What is, and what should be, local structure of symmetries?

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One of the key ideas of 19th century mathematics, emerging gradually over several decades, was that, after starting with an object that has some symmetry, one can usefully forget the object and retain only the abstract symmetry, which is called a group. At present, one of the aims in group theory is to try to forget the group and to retain only its so-called local structure. It is an open question as to quite what the local structure ought to be. We shall be indicating one approach to this.

This account, based on a casual talk, is intended to be accessible to all mathematics undergraduates who have had some experience of the notion of a mathematical object as a set equipped with some structure such as a multiplication operation or a partial ordering relation. My intention is just to give an impressionistic indication of what kind of thing we study and how we study it when doing mathematical research.

1: Groups, and how we wish to forget them

The title of this account is a reference to a celebrated 1888 paper by Dedekind, *Was sind und sollen die Zahlen?* literally translating a little clumsily into English as, *What are, and what should be, the numbers?* That paper was an important step in a revolution that was taking place in pure mathematics. The new idea was that we should pay attention to the structures that arise, and we are free to use the technology of set theory to create expressions of those structures. Thus, the distinction between discovery and invention fades or even disappears.

Two of the paradigms of that new way of thinking were the notion of a group and the notion of a vector space. Both of those notions materialized very slowly over the course of the 19th century.

A vector space, roughly, is a set of points equipped with some accessories that allow us to capture some but not all of the features of physical space.

When symmetries arise in mathematics, they often take the form of a set G which acts on some structured set X in such a way as to preserve the structure. The action is an operation whereby each element $g \in G$ sends each element $x \in X$ to an element $gx \in X$. The structure of G itself comes from that action. One of the fundamental principles of group theory is that we can abstract the structure of G, forgetting about the action of G on X and, in fact, forgetting about X entirely.

We now fix a prime number p. Some of the theorems that one learns about in a first course on group theory, such as Lagrange's Theorem and the three Isomorphism

Theorems, were just emergent along with the development of the notion of a group. People knew those theorems before they knew what a group is. Historically, the first genuine theorem in group theory is Sylow's Theorem, the original version of it obtained in the 1860s (prior to the modern definition of a group). A group contained in G is called a **subgroup** of G. When the size of a subgroup is a power of p, we call it a **p-subgroup** of G. Sylow's Theorem tells us that, when p divides the size of G, there are many p-subgroups. The theorem also tells us something about how G acts as symmetries on the p-subgroups and also on the set of p-subgroups. This was the beginning of what is called **p-local group theory**.

Generally, when studying mathematical objects of some particular kind, one approach is to classify those objects which are simple in the sense that they cannot be broken down into smaller objects. One may also have to examine the way the simple objects can be glued together. Thus, the study becomes something like chemistry, as a study of atoms and the way atoms can be combined to form molecules. The classification of the simple finite groups, arguably the theorem with the longest dedicated proof in all of mathematics, says that the simple finite groups consist of several infinite families, together with 26 extra cases called the sporadic groups. But there remains some doubt over whether that theorem is correct. Efforts continue in the hope of making the theorem and its proof more comprehensible, and p-local group theory is one of the important approaches.

Another motive for p-local theory is that it sometimes specializes in interesting ways for various kinds of group, such as the symmetric groups and groups of Lie type, that frequently appear in various applications in pure mathematics, physics and engineering.

One of the ideals of p-local group theory would be to continue the process of abstraction, that is, to continue the process of forgetting. We have already learned how to forget the object upon which the group acts. The next step must be to forget the group. All that is to be remembered is a thing that expresses the p-local structure of the group. The properties of that p-local thing ought to be the p-local properties of the group. Then, when the time comes for applications, the group can be recollected and, if necessary, the object upon which the group acts can be recollected too.

Arguably, that ideal has already been achieved in the pure theory of finite groups. Fixing a prime p, then there are things called **fusion systems** which are studied in abstract. If, on some later occasion, one does feel inclined to introduces a finite group G, then G has a well-defined fusion system $\mathcal{F}$, and some of the p-local properties of G are indeed properties of $\mathcal{F}$.

