MATH 323, Algebra I, Fall 2020

Course notes, Chapter 7, Permutation sets and Sylow's Theorem

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These notes, updated as the course progresses, are a record of the prepared text of the lectures, with a little more detail added, but they cannot cover much of the oral component of the lectures.

Summary of contents

The notion of a permutation set is fundamental to the way groups arise as expressions of symmetry in contexts of application. Below, we shall be saying very little about external applications. Even within group theory itself, though, much use is made of the way groups express symmetries of structures within themselves, and the notion of a permutation set is still tremendously useful.

Fuller details on the material we shall be summarizing can be found in Judson, primarily Chapters 5, 14, 15. These notes are independent of that text.

We shall be discussing:

- permutation sets,
- the Orbit-Stabilizer Equation,
- conjugacy classes of group elements,
- Cauchy's special case of Sylow's First Theorem,
- properties of finite p-groups,
- Sylow's First, Second and Third Theorems,
- a strong version of Sylow's Theorem.

The notion of a permutation set: In the four chapters before this one, we have been discussing what groups are, but now we shall be considering groups in a different way, from the perspective of what groups do. Or rather, to put it more accurately, we shall now be looking at the kind of role non-abelian groups tend to play when they appear in other areas of mathematics. We mention that the theory and applications of abelian groups is somewhat separate.

Recall that, for a set Ω, the symmetric group $\mathrm{Sym}(\Omega)$ is defined to be the group whose elements are the permutations of Ω, the group operation being composition. For $\theta, \phi \in \mathrm{Sym}(\Omega)$, we often write the composite as $\theta\phi$ instead of $\theta \circ \phi$. For $w \in \Omega$, we often write ϕw instead of $\phi(w)$. Since $(\theta\phi)w = \theta(\phi w)$, we can write $\theta\phi w$ without ambiguity.

Given a group G, we define a **permutation representation** of G on Ω to be a group homomorphism $\rho : \mathrm{Sym}(\Omega) \leftarrow G$ where Ω is a set. We say that G **acts** on Ω

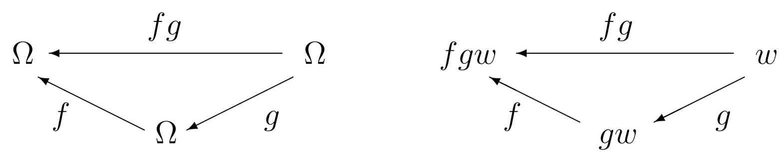
via ρ. The pair (Ω, ρ) is called a **permutation set** for G or, more briefly, a G**-set**. In normal usage of the terminology, we speak of the G-set Ω. When that casual language is used, it is to be understood that we are referring not to Ω merely as a set, but rather to Ω as a set equipped with ρ.

For $g \in G$ and $w \in \Omega$, we say that g sends w to $\rho(g)w$ via ρ. Often, we omit the symbol ρ from the notation, writing gw instead of $\rho(g)w$. That brief notation, with the symbol ρ omitted and with the brackets omitted too, gives us another way of presenting the notion of a G-set. We can understand a G-set to be a set Ω equipped with a function $\Omega \leftarrow G \times \Omega$ written $gw \leftarrow\!\shortmid (g, w)$, such that the following two conditions hold:

Identity: for all $w \in \Omega$, we have $1_G w = w$.

Associativity: for all $f, g \in G$ and $w \in \Omega$, we have $(fg)w = f(gw)$, in other words, the expression fgw is unambiguous.

The next diagram illustrates the associativity condition.



As a convenient jargon, given a G-set Ω and an element $g \in G$, we say that g **permutes** the elements of Ω, and we say that g sends an element $w \in \Omega$ to the element $gw \in \Omega$.

A typical example: Recall, again from Chapter 4, that D_8 is the group with order 8 and elements 1, c, c^2, c^3, d, cd, c^2d, c^3d where $c^4 = d^2 = 1$ and $dc = c^3d$. We can view D_8 as the group of rotations and reflections of a square, with c acting as a rotation through 1/4 of a full revolution, d acting as a reflection. Then the set consisting of the 4 vertices of the square is a D_8-set. In an evident way, the set consisting of the 4 edges of the square is also a D_8-set.