

Rudimentary Linear Algebra Exercises and Solutions

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In its present form, these notes are supplementary material for the course

MATH 225 Linear Algebra and Differential Equations

However, a version of the notes was originally produced as supplementary material for a much easier course. So, as regards MATH 225, they may be useful only as a preliminary, prior to tackling more difficult exercises. A collection of more difficult exercises can be found on the course Moodle page.

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1: Gaussian elimination and Gauss–Jordan elimination

Question 1.1: Find all the solutions to

$$2x + y + z = 7, \quad 4x + 3y + z = 13, \quad 6x - y - 2z = 7,$$

- (a) Using Gaussian elimination,
- (b) Using Gauss–Jordan elimination.

Question 1.2: Find all the solutions to

$$2x + y + z = 7, \quad 4x + 3y + z = 13, \quad 8x + 5y + 3z = 27,$$

- (a) Using Gaussian elimination,
- (b) Using Gauss–Jordan elimination.

Question 1.3: By any method, find all the solutions to

$$3x + 2y + z = 14, \quad x + 6y + 5z = 20, \quad 9x + 22y + 17z = 88.$$

Question 1.4: Find all the real numbers a, b, c, d such that

$$a + b + c + d = 10, \quad 2a + 3b + 4c + 5d = 40, \quad 4a + 5b + 6c + 7d = 60.$$

Solutions

1.1: Part (a). The augmented matrix is $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 7 \\ 4 & 3 & 1 & 13 \\ 6 & -1 & -2 & 7 \end{array} \right]$.

Subtracting multiples of row 1 from rows 2 and 3 gives $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 7 \\ 0 & 1 & -1 & -1 \\ 0 & -4 & -5 & -14 \end{array} \right]$.

Adding 4 times row 2 to row 3 gives $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 7 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -9 & -18 \end{array} \right]$.

Multiplying rows 1 and 3 by factors, $\left[\begin{array}{ccc|c} 1 & 1/2 & 1/2 & 7/2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{array} \right]$.

The equations are $z = 2$ and $y = z - 1 = 1$ and $x = 7/2 - y/2 - z/2 = 7/2 - 1/2 - 1 = 2$. In conclusion, $(x, y, z) = (2, 1, 2)$.

Part (b). Subtracting multiples of row 3 from rows 1 and 2 gives $\left[\begin{array}{ccc|c} 1 & 1/2 & 0 & 5/2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array} \right]$.

Subtracting half of row 2 from row 3 gives $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array} \right]$.

Thus, again, $(x, y, z) = (2, 1, 2)$.

1.2: Part (a). We code the problem as $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 7 \\ 4 & 3 & 1 & 13 \\ 8 & 5 & 3 & 27 \end{array} \right]$.

Subtracting multiples of row 1 from rows 2 and 3 gives $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 7 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & -1 \end{array} \right]$.

Subtracting row 2 from row 3 gives $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 7 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right]$.

Dividing row 1 by 2 gives $\left[\begin{array}{ccc|c} 1 & 1/2 & 1/2 & 7/2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right]$.

Introducing a parameter r , we put $z = r$. Then $y = -1 + z = r - 1$ and

$$x = 7/2 - y/2 - z/2 = 7/2 - (r - 1)/2 - r/2 = 4 - r.$$

Hence $(x, y, z) = (4 - r, r - 1, r)$ where r is a parameter.

Part (b). Subtracting half of row 2 from row 1 gives $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right]$.

Hence, again, $(x, y, z) = (4 - r, r - 1, r)$.

1.3: The augmented matrix is $\left[\begin{array}{ccc|c} 3 & 2 & 1 & 14 \\ 1 & 6 & 5 & 20 \\ 9 & 22 & 17 & 88 \end{array} \right]$.

Interchanging rows 1 and 2 gives $\left[\begin{array}{ccc|c} 1 & 6 & 5 & 20 \\ 3 & 2 & 1 & 14 \\ 9 & 22 & 17 & 88 \end{array} \right]$.

Subtracting multiples of row 1 from the others, $\left[\begin{array}{ccc|c} 1 & 6 & 5 & 20 \\ 0 & -16 & -14 & -46 \\ 0 & -32 & -28 & -92 \end{array} \right]$.

Dividing row 2 by -16 gives $\left[\begin{array}{ccc|c} 1 & 6 & 5 & 20 \\ 0 & 1 & 7/8 & 23/8 \\ 0 & -32 & -28 & -92 \end{array} \right]$.

