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MATH 104, Thinking Mathematically 2

Bilkent University, Spring 2019

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Course specification

MATH 104, Sections 1, 2, 3, *Thinking Mathematically 2*, Spring 2019

Version: 22 April 2019.

Course Aims: To aquire skill and knowledge in some practical areas of mathematics.

Course Description: This is an introductory course with an emphasis on methods of calculation, but also with some attention to underlying theory.

Course Requirements: A prerequisite is MATH 103 *Thinking Mathematically 1*.

Teachers:

Section 1: Arzu Sibel Ikinci, Office RA-105, aikinci at bilkent dot edu dot tr.
Section 2: Ahmet Durukal, Office EA-211, durukal at bilkent dot edu dot tr
Section 3: Laurence Barker, Office SAZ 129, barker at fen dot bilkent dot edu dot tr.

Course text: Robert Blitzer, "Thinking Mathematically", 7th edition, 2019 Pearson. ISBN: 978-0-134-68371-3.

Classes:

Section 1: Mondays 10:40 - 12:30, Thursdays 09:40 - 10:30, A-Z27.
Section 2: Mondays 15:40 - 16:30, Thursdays 13:40 - 15:30, V-02.
Section 3: Mondays 09:40 - 10:30, Wednesdays 10:40 - 12:30, SB-Z03.

Office Hours:

Section 1: Thursdays 08:40 - 09:30, RA-105.
Section 2: Mondays 16:40 - 17:30, EA-211.
Section 3: Mondays 08:40 - 09:30, SA-129.

Office hours is for *all* the students on the course, not just the proficient. If you are having difficulty with the course, then that is an especially good reason to visit your course teacher for help.

Weekly Syllabus

The format below is, *Week number: Monday date: Subtopics.*

1: 4 Feb: Personal finance. Percentages. Simple interest.

2: 11 Feb: Compound interest. Nominal and effective interest rates.

3: 18 Feb: Annuities. Investments. Regular repayments.

4: 25 Feb: Measurement. The metric system. Area, volume, weight, temperature.

5: 4 Mar: Geometry. Lines, angles, triangles, polygons, tessellations, area, circumference.

6: 11 Mar: Volume, surface area, trigonometry.

7: 18 Mar: Counting. Permutations and combinations. Fundamentals of probability.

8: 25 Mar: Enumerative probability. Conditional probability. Expectation.

9: 1 Apr: Statistics. Sampling, distributions. Measures of central tendency. Measures of dispersion.

10: 8 Apr: Normal distribution, Correlation.

11: 15 Apr: Voting methods and their flaws.

12: 22 Apr: Apportionment methods and their flaws.

13: 29 Apr: Graphs, paths, circuits.

14: 6 May: Trees, Euler paths, Euler circuits

15: 13 May: Review for Final Exam.

Assessment:

- Homework 18%.
- Midterm I, 24%, Monday, 11 March, 18:00.
- Midterm II, 24%, Wednesday, 17 April, 18:00.
- Final, 34%, Monday, 20 May, 18:00.

Criteria for FZ: (1) Less than 20% in the sum of the Midterm 1 and Midterm 2 marks, or (2) less than 40% in that sum and less than 50% attendance.

Homework policy:

All six homework assignements are compulsory.
There are no homework makeups.
Late homework assignments will not be accepted.

Classroom policy:

Proper academic behaviour is expected.
In particular, all talking must be addressed to the whole class.
Latecomers will not be allowed to attend.
Food or drink may not be consumed in the classroom.

Class Announcements: All students, including any absentees from a class, will be deemed responsible for awareness of class announcements.

MATH 104: Thinking Mathematically 2 Spring 2019. Midterm 1

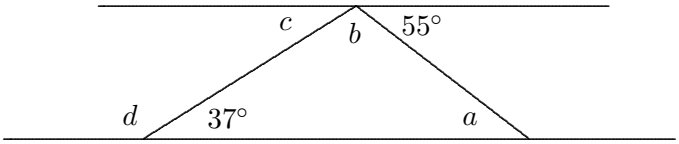
11 March 2019, Bilkent University.