An illustration of the approach is given by an early 20th century theorem called the Frobenius Normal Complement Theorem, which says that, when the fusion system of G has a trivial structure, the group G is built up from three parts: a p-subgroup D, a subgroup K whose size is not divisible by p, and an action of D on K.

2: Fusion systems and Puig categories for A_4 and A_5

One of the most fundamental concepts in mathematics is that of a category. In fact, it has sometimes been suggested that category theory has a better right than set theory to be called the foundation of mathematics.

A category consists of:

- some things called the **objects**,
- for any two of the objects X and Y, some things called the **morphisms** to X from Y.

Quite often, the objects are structured sets, and the morphisms are structure-preserving functions between the sets. For example, there is:

- the category of groups, whose morphisms are the group homomorphisms.
- the category of vector spaces over a given field, the morphisms being the linear maps.

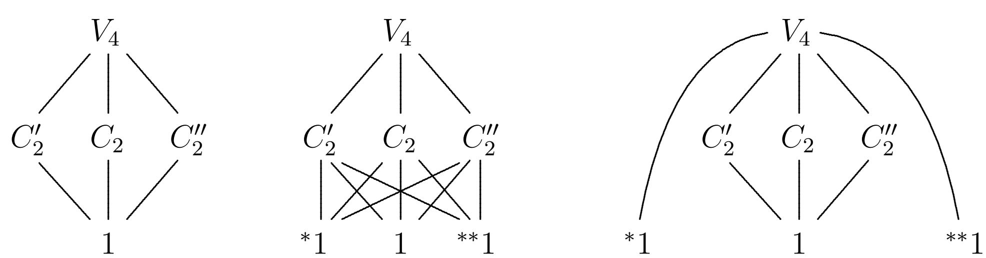
The most important kind of morphism is called an **isomorphism**. When there exists an isomorphism $X \leftarrow Y$, we say that X and Y are **isomorphic**, we understand X and Y to have the same structure and we treat X and Y as theoretically equivalent.

From the perspective of category theory, a major motivation for the notion of a group is that the isomorphisms $X \leftarrow X$, called the **automorphisms** of X, form a group. Indeed, when we say that G acts on X, what we often mean is that there is a homomorphism from G to the automorphism group of X.

Ironically, when category theory was first introduced, in the 1940s, its purpose was to describe situations where the objects under consideration were no longer structured sets, and the morphisms were no longer functions between the objects. Over time, mathematicians have become more and more inventive over the categories that they consider. Or one could say that they have become more perceptive at discovering the categories involved in the topics they study.

The fusion system of a finite group G has, as objects, the subgroups of any chosen and fixed maximal p-subgroup of G. (Part of Sylow's Theorem guarantees that, up to isomorphism, the fusion system is well-defined, independently of the choice.) The morphisms come from the way G expresses symmetry of that structure within itself.

To give an illustration, let us consider the two groups A_4 and A_5, which have sizes 12 and 60, respectively. Generally, A_n can be described as the group of rotational symmetries of a regular n-simplex, that is, a figure consisiting of n equidistant points in a real vector space of dimension $n-1$. Incidentally, A_5 can also be regarded as the group of rotational symmetries of an icosahedron. And A_5 is, famously, the symmetry group of those polynomial equations which are as simple as possible subject to the condition that the solutions cannot be expressed using addition, multiplication, and extraction of square roots, cube roots and higher roots. Despite the diversity of all those realizations of A_5, they are all contexts with the same essential symmetry, and that symmetry is nothing more nor less than A_5, as a group well-defined up to isomorphism.



The diagram on the left shows the organization of the subgroups of a maximal 2-

subgroup of A_4 or A_5. Putting $p = 2$, then A_4 and A_5 have the same fusion system, and the diagram can be viewed as a sketch of that fusion system. The morphisms of the fusion system permute the three objects denoted C_2, C_2', C_2''. Those three objects, by the way, are all isomorphic copies of group called C_2. The larger group V_4 is the group of rotations and reflections of a non-square rectangle.

To explain what the other two diagrams depict, we have to narrow down to my own specialist area, representation theory.