Adding 32 times row 2 to row 3 gives $\left[\begin{array}{ccc|c} 1 & 6 & 5 & 20 \\ 0 & 1 & 7/8 & 23/8 \\ 0 & 0 & 0 & 0 \end{array} \right]$.

Subtracting 6 times row 2 from row 1 gives $\left[\begin{array}{ccc|c} 1 & 0 & -1/4 & 11/4 \\ 0 & 1 & 7/8 & 23/8 \\ 0 & 0 & 0 & 0 \end{array} \right]$.

The equations are $x - z/4 = 11/4$ and $y + 7z/8 = 23/8$. Introducing a parameter s , we can put $z = s$. The solutions are

$$(x, y, z) = (s/4 + 11/4, -7s/8 + 23/8, s)$$

where s is any real number.

Comment: That was by Gauss–Jordan elimination. The Gaussian method would be to omit the last row operation, and to write down the equations $x + 6y + 5z = 20$ and $y + 7z/8 = 23/8$. Again putting $z = s$, we would immediately obtain $y = -7s/8 + 23/8$. Then we would calculate x from

$$x = -6y - 5z + 20 = -6 \left(\frac{-7s}{8} + \frac{23}{8} \right) - 5s + 20 = \frac{6.7 - 5.8}{8}s + \frac{-6.28 + 20.8}{8} = \frac{s}{4} + \frac{11}{4}.$$

1.4: The augmented matrix is $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 10 \\ 2 & 3 & 4 & 5 & 40 \\ 4 & 5 & 6 & 7 & 60 \end{array} \right]$.

Subtracting multiples of row 1 from the other rows gives $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 10 \\ 0 & 1 & 2 & 3 & 20 \\ 0 & 1 & 2 & 3 & 20 \end{array} \right]$.

Subtracting row 2 from the other rows, we obtain $\left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & -10 \\ 0 & 1 & 2 & 3 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$.

The linear system has been reduced to

$$a = c + 2d - 10, \quad b = -2c - 3d + 20.$$

So the solutions are

$$(a, b, c, d) = (c + 2d - 10, -2c - 3d + 20, c, d)$$

where c and d can be any real numbers.

2: Inverting matrices by Gauss–Jordan elimination

Question 2.1: By Gauss–Jordan elimination, find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 1 \\ 6 & -1 & -2 \end{bmatrix}.$$

Question 2.2: Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 16 \end{bmatrix}$. Evaluate A^{-1} by Gauss–Jordan elimination.

Question 2.3: By Gauss–Jordan elimination, find the inverse of $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \end{bmatrix}$.

Solutions

2.1: We set up the problem as $\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & 1 & 0 & 1 & 0 \\ 6 & -1 & -2 & 0 & 0 & 1 \end{array} \right]$.

Subtracting multiples of row 1 from the other two rows, $\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & -4 & -5 & -3 & 0 & 1 \end{array} \right]$.

Adding 4 times row 2 to row 3 gives $\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -9 & -11 & 4 & 1 \end{array} \right]$.

Dividing row 3 by -9 gives $\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 11/9 & -4/9 & -1/9 \end{array} \right]$.

Adding multiples of row 3 to the other two rows, $\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & -2/9 & 4/9 & 1/9 \\ 0 & 1 & 0 & -7/9 & 5/9 & -1/9 \\ 0 & 0 & 1 & 11/9 & -4/9 & -1/9 \end{array} \right]$.

Subtracting row 2 from row 1 yields $\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 5/9 & -1/9 & 2/9 \\ 0 & 1 & 0 & -7/9 & 5/9 & -1/9 \\ 0 & 0 & 1 & 11/9 & -4/9 & -1/9 \end{array} \right]$.

Finally, dividing row 1 by 2 gives $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5/18 & -1/18 & 2/18 \\ 0 & 1 & 0 & -7/9 & 5/9 & -1/9 \\ 0 & 0 & 1 & 11/9 & -4/9 & -1/9 \end{array} \right]$.

So $A^{-1} = \frac{1}{18} \begin{bmatrix} 5 & -1 & 2 \\ -14 & 10 & -2 \\ 22 & -8 & -2 \end{bmatrix}$.

2.2: We set up the problem as $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 1 & 0 \\ 1 & 4 & 16 & 0 & 0 & 1 \end{array} \right]$.

Subtracting row 1 from rows 2 and 3 gives $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 3 & 15 & -1 & 0 & 1 \end{array} \right]$.

Subtracting 3 times row 2 from row 3 gives $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 6 & 2 & -3 & 1 \end{array} \right]$.

