Tabak: Map colouring.

Süleyman Taşkafa: Karnaugh maps.

Other suggested presentation topics:

- Ford–Fulkerson algorithm.
- The Euclidian algorithm and its complexity.

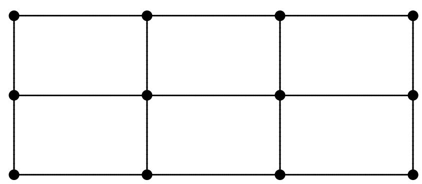
MATH 110: DISCRETE MATHEMATICS. Midterm 1, Fall 12

Time allowed: two hours. Please put your name on EVERY sheet of your manuscript. The use of telephones and other electronic devices is prohibited. The use of very faint pencils is prohibited too. You may take the question-sheet home.

LJB, 9 November 2012, Bilkent University.

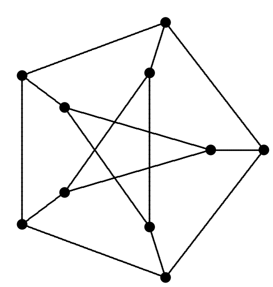
1: 25% Show that $\sum_{i=0}^{n}(2i+3)3^i = (n+1)3^{n+1}$. *(Explain your reasoning carefully. Marks will be deduced for arguments that could be adapted to obtain false equations.)*

2: 25% For integers $1 \leq m \leq n$, let $G_{m,n}$ be the graph defined as follows. The vertices are $v_{i,j}$, where i and j are integers such that $1 \leq i \leq m$ and $1 \leq j \leq n$. There is an edge between two vertices $v_{i,j}$ and $v_{i',j'}$ if and only if $|i - i'| + |j - j'| = 1$. For example, the graph $G_{3,4}$ has a diagram as shown.



(a) For which m and n does $G_{m,n}$ have an Euler circuit? (Say what general criterion you are using.)
(b) For which m and n does $G_{m,n}$ have an Euler trail that is not a circuit? (Say what general criterion you are using.)
(c) What is the minimum number of edges that can be removed from $G_{3,4}$ to produce a graph that has an Euler trail that is not a circuit?

3: 30% Let G be a planar graph with n vertices, e edges, f faces. Suppose that G is not a tree, and that every cycle in G has at least 5 edges.
(a) State, without proof, an equation involving n and e and f. *(No marks for proving it!)*
(b) Let P be the set of pairs (F, ϵ) such that F is a face and ϵ is one of the edges of F. Find a bound for $|P|$ in terms of f.
(c) Find an equality for $|P|$ in terms of e.
(d) Using parts (a), (b), (c), explain why $e \leq 5(n-2)/3$.
(*You may not assume the theorem asserting that $e \leq c(n-2)/(c-2)$ where c is the minumum number of edges of a face. The whole point of this part is to prove a special case of a variant of that theorem.)*
(e) Deduce that the depicted graph is not planar.



4: 20% Let V be a set with size $|V| = 101$.
(a) How many irreflexive relations on V are there?
(b) How many symmetric relations on V are there?
(c) How many (loop-free undirected) graphs with vertex set V are there?

MATH 110: DISCRETE MATHEMATICS. Midterm 2, Fall 2012

Time allowed: 110 minutes. Please put your name on EVERY sheet of your manuscript. The use of telephones and other electronic devices is prohibited. The use of very faint pencils is prohibited too. Do not hand in the question sheet. You may take the question-sheet home.

LJB, 7 December 2012, Bilkent University.

1: 16% Which of the following assertions are true? In each case, justify your answer with a proof or a counter-example. (In cases where the assertion is false, you must give a specific counter-example, not just a vague comment about how something might perhaps go wrong.)
(a) Given any finite sets X and Y and functions $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ is injective, then f and g are injective.
(b) Given any finite sets X and Y and functions $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are injective, then f and g are injective.
(c) Given finite sets X and Y and functions $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are injective, then f and g are bijective.
(d) Given finite sets X and Y and functions $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are bijective, then f and g are bijective.

2: 14% Consider the coding scheme with encoding function $\mathbb{Z}_2^3 \to \mathbb{Z}_2^6$ given by generating matrix

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} .$$

(a) Write down all 8 message words and their corresponding codewords.
(b) How many errors of transmission can be detected? How many errors of transmission can be corrected? Briefly justify your answers.