Time allowed: 2 hours. Please put your name on EVERY sheet of your manuscript. The use of telephones, calculators or other electronic devices is prohibited. The use of red pens or very faint pencils is prohibited too. You may take the question sheet home.

Remember to show your working, except where your answers are obvious.

1: 17 marks. For this question, you may make use of the following facts: 1 inch is approximately 2.54 centimeters; 1 yard is 36 inches.
(a) Convert 2540 meters to inches.
(b) Convert 9144 centimeters to yards.

2: 17 marks. In the following diagram, the two lines running left-to-right are parallel. Find the angles a, b, c, d.

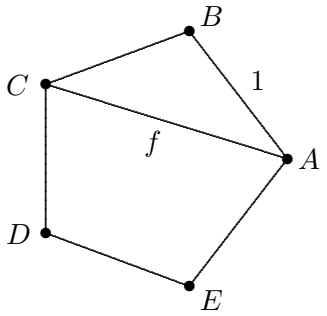


3: 17 marks. A 6-foot tree is planted 15 feet from an 18-foot lamppost. How long is the shadow of the tree?

4: 17 marks. Which interest rate is higher:
(a) a (nominal) rate of 10% compounded every 3 months,
(b) a (nominal) rate of 10.2% compounded every 6 months?

5: 17 marks. Ayse and Baran both start with 1000 TL. They both make savings that have an interest rate of 10 percent. After 2 years, Ayşe receives a present of 2000 TL. After 4 years from the start, she receives present of 2500 TL. After 5 years from the start, Baran receives a present of 5000 TL. Assuming no expenditure and no other income, which of the two is the richer at the end of 7 years?

6: 15 marks. (This question is extremely difficult, deliberately. It is to test the strongest few students taking the course.) Consider a regular pentagon (a five-sided figure with all five vertices A, B, C, D, E lying equally spaced on a circle). Suppose the distance between any two adjacent vertices, for instance A and B, is 1. Let f be the distance between any two non-adjacent vertices, for instance, $f = AC$. Find a quadratic equation with solution f. (Hint: One way is to find two right-angled triangles and to apply Pythagoras' Theorem.)



Solutions to Midterm 1

1: Part (a): 100 000 inches.
Part (b): We have $254 \times 36 = 7620 + 1524 = 9144$. So 9144 cm = 3600 in = 100 yards.

2: $a = 55°$, $b = 180 - 37 - 55 = 88°$, $c = 37°$, $d = 180 - 37 = 143°$.

3: By considering two similar triangles, the length x of the shadow satisfies $(x+15)/18 = x/6$. So $x + 15 = 3x$. So $2x = 15$. Therefore, $x = 7.5$ feet.

4: We have $1.025^2 = 1 + 0.05 + (0.025)^2$. So option (a) has a 6-month interest rate less than 5.1%. On the other hand, (b) has a 6-month interest rate of exactly 5.1%. We have shown that the higher rate is option (b).

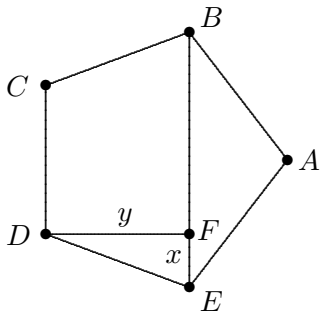
5: After 5 years, Ayşe's present of 2000 has grown to $2000 \times 1.1^3 = 2000 \times 1.331 = 2662$. At the same time, her present of 2500 has grown to $2500 \times 1.1 = 2750$. The sum $2662 + 2750 = 5412$ is greater than 5000. So Ayşe is the richer after 5 years. It follows that Ayşe is the richer after 7 years.

Comment: Alternatively, one can calculate Ayşe's growing savings by directly calculating the interest for each year: $2000 + 200 + 220 + 242 = 2662$ and $2500 + 250 = 2750$.

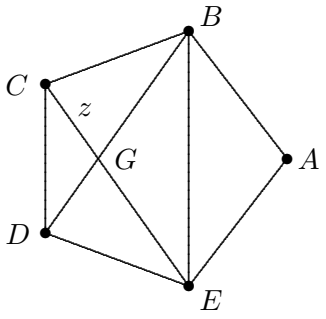
6: *Solution 1, using the hint:* Let x, y, F be as illustrated. Note that $f = 1 + 2x$. By considering the right-angled triangle DFE and applying Pythagoras' Theorem, we have $x^2 + y^2 = 1$. By considering the right-angled triangle DFB and applying the same theorem, $(1 + x)^2 + y^2 = f^2$. So

$$f^2 = 1 + 2x + x^2 + y^2 = 2 + 2x = f + 1.$$

In other words, $f^2 - f - 1$.