Dividing row 3 by 6 gives $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1/3 & -1/2 & 1/6 \end{array} \right]$.

Subtracting multiples of row 3 from rows 1 and 2 gives $\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 2/3 & 1/2 & -1/6 \\ 0 & 1 & 0 & -2 & 5/2 & -1/2 \\ 0 & 0 & 1 & 1/3 & -1/2 & 1/6 \end{array} \right]$.

Subtracting row 2 from row 1 gives $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8/3 & -2 & 1/3 \\ 0 & 1 & 0 & -2 & 5/2 & -1/2 \\ 0 & 0 & 1 & 1/3 & -1/2 & 1/6 \end{array} \right]$.

Therefore, $A^{-1} = \frac{1}{6} \begin{bmatrix} 16 & -12 & 2 \\ -12 & 15 & -3 \\ 2 & -3 & 1 \end{bmatrix}$.

2.3: We set up the problem as $\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 3 & 9 & 0 & 0 & 1 \end{array} \right]$.

Subtracting multiples of row 1 from the other two rows, we obtain $\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 4 & 8 & -1 & 0 & 1 \end{array} \right]$.

Dividing row 2 by 2 gives $\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 4 & 8 & -1 & 0 & 1 \end{array} \right]$.

Subtracting 4 times row 2 from row 3 gives $\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 8 & 1 & -2 & 1 \end{array} \right]$.

Dividing row 3 by 8 gives $\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 1/8 & -1/4 & 1/8 \end{array} \right]$.

Subtracting row 3 from row 1 gives $\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 7/8 & 1/4 & -1/8 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 1/8 & -1/4 & 1/8 \end{array} \right]$.

Adding row 2 to row 1 gives $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/8 & 3/4 & -1/8 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 1/8 & -1/4 & 1/8 \end{array} \right]$.

We obtain $A^{-1} = \frac{1}{8} \begin{bmatrix} 3 & 6 & -1 \\ -4 & 4 & 0 \\ 1 & -2 & 1 \end{bmatrix}$.

3: Evaluating determinants

Question 3.1: Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 1 \\ 6 & -1 & -2 \end{bmatrix}$.

- (a) Evaluate $\det(A)$ by calculating suitable cofactors.
- (b) Evaluate $\det(A)$ using row or column operations.

Solutions

3.1: Part (a). We have

$$\begin{aligned}\det(A) &= \begin{vmatrix} 2 & 1 & 1 \\ 4 & 3 & 1 \\ 6 & -1 & -2 \end{vmatrix} = 2 \begin{vmatrix} 3 & 1 \\ -1 & -2 \end{vmatrix} - \begin{vmatrix} 4 & 1 \\ 6 & -2 \end{vmatrix} + \begin{vmatrix} 4 & 3 \\ 6 & -1 \end{vmatrix} \\ &= 2(-6 + 1) - (-8 - 6) + (-4 - 18) = -10 + 14 - 22 = -18 .\end{aligned}$$

Part (b). Adding multiples of row 1 from the other rows, $\det(A) = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & -4 & -5 \end{vmatrix}.$

Adding 4 times row 2 to row 3 gives $\det(A) = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -9 \end{vmatrix} = 2 \cdot 1 \cdot (-9) = -18.$

4: Inverting matrices by the method of minors and cofactors

Question 4.1: Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 1 \\ 6 & -1 & -2 \end{bmatrix}$.

(a) By the method of minors and cofactors, evaluate A^{-1} .

(b) Using A^{-1} , give another method for solving the linear system in Question 1.1.

Question 4.2: Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$. By the method of minors and cofactors, evaluate A^{-1} .

Question 4.3: By the method of minors and cofactors, find the inverse of $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \end{bmatrix}$.

Solutions

4.1: The matrix of minors is $\begin{bmatrix} -6+1 & -8-6 & -4-18 \\ -2+1 & -4-6 & -2-6 \\ 1-3 & 2-4 & 6-4 \end{bmatrix} = \begin{bmatrix} -5 & -14 & -22 \\ -1 & -10 & -8 \\ -2 & -2 & 2 \end{bmatrix}$.

The matrix of cofactors is $\begin{bmatrix} -5 & 14 & -22 \\ 1 & -10 & 8 \\ -2 & 2 & 2 \end{bmatrix}$.

The adjugate matrix is $\text{adj}(A) = \begin{bmatrix} -5 & 1 & 2 \\ 14 & -2 & 2 \\ -22 & 8 & 2 \end{bmatrix}$. We have $\det(A) = 2(-5) + 14 - 22 = -18$

from the (1, 1) entry of the equality $A \text{adj}(A) = \det(A)I$. So $A^{-1} = \frac{1}{18} \begin{bmatrix} 5 & -1 & 2 \\ -14 & 10 & -2 \\ 22 & -8 & -2 \end{bmatrix}$.