3: 35% Consider the coding scheme in Question 2.
(a) Write down the parity-check matrix H.
(b) Find the syndromes for each of the words

000001 , 000010 , 000100 , 001000 , 010000 , 100000 , 000110 , 000101 .

(c) Using part (b), without calculating entries of the decoding table, explain why there must exist a decoding table with coset leaders

000000 , 000001 , 000010 , 000100 , 010000 , 100000 , 000110 , 000101 .

(Warning: there are no marks for doing this by calculating entries of the decoding table.)
(d) Using the syndromes in part (b), decode the message 000111, 111000, 111101. (Warning: there are no marks, here, for doing this using the decoding table.)

4: 25% Consider, once again, the coding scheme in Questions 2 and 3. Now write out the decoding table with the coset leaders specified in part (c) of Question 3. Use it to check your decoding of the message in part (d) of Question 3.

5: 10% Let n be a positive integer. Let S be the set of linear codes in $\mathbb{Z}_2^n$. We define a relation $\equiv$ on S such that, given two linear codes C and C' in S, then $C \equiv C'$ if and only if there exists a bijection $f : C \to C'$ with the property that $d(c_1, c_2) = d(f(c_1), f(c_2))$ for all $c_1, c_2 \in C$. Show that $\equiv$ is an equivalence relation.

MATH 110: DISCRETE MATHEMATICS. Final, Fall 2012

Time allowed: Two hours. Please put your name on EVERY sheet of your manuscript. The use of telephones and other electronic devices is prohibited. The use of very faint pencils is prohibited too. Please do not hand in your question sheet.

LJB, 11 January 2013, Bilkent University.

1: 15% Let n be an integer with $n \geq 2$. Consider a building in the shape of a cube. The inside of the building is divided up into n^3 rooms, all of them in the shape of a cube and all of them having the same size. Between any two adjacent rooms, there is a door. (Thus, each room has 3, 4, 5 or 6 doors. The rooms with 6 doors have one door in the ceiling to the room above, one door in the floor to the room below, and four doors in the horizontal walls to the rooms on the same level.) Is it possible to make a tour of the building, passing through each door exactly once? Does the answer depend on n? Justify your answer.

2: 30% A graph is said to be **4-regular** provided every vertex has degree 4.

(a) Draw a 4-regular planar graph with 6 vertices.

(b) Let G be a 4-regular planar graph with n vertices, e-edges and f faces. Express n and e in terms of f.

(c) Show that G has a cycle with length 3. (Hint: first explain why the average number of edges per face is $2e/f$.)

3: 15% Let X and Y be sets and let $f : X \to Y$ be a function. Are any two of the following three conditions (A), (B), (C) equivalent to each other? Justify your answer carefully.

(A) For all sets Z and all functions $g : Y \to Z$ and $h : Y \to Z$ satisfying $h \circ f = g \circ f$, we have $h = g$.

(B) f is injective.

(C) f is surjective.

4: 20% Consider the linear encoding functions $E : \mathbb{Z}_2^2 \to \mathbb{Z}_2^3$. (These are the linear maps given by an arbitrary matrix G. Thus, $E(w) = wG$ for a message word w.) How many of those encoding functions can detect a single error of transmission?

5: 20% Let n be a positive integer. A **linear code** in $\mathbb{Z}_2^n$, recall, is a non-empty subset C of $\mathbb{Z}_2^n$ such that, for all $c_1, c_2 \in C$, we have $c_1 + c_2 \in C$.

(a) Show that $\underline{0} \in C$, where $\underline{0}$ denotes the binary string whose digits are all 0.

(b) Let $C_0 = \{0\}$. We choose elements b_1, b_2, ... of C as follows. If $C_0 \neq C$, choose an element $b_1 \in C$ such that $b_1 \neq \underline{0}$. Let $C_1 = \{\underline{0}, b_1\}$. Generally, when we have chosen b_i, we let C_i be the smallest linear code in $\mathbb{Z}_2^n$ such that $\{b_1, ..., b_i\} \in C_i$. If $C_i = C$, we stop, otherwise we choose an element $b_{i+1} \in C$ such that $b_{i+1} \notin C_i$. By considering the codes C_0, C_1, ..., show that there exists a natural number m such that $|C| = 2^m$.