Solution 2, not using the hint: Let z and G be as shown. The lines BD and AE are parallel, likewise the lines AB and EC. So the figure $ABGE$ is a rhombus, in other words, all four of its edges have the same length. That length is 1. So $f = z + 1$. The lines BE and CD are parallel, so the triangles BGE and CGD are similar. Hence $BE/BG = CD/CG$, that is, $f/1 = 1/z = 1/(f - 1)$. Therefore, $f^2 - f - 1 = 0$.



Comment: Using the formula, in MATH 103, for the solution to a quadratic equation, we deduce that $f = (1+\sqrt{5})/2$ or $f = (1-\sqrt{5})/2$. But f is a positive number. So $f = (1+\sqrt{5})/2$. This irrational number, called the **golden ratio**, arises in many different areas of mathematics.

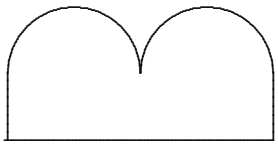
MATH 104: Thinking Mathematically 2 Spring. Midterm 2

17 April 2019, Bilkent University.

Time allowed: 2 hours. Please put your name on EVERY sheet of your manuscript. The use of telephones, calculators or other electronic devices is prohibited. The use of red pens or very faint pencils is prohibited too. You may take the question sheet home.

Remember to show your working, except where your answers are obvious.

1: 17 marks. The following shape consists of a rectangle and two half-circles of equal radius. The shorter side of the rectangle has length x. The longer side of the rectangle has length $4x$. Give formulas for: **(a)** the perimeter of the shape, **(b)** the area.



2: 17 marks. Let θ be an acute angle (an angle between 0 and 90°) such that $\tan(\theta) = 5/12$. Evaluate: **(a)** $\cos(\theta)$. **(b)** $\cos(90^\circ - \theta)$.

3: 17 marks. 1000 car drivers were asked whether they sometimes use a phone while driving and whether they have had a speeding ticket during the last year. The results were as follows:

	speeding ticket	no speeding ticket
sometimes uses phone	30	200
never uses phone	70	700

Assuming the data is representative of car drivers in general, find:
(a) the probability that a car driver sometimes uses a phone while driving,
(b) the probability that a car driver sometimes uses a phone and has had a speeding ticket during the last year,
(c) the probability that a car driver sometimes uses a phone while driving given that the driver has had a speeding ticket during the last year.

4: 17 marks. A bag contains 10 differently coloured balls, exactly one of them red, exactly one of them blue. How many ways are there of picking 3 (distinct) balls from the bag if:
(a) the order is important?
(b) the order is not important?
(c) the order is important, the red ball is chosen and the blue ball is not chosen?
(d) the order is not important, the red ball is chosen and the blue ball is not chosen?

5: 17 marks. The weights of 8 cats, in kilograms, are approximately: 3, 3, 4, 4, 4, 5, 5, 6.
(a) What is the mean for this data set?
(b) What is the median?
(c) Is there a mode and, if so, what is it?

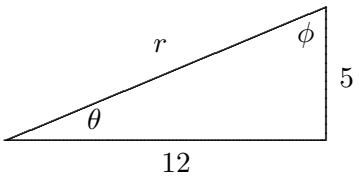
6: 15 marks. Evaluate ${}_{20}C_0 + {}_{20}C_1 + {}_{20}C_2 + ... + {}_{20}C_{20}$.

Solutions to Midterm 2

1: Part (a). The perimeter is $(2\pi + 6)x$. Part (b). The area is $(\pi + 4)x^2$.

2: Let r and ϕ be as depicted. By Pythagoras' Theorem, $r^2 = 12^2 + 5^2 = 169$, so $r = 13$ and the answer to (a) is $12/13$.