Part (b). We have $A^{-1} \begin{bmatrix} 7 \\ 13 \\ 7 \end{bmatrix} = \frac{1}{18} \begin{bmatrix} 35 - 13 + 14 \\ -98 + 130 - 14 \\ 154 - 104 - 14 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

4.2: The matrix of minors is $\begin{bmatrix} 2.9-4.3 & 1.9-4.1 & 1.3-2.1 \\ 1.9-1.3 & 1.9-1.1 & 1.3-1.1 \\ 1.4-1.2 & 1.4-1.1 & 1.2-1.1 \end{bmatrix} = \begin{bmatrix} 6 & 5 & 1 \\ 6 & 8 & 2 \\ 2 & 3 & 1 \end{bmatrix}$.

Taking the transpose and introducing alternating $\pm$ signs, $\text{adj}(A) = \begin{bmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{bmatrix}$.

We have $\det(A) = 1.6 + 1(-5) + 1.1 = 2$. So $A^{-1} = \frac{\text{adj}(A)}{\det(A)} = \frac{1}{2} \begin{bmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{bmatrix}$.

4.3: The matrix of minors for A is $\begin{bmatrix} 9-3 & 9-1 & 3-1 \\ -9-3 & 9-1 & 3+1 \\ -1-1 & 1-1 & 1+1 \end{bmatrix} = \begin{bmatrix} 6 & 8 & 2 \\ -12 & 8 & 4 \\ -2 & 0 & 2 \end{bmatrix}$.

Hence, $\text{adj}(A) = \begin{bmatrix} 6 & 12 & -2 \\ -8 & 8 & 0 \\ 2 & -4 & 2 \end{bmatrix}$.

From the (1, 1) entry of the equality $A \text{adj}(A) = \det(A)I$, we have $\det(A) = 6 + 8 + 2 = 16$. Therefore, $A^{-1} = \text{adj}(A)/16$, and we obtain the same value of A^{-1} as in part (a).

5: Rank and nullity

Question 5.1: Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 4 & 5 & 6 & 7 \end{bmatrix}$.

- (a) Find a basis for the kernel of A . (The kernel of a matrix is also called the null space.)
- (b) Evaluate the rank and nullity of A .

Question 5.2: Let $A = \begin{bmatrix} 2 & 1 & 3 & 2 \\ 3 & 4 & 4 & 5 \\ 7 & 6 & 10 & 9 \end{bmatrix}$.

- (a) Find a basis for the image space $\text{Im}(A)$.
- (b) What are the rank and nullity of A ?

5.3: Let $M = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 4 & 3 & 2 & 1 \\ 1 & 3 & 5 & 7 \end{bmatrix}$.

- (a) Find a basis for the image of M .
- (b) Find a basis for the null space of M .
- (c) What are the rank and nullity of M ?

Solutions

5.1: Part (a). Subtracting multiples of row 1 from the other rows gives $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix}$.

Subtracting row 2 from the other rows, we obtain $\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

So $(a, b, c, d) \in \ker(A)$ if and only if $a = c + 2d$ and $b = -2c - 3d$, in other words,

$$(a, b, c, d) = (c + 2d - 10, -2c - 3d + 20, c, d) = c(1, -2, 1, 0) + d(2, -3, 0, 1).$$

We now see that $\ker(A)$ has basis

$$\{(1, -2, 1, 0), (2, -3, 0, 1)\},$$

indeed, we have shown the spanning property, and the linear independence is clear by considering the 3rd and 4th coordinates.

Part (b). By part (a), $\text{null}(A) = 2$. By the rank-nullity formula, $\text{rank}(A) = 4 - \text{null}(A) = 2$.

5.2: Part (a). Rearranging the columns, we obtain $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 4 & 3 & 4 & 5 \\ 6 & 7 & 10 & 9 \end{bmatrix}$.

Subtracting multiples of column 1 from the other columns, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & -5 & -8 & -3 \\ 6 & -5 & -8 & -3 \end{bmatrix}$.

Multiplying the columns by scalars, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 1 & 1 \\ 6 & 1 & 1 & 1 \end{bmatrix}$.

Subtracting column 2 from columns 3 and 4 yields $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 6 & 1 & 0 & 0 \end{bmatrix}$.