In applications, when a group is expressing the symmetry of an object, it often happens that the object is a vector space. Even if our primary interest is in finite groups, it sometimes helps to reinstate the device whereby a given group is allowed to act on something. When that something is a vector space, the action is called a **representation**, because it allows us to represent the group as a set of linear maps or, if we wish, as a set of matrices.

In the context of p-local group theory, some vector spaces of especial interest are those where the underlying field contains a copy of the integers modulo p. In those vector spaces, a succession of p identical jumps always returns to the starting point.

Fixing a prime p and a group G, taking the vector spaces to be of that kind where p identical jumps have no effect, the category of representations breaks down into separate components called blocks. Fixing a block b, then we have a category of representations associated with the triple (p, G, b). Some of the properties of that category are determined by an associated fusion system.

Alas, the fusion system does not determine the number ℓ of simple objects in that category of representations. To include ℓ in the p-local structure, we can pass to a refinement of the fusion system called the **Puig category**. Here, the objects are no longer subgroups P of some p-subgroup of G, rather, they are pairs (P, γ), where γ is something called a **point**. It turns out that, for the Puig category, ℓ is the number of vertices that appear at the bottom of the diagram. That is to say, ℓ is the number of minimal elements that appear when the Puig category is regarded merely as a partially ordered set.

The middle diagram above and the right-hand diagram, respectively, indicate the Puig categories for certain blocks of A_4 and A_5, again in the case $p = 2$. In both cases, one can read off the diagram that $\ell = 3$.

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Fusion systems were introduced in 1990, by Lluís Puig. In 1992, I had a postdoc hosted by him, and he explained the idea in the context of a conjecture, called Alperin's Conjecture, which still remains open. Recall, ℓ is the number of simple objects in the category of representations associated with (P, G, b). The conjecture, asserting a formula for ℓ, was considered to be a paradigm of p-local group theory.

Yet Puig started his explanation to me by saying "You do know, don't you, that the usual way of expressing Alperin's Conjecture is not really p-local?"

What he meant by that surprising assertion was that, in his own radical view, the p-local structure of a group or a block ought to be determined by just p-subgroups together with only a finite amount of further information. All the previously known formulations of Alperin's conjecture involved entities that had infinite possible variation beyond the specification of the p-subgroups. But the fusion system consists of p-subgroups together with only the finite amount of data specifying the morphisms. Puig found a way of reformulating the conjecture in a way that involved the fusion system and only a finite amount of data beyond that.

The Puig category, despite being a refinement of the fusion system, was actually one of his earlier inventions. Again, it consists of p-subgroups plus only a finite amount of information. As we noted above, one advantage of the Puig category over the fusion system is that it does determine ℓ, indeed, ℓ is the number of minimal objects in the Puig category.

At present, the term "p-local data" is vague. Perhaps when some conjectures such as Alperin's Conjecture have been resolved, we shall finally be in a position to say exactly what the term should mean. But if that conjecture is true, then ℓ must be determined by the p-local data. That raises the suggestion that perhaps the p-local data ought to be deemed to be the Puig category, or something closely related to the Puig category.

There is a snag, though. The Puig category is mysterious. We cannot base our p-local theory on it because we do not, at present, have much of an understanding of it. Those things γ called points, which occur in the pairs (P, γ), are infinite sets, and it is usually a practical impossibility to explicitly describe a single one the elements of a point.

In fact, we do not even have an explicit classification of the points. A conjecture in a recent paper by Matthew Gelvin and me implies a classification of the points, and we have verified the conjecture in certain cases.