We have $90° - \theta = \phi$. So the answer to (b) is $\cos(\phi) = 5/13$.



3: Part (a), $(200 + 30)/1000 = 23/100$.

Part (b), $30/1000 = 3/100$. Part (c), $30/(30 + 70) = 3/10$.

4: Part (a), ${}_{10}P_3 = 10.9.8 = 720$.

Part (b): ${}_{10}C_3 = 10.9.8/3.2.1 = 720/6 = 120$.

Part (c): Removing the blue ball, there are 9 left. The number of possibilities with red first, RXX, is $8.7 = 56$. For the same reason, the number of possibilities of the form XRX is 56, likewise for XXR. So the answer is $3.56 = 168$.

Part (d): For each unordered possibiliy here, there are 6 possibilities counted in part (c). So the answer is $168/6 = 28$.

Comment: Having answered parts (a) and (b) of this question, we can then do (c) and (d) using probability. For the possibilities in (a), the probability of a red but no blue ball is $3/10 \times 7/9 = 7/30$. So the answer to (c) is the answer to (a) times $7/30$, which is 168. Similarly, the answer to (d) is the answer to (b) times $7/30$, which is 28.

5: **(a)** The mean is $(3 + 3 + 4 + 4 + 4 + 5 + 5 + 6)/8 = 34/8 = 17/4 = 4.25$.

(b) The median is 4.

(c) The mode is 4.

6: Each ${}_{20}C_r$ is the number of subsets of $\{1, 2, ..., 19, 20\}$ that have size r. To choose a subset S of $\{1, ..., 20\}$, we run through the numbers 1, ..., 20 and, for each of those numbers i, we choose whether or not i is in S. So the answer is the number of subsets of $\{1, ..., 20\}$, which is

$$2^{20} = 1024^2 = 1000^2 + 2.24.1000 + 24^2 = 1\,048\,576 \ .$$

Comment: Some candidates gave pages of calculations for this question, but concluded with an incorrect answer. In most cases, no marks were awarded for that. Unless one is a calculating wizard, the approach by direct calculation, under exam time-pressure, is very unreliable. The course is, after all, called "Thinking Mathematically", not "Calculating Robotically". But it turns out that a few students on the course actually are calculating wizards! Correct or almost correct answers obtained by diabolical calculation did gain full or almost full marks.

MATH 104: Thinking Mathematically 2 Spring 2019. Makeup 2

13 May 2019, Bilkent University.

Time allowed: 2 hours. Please put your name on EVERY sheet of your manuscript. The use of telephones, calculators or other electronic devices is prohibited. The use of red pens or very faint pencils is prohibited too. You may take the question sheet home.

Remember to show your working, except where your answers are obvious.

1: 17 marks. Consider a regular hexagon whose sides all have length 1 kilometer.
(a) What is the perimeter of the hexagon?
(b) What is the area of the hexagon?
(c) What is the perimeter of the circle passing through the six vertices of the hexagon?
(d) What is the area of that circle?

2: 17 marks. In this question, you may assume that $\cos(72^\circ) = (-1+\sqrt{5})/4$.
(a) Express $\sin(72^\circ)$ in terms involving $\sqrt{5}$.
(b) Express $\tan(72^\circ)$ in terms involving $\sqrt{5}$.

3: 17 marks. Exams in Biology and Geography were taken by 60 students; exactly 45 passed Biology, exactly 40 passed Geography, exactly 10 failed both exams.
(a) How many students passed both exams?
(b) A student is chosen at random from those who passed Geography. What is the probability that he or she also passed Biology?

4: 17 marks. (a) How many numbers betwee 1000 and 9999 have no repeated digits? (For example, 2758 has no repeated digits, whereas 2787 has repeated digit 7.)
(b) Among the numbers in part (a), for how many are the digits in ascending order? (For example, the digits of 2578 are in ascending order, but the digits of 2758 are not.)

5: 17 marks. Consider the data set 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5.
(a) What is the mean for this data set?
(b) What is the median?
(c) What is the mode?

6: 15 marks. There are 30 balls and two empty bags. Each ball is placed in one of the two bags, with a probability of 1/2 for each bag. What is the probability that, at the end, each bag will contain an even number of balls? (No marks will be awarded for long arithmetic calculations.)