The latest matrix is in column echelon form, so a basis for $\text{Im}(A)$ is $\left\{ \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Part (b). By part (a), $\text{rank}(A) = 2$. By the rank-nullity formula, $\text{null}(A) = 4 - 2 = 2$.

5.3:Part (a). Subtracting multiples of the first column from the others, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & -2 & -3 \\ 4 & -5 & -10 & -15 \\ 1 & 1 & 2 & 3 \end{bmatrix}$.

Further column operations yield $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 4 & 5 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$.

which is in column echelon form. It follows that $\text{Im}(M)$ has basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 5 \\ -1 \end{bmatrix} \right\}$.

Part (b). We obtain $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -5 & -10 & -15 \\ 0 & 1 & 2 & 3 \end{bmatrix}$ by subtracting multiples of row 1.

Further row operations yield $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, then $\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

The null space of A consists of the vectors (x_1, x_2, x_3, x_4) such that

$$x_1 - x_3 - 2x_4 = x_2 + 2x_3 + 3x_4 = 0 .$$

Introducing parameters s and t , putting $x_3 = s$ and $x_4 = t$, then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} s + 2t \\ -2s - 3t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} .$$

So the null space of A has basis $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Part (d). Evidently, $\text{rank}(A) = \text{null}(A) = 2$.

6a: Gram–Schmidt orthogonalization process, theory

The Gram-Schmidt process is as follows. Let V be a real or complex inner product space. Let $\{u_1, u_2, \dots\}$ be a finite or infinite linearly independent set in V .

- We can construct an **orthogonal set** $\{v_1, v_2, \dots\}$ in V as follows. We put

$$v_1 = u_1 .$$

For $m \geq 2$, we put

$$v_m = u_m - \sum_{j=1}^{m-1} \frac{\langle u_m | u_j \rangle}{\|u_j\|^2} u_j .$$

Thus, in particular,

$$v_2 = u_2 - \frac{\langle u_2 | u_1 \rangle}{\|u_1\|^2} u_1 , \quad v_3 = u_3 - \frac{\langle u_3 | u_1 \rangle}{\|u_1\|^2} u_1 - \frac{\langle u_3 | u_2 \rangle}{\|u_2\|^2} u_2 .$$

- We can construct an **orthonormal set** $\{w_1, w_2, \dots\}$ in V by putting $w_n = v_n / \|v_n\|$ for all n .

Note that, if we calculate in the order v_1, w_1, v_2, w_2 , then the formula for v_m can be written more simply as

$$v_m = u_m - \sum_{j=1}^{m-1} \langle v_m | w_j \rangle w_j .$$

As a little calculating trick, we mention that, given any nonzero vector v and a positive real number c , then the normalization of v is

$$v / \|v\| = cv / \|cv\| .$$

Similarly, a term in the formula for v_m satisfies

$$\frac{\langle u_m | u_j \rangle}{\|u_j\|^2} u_j = \frac{\langle u_m | cu_j \rangle}{\|cu_j\|^2} cu_j .$$

Thus, replacing any v_n with any positive scalar multiple of v_n will not change the calculated orthonormal set $\{w_1, \dots\}$.

6b: Gram–Schmidt orthogonalization process, the exercises

Question 6.1: Let $u_1 = (1, -1, -1)$ and $u_2 = (-1, 1, -1)$ and $u_3 = (-1, -1, 1)$ in $\mathbb{R}^3$. Let $\{w_1, w_2, w_3\}$ be the orthonormal basis for $\mathbb{R}^3$ obtained from $\{u_1, u_2, u_3\}$ by the Gram–Schmidt process. Evaluate w_1 and w_2 and w_3 .

Question 6.2: Let $e_1 = (1, 1, 1)$ and $e_2 = (1, 2, 0)$ and $e_3 = (0, 1, 2)$. Let $\{f_1, f_2, f_3\}$ be the orthonormal basis for $\mathbb{R}^3$ obtained from $\{e_1, e_2, e_3\}$ by the Gram–Schmidt process. Evaluate f_1 and f_2 and f_3 .

Question 6.3: Find an orthogonal basis for the real vector space P_2 of polynomials of degree at most 2 with the inner product

$$(p, q) = \int_0^2 p(t)q(t) \, dt .$$

Question 6.4: Let V be the vector space consisting of the functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that there exist real numbers a, b, c satisfying $f(x) = ax^4 + bx^3 + cx^2$. We make V become an inner product space with inner product given by

$$(f, g) = \int_{-1}^1 f(x)g(x) \, dx .$$

Apply the Gram–Schmidt process to find an orthogonal basis for V starting from the basis $\{u_1, u_2, u_3\}$ where $u_1(x) = x^2$ and $u_2(x) = x^3$ and $u_3(x) = x^4$.