3: Fusion systems and Puig categories for S_4 and S_5

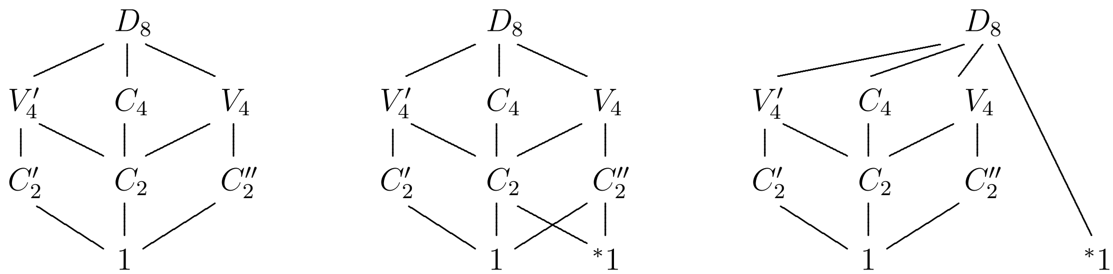
The above diagrams of the Puig categories for some particular blocks of A_4 and A_5 were calculated using a method that is philosophically satisfactory in that it involves only p-subgroups and some further information that has traditionally been considered to be p-local (though not p-local in Puig's stronger sense).

Some years ago, by a less satisfactory method — easier to carry out for small examples but not p-local even in any weak traditional sense — I obtained the following analogous diagrams associated with the groups S_4 and S_5, which have sizes 24 and 120, respectively. In general, S_n is the symmetry group obtained from the rotations and reflections of a regular n-simplex. Just to give a further illustration of the power of isomorphism, S_4 can also be described as the group of rotational symmetries of a cube, and S_4 is also the symmetry group of the most complicated polynomial equations of degree 4.

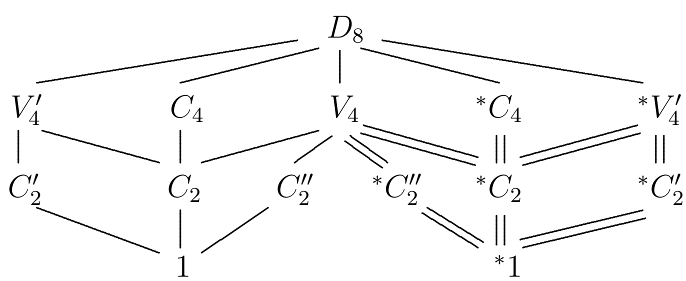
The diagram on the left, below is a picture of the fusion system for a certain block of both S_4 and a certain block of S_5, still with $p = 2$. The largest of the subgroups, D_8, is the group of rotations and reflections of a square. It contains 5 copies of C_2, but that becomes 3 upon treating copies of C_2 as the same when they are isomorphic via a symmetry coming from D_8. (Thus, in a more sophisticated language, the diagram depicts the lattice of conjugacy classes of subgroups of D_8).

The middle diagram below indicates the Puig category for the S_4 block, the right-hand diagram, for the S_5 block. I have not yet finished recovering those diagrams using the p-local method.

The two diagrams of the Puig categories show that, for these two cases, the number of minimal objects is $\ell = 2$.



Actually, both methods of calculation require the introduction of a larger category with some further objects. Part of the structure of this larger category comes from something like a partial ordering, but with a multiplicity associated with every pair of objects X and Y such that $X < Y$. The next diagram depicts that larger category for S_5, with the single lines indicating multiplicity 1 and the double lines indicating multiplicity 2.



If these diagrams fill you with horror, and if you feel you would prefer just to prove elegant general theorems using elegant general arguments, then I have some bad news for you. General arguments tend to come at the end of the process. First one has to guess the theorems. Not always, but often, the guesses have to be gleaned from particular cases or from messy special cases.

To calculate the multiplicities, part of the process involves inverting some matrices. Thus, packages of multiplicities are all worked out simultaneously. The features of the matrices involved are quite remeniscent of a scenario in combinatorics called Möbius inversion. As yet, I am unable to use the theory of Möbius inversion here, because I lack a formula for one aspect of that scenario, called the Möbius function. (You may have encountered the original Möbius function μ in classical number theory: $\mu(n) = 0$ if n is divisible by a square, otherwise $\mu(n) = (-1)^r$, where r is the number of prime fctors of n.) But the resemblance tells me that I ought to be looking for some such formula.

Thus, sure enough, the calculations are something of a chore. But what makes that task interesting is that it affords glimpses of some coherent organization underneath. When one succeeds in nailing down something of that organization, when one can characterize something of the underlying structure, then what one has is a theorem.