MATH 104: Thinking Mathematically 2 Spring. Final

20 May 2019, Bilkent University.

Time allowed: 2 hours. Please put your name on EVERY sheet of your manuscript. The use of telephones, calculators or other electronic devices is prohibited. The use of red pens or very faint pencils is prohibited too. You may take the question sheet home.

Remember to show your working, except where your answers are obvious.

1: 20 marks. In this question, you may use the fact that, for the normal distribution, the probability of a sample lying within one standard deviation of the mean is 68%, and the probabiliy of a sample lying within 2 standard deviations is 95%.

The lengths of adult tree pythons are normally distributed with a mean of 2.5 meters and a standard deviation of 0.25 meters.

(a) A tree python is chosen at random. What is the probability that its length is greater than 3 meters?

(b) Two tree pythons are chosen at random. What is the probability that at least one of them has length less than 2.25 meters?

2: 10 marks. In a particular neighbourhood, the average price of a house is 600 000 euros. The percentage of houses more expensive than 700 000 euros is 2.5. Assuming that the house prices are normally distributed, what is the standard deviation of the house prices?

3: 20 marks. The ballots for four candidates A, B, C, D are as follows. Answer the following questions, showing your working.

number of voters	12	10	8	6
first choice	A	B	C	D
second choice	B	C	D	A
third choice	C	D	A	B
fourth choice	D	A	B	C

(a) Which candidate wins by the plurality method?

(b) Which candidate wins by the Borda count method?

4: 20 marks. For each positive integer n, the graph K_n has n vertices and each vertex has degree $n-1$.

(a) How many edges does K_n have?

(b) For which values of n does K_n have an Euler path?

5: 30 marks. **(a)** The graph of a cube is connected, with 8 vertices, 12 edges, every vertex having the same degree. How many edges need to be added to produce a graph with an Euler circuit?

(b) The graph of a tesseract is connected, with 16 vertices, 32 edges, every vertex having the same degree. What is that degree?

(c) For the graph of a tesseract, how many edges need to be added to produce a graph with an Euler circuit?

Solutions to Final

1: Part (a). The probability of the length being less than 2 meters or greater than 3 meters is 5%. By symmetry, the answer is 2.5%.

Part (b). The probability that a given python has length less than 2.25 meters or more than 2.75 meters is 0.32. So the probability of it having length less than 2.25 meters is 0.16. Since $0.16^2 = 0.0256$, the probability that both pythons have length less than 2.25 meters is 0.0256. So the answer is $0.16 + 0.16 - 0.0256 = 0.2944$.

2: The difference between 700 000 and 600 000 is 2 standard deviations. So the standard deviation is 50 000 euros.

3: Part (a). Since A has the most first-choice votes, A wins by this method.

Part (b). The Borda counts are as follows:

$$\begin{aligned} A &: 4.12 + 1.10 + 2.8 + 3.6 = 48 + 10 + 16 + 18 = 92, \\ B &: 3.12 + 4.10 + 1.8 + 2.6 = 36 + 40 + 8 + 12 = 96, \\ C &: 2.12 + 3.10 + 4.6 + 1.6 = 24 + 30 + 32 + 6 = 92, \\ D &: 1.12 + 2.10 + 3.8 + 4.6 = 12 + 20 + 24 = 24 = 80. \end{aligned}$$

So B wins by this method.

4: Part (a). Any two distinct vertices are linked by an edge. So the number of edges is the number of unordered pairs of distinct vertices. That number is ${}_nC_2 = n(n-1)/2$.

Part (b). The graph K_n has an Euler path if and only if $n = 2$ or n is odd. Indeed, by the Euler Path Theorem, K_n has an Euler circuit if and only if n is odd, while K_n has an Euler path with distinct end-points if and only if $n = 2$.

5: Part (a). By the Euler Path Theorem, we must add 4 edges. For instance, we can add the 4 diagonals.

Part (b). Cutting each edge in half then, since each edge has two-end-points, the number of half-edges is 64. So each vertex has 4 half-edges. In other words, each vertex has degree 4.

Part (c). By the Euler Path Theorem, the graph of a tesseract already has an Euler circuit. The number of edges that need to be added is 0.