Solutions

6.1: Recall, $w_k = v_k / \|v_k\|$ where $v_1 = u_1$ and, generally,

$$v_k = u_k - \langle w_1 | u_k \rangle w_1 - \dots - \langle w_{k-1} | u_k \rangle w_{k-1} .$$

We have $v_1 = (1, -1, -1)$. So $w_1 = \frac{1}{\sqrt{3}}(1, -1, -1)$. We have

$$\begin{aligned} v_2 &= (-1, 1, -1) - \frac{1}{3}((1, -1, -1) \cdot (-1, 1, -1))(1, -1, -1) \\ &= (-1, 1, -1) + \frac{1}{3}(1, -1, -1) = \frac{2}{3}(-1, 1, -2) . \end{aligned}$$

So $w_2 = \frac{(-1, 1, -2)}{\|(-1, 1, -2)\|} = \frac{1}{\sqrt{6}}(-1, 1, -2)$. We have

$$\begin{aligned} v_3 &= u_3 - \frac{1}{3}((1, -1, -1) \cdot (-1, -1, 1))(1, -1, -1) - \frac{1}{6}((-1, 1, -2) \cdot (-1, -1, 1))(-1, 1, -2) \\ &= (-1, -1, 1) + \frac{1}{3}(1, -1, -1) + \frac{1}{3}(-1, 1, -2) = (-1, -1, 0) . \end{aligned}$$

So $w_3 = \frac{(-1, -1, 0)}{\|(-1, -1, 0)\|} = \frac{1}{\sqrt{2}}(-1, -1, 0)$. In conclusion,

$$w_1 = \frac{1}{\sqrt{3}}(1, -1, -1) , \quad w_2 = \frac{1}{\sqrt{6}}(-1, 1, -2) , \quad w_3 = \frac{1}{\sqrt{2}}(-1, -1, 0) .$$

6.2: Put $f_1 = e_1 / \|e_1\| = (1, 1, 1) / \sqrt{3}$. Let $f'_2 = e_2 - \langle f_1 | e_2 \rangle f_1$. Then

$$f'_2 = (1, 2, 0) - \frac{1}{3}((1, 1, 1) \cdot (1, 2, 0))(1, 1, 1) = (1, 2, 0) - (1, 1, 1) = (0, 1, -1) .$$

Put $f_2 = f'_2 / \|f'_2\| = (0, 1, -1) / \sqrt{2}$. Let $f'_3 = e_3 - \langle f_1 | e_3 \rangle f_1 - \langle f_2 | e_3 \rangle f_2$. Then

$$\begin{aligned} f'_3 &= (0, 1, 2) - \frac{1}{3}((1, 1, 1) \cdot (0, 1, 2))(1, 1, 1) - \frac{1}{2}((0, 1, -1) \cdot (0, 1, 2))(0, 1, -1) \\ &= (0, 1, 2) - (1, 1, 1) + \frac{1}{2}(0, 1, -1) = \frac{1}{2}(-2, 1, 1) . \end{aligned}$$

Put $f_3 = f'_3 / \|f'_3\| = (-2, 1, 1) / \sqrt{6}$. In conclusion,

$$f_1 = (1, 1, 1) / \sqrt{3} , \quad f_2 = (0, 1, -1) / \sqrt{2} , \quad f_3 = (-2, 1, 1) / \sqrt{6} .$$

6.3: Putting $u_0(t) = 1$ and $u_1(t) = t - 1$ and $u_2(t) = (t - 1)^2$, we shall apply the Gram-Schmidt process to the basis $\{u_0, u_1, u_2\}$ of P_2 , obtaining an orthonormal basis $\{v_0, v_1, v_2\}$. The formula to be used is $v_0 = u_0$ and

$$v_i = u_i - \frac{(v_{i-1}, u_i)}{\|v_{i-1}\|^2} v_{i-1} - \dots - \frac{(v_0, u_i)}{\|v_0\|^2} v_0 .$$

Thus, $v_0(t) = u_0(t) = 1$. We remark that $\int_{-1}^1 f(t) dt = 0$ for any integrable $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfying $f(t) = -f(t)$. In particular, $(v_0, u_1) = \int_0^2 (t-1) dt = \int_{-1}^1 s ds = 0$. So $v_1(t) = u_1(t) = t-1$. Similarly, $(v_1, u_2) = 0$. Since $(v_0, u_2) = \int_0^2 (t-1)^2 dt = (t-1)^3/3|_0^2 = 2/3$ and $\|v_0\|^2 = \int_0^2 dt = 2$, we have

$$v_2(t) = u_2(t) - \frac{(v_1, u_2)}{\|v_1\|^2} v_1 - \frac{(v_0, u_2)}{\|v_0\|^2} v_0 = (t-1)^3 - 1/3 = t^3 - 3t^2 + 2t - 1/3.$$

So an orthogonal basis for P_2 is $\{1, t-1, t^3 - 3t^2 + 2t - 1/3\}$ as a set of functions of t .

Comment: A more obvious choice of basis elements is with $u_0(t) = 1$ and $u_1(t) = t$ and $u_2(t) = t^2$. Then the calculations are a little more complicated but the obtained orthogonal basis is the same.

6.4: We have

$$v_1(x) = u_1(x), \quad v_2(x) = u_2(x) - \frac{\langle v_1 | u_2 \rangle}{\|v_1\|^2} v_1(x), \quad v_3(x) = u_3(x) - \frac{\langle v_1 | u_3 \rangle}{\|v_1\|^2} v_1(x) - \frac{\langle v_2 | u_3 \rangle}{\|v_2\|^2} v_2(x).$$

Trivially, $v_1(x) = x^2$. For n odd, $\int_{-1}^1 x^n dx = 0$ while, for n even, $\int_{-1}^1 x^n dx = \frac{2}{n+1}$. We have

$$\langle v_1 | u_2 \rangle = \int_{-1}^1 x^5 dx = 0. \quad \text{Therefore, } v_2(x) = u_2(x) = x^3. \quad \text{We have } \langle v_2 | u_3 \rangle = \int_{-1}^1 x^7 dx = 0.$$

Also, $\langle v_1 | u_3 \rangle = \int_{-1}^1 x^6 dx = 2/7$ and $\|v_1\|^2 = \int_{-1}^1 x^4 dx = 2/5$. So $\frac{\langle v_1 | u_3 \rangle}{\|v_1\|^2} = 5/7$, whence $v_3(x) = x^4 - 5x^2/7$. In conclusion,

$$v_1(x) = x^2, \quad v_2(x) = x^3, \quad v_3(x) = x^4 - 5x^2/7.$$

7: Diagonalization

Question 7.1: Let $B = \begin{bmatrix} 3 & 0 & 0 \\ 6 & -6 & 6 \\ 8 & -12 & 11 \end{bmatrix}$.

Find invertible P and diagonal D such that $B = PDP^{-1}$.

Question 7.2: Diagonalize the matrix $A = \begin{bmatrix} 4 & 0 & 3 \\ -4 & 1 & -4 \\ -2 & 0 & -1 \end{bmatrix}$.

Question 7.3: Let $A = \begin{bmatrix} -7 & -4 & 6 \\ -9 & -2 & 6 \\ -19 & -9 & 15 \end{bmatrix}$.

Note, you do not need to consider the characteristic equation in this question.

(a) Show that $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 3 \\ 7 \end{bmatrix}$ are eigenvectors of A , and find their corresponding eigenvalues.

(b) Let $P = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 2 & 5 & 7 \end{bmatrix}$. Find a diagonal matrix D such that $A = PDP^{-1}$.

(c) By any method, evaluate P^{-1} .

(d) Using parts (b) and (c), give a formula for the entry in row 2 and column 2 of A^n .

Question 7.4: Let $A = \begin{bmatrix} 1 & -8 & 4 \\ -8 & 1 & 4 \\ 4 & 4 & 7 \end{bmatrix}$.

(a) Show that the characteristic equation for A is $\lambda^3 - 9\lambda^2 - 9\lambda + 9 = 0$.

(b) Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Question 7.5 Find all the eigenvalues and all the eigenvectors for the matrix

$$\begin{bmatrix} i & i \\ i & i \end{bmatrix}$$

where i denotes a square root of -1 . (By convention, we do not consider the zero vector $(0, 0)$ to be an eigenvector.)

Solutions

7.1: The characteristic equation is

$$0 = \det(\lambda I - B) = \begin{vmatrix} \lambda - 3 & 0 & 0 \\ -6 & \lambda + 6 & 0 \\ -8 & 12 & \lambda - 11 \end{vmatrix} = (\lambda - 3) \begin{vmatrix} \lambda + 6 & -6 \\ 12 & \lambda - 11 \end{vmatrix}$$

$$= (\lambda - 3)((\lambda + 6)(\lambda - 11) + 72) = (\lambda - 3)(\lambda^2 - 5\lambda + 6) = (\lambda - 2)(\lambda - 3)^2.$$

So the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = \lambda_3 = 3$. We are to find a basis $\{e_1, e_2, e_3\}$ such that $Be_i = \lambda_i e_i$.

Write $e_1 = (x, y, z)$. We have

$$3x = 2x, \quad 6x - 6y + 6z = 2y, \quad 8x - 12y + 11z = 2z.$$

Hence $x = 0$ and $3z = 4y$. Putting $y = 3$, then $z = 4$. Thus, $e_1 = (0, 3, 4)$.

Now write $e_2 = (x, y, z)$. We have

$$3x = 3x, \quad 6x - 6y + 6z = 3y, \quad 8x - 12y + 11z = 3z.$$

Hence $2x + 2z = 3y$. Putting $x = 0$ and $z = 3$, then $y = 2$. Thus, $e_2 = (0, 2, 3)$.

Finally, write $e_3 = (x, y, z)$. The equations for x, y, z are the same as for e_2 . Putting $x = 1$ and $z = 2$, then $y = 2$. Thus, $e_3 = (1, 2, 2)$. We have

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 2 & 2 \\ 4 & 3 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

7.2: The characteristic equation is

$$\det(A - \lambda I) = \begin{vmatrix} 4 - \lambda & 0 & 3 \\ -4 & 1 - \lambda & -4 \\ -2 & 0 & -1 - \lambda \end{vmatrix} = (4 - \lambda)(1 - \lambda)(-1 - \lambda) + 6(1 - \lambda)$$

$$= (1 - \lambda)(2 - 3\lambda + \lambda^2) = (1 - \lambda)^2(2 - \lambda).$$

So the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 1$ and $\lambda_3 = 2$. We shall find corresponding eigenvectors f_1 and f_2 and f_3 , respectively. Any eigenvector (x, y, z) with eigenvalue 1 satisfies

$$\begin{bmatrix} 3 & 0 & 3 \\ -4 & 0 & -4 \\ -2 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

So we can put $f_1 = (0, 1, 0)$ and $f_2 = (1, 0, -1)$. Meanwhile, any eigenvector (x, y, z) with eigenvalue 2 satisfies

$$\begin{bmatrix} 2 & 0 & 3 \\ -4 & -1 & -4 \\ -2 & 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

So we can put $f_3 = (-3, 4, 2)$. Therefore, a diagonalizing basis is $\{(0, 1, 0), (1, 0, -1), (-3, 4, 2)\}$, furthermore, $A = PDP^{-1}$, where

$$P = \begin{bmatrix} 0 & 1 & -3 \\ 1 & 0 & 4 \\ 0 & -1 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Comment: This was an exam question for MATH 220. Many examinees were unable to deal with a repeated eigenvalue. For the eigenvalue 1, one has to choose 2 linearly independent eigenvectors. And the zero vector $(0, 0, 0)$ is not an eigenvector. If your purported invertible matrix P has a column whose entries are all 0 then, of course, your P cannot be invertible.

7.3: Part (a). We have

$$\begin{aligned} A \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} &= \begin{bmatrix} -7 - 4 + 12 \\ -9 - 2 + 12 \\ -19 - 9 + 30 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \\ A \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} &= \begin{bmatrix} -14 - 12 + 30 \\ -18 - 6 + 30 \\ -38 - 27 + 74 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 10 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \\ A \begin{bmatrix} 3 \\ 3 \\ 7 \end{bmatrix} &= \begin{bmatrix} -21 - 12 + 42 \\ -27 - 6 + 42 \\ -37 - 27 + 105 \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ 21 \end{bmatrix} = 3 \begin{bmatrix} 3 \\ 3 \\ 7 \end{bmatrix}. \end{aligned}$$

So the three specified vectors are eigenvectors with eigenvalues 1, 2, 3, respectively.

Part (b). By part (a), $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

Part (c). We set up the problem as $\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 3 & 3 & 0 & 1 & 0 \\ 2 & 5 & 7 & 0 & 0 & 1 \end{array} \right]$.

Subtracting multiples of row 1 from the other rows, $\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 & 1 \end{array} \right]$.

Subtracting row 2 from row 3 gives $\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array} \right]$.

Subtracting 3 times row 3 from row 1 gives $\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 4 & 3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array} \right]$.

Subtracting 2 times row 2 from row 1 gives $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & 1 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array} \right]$.

So $P^{-1} = \begin{bmatrix} 6 & 1 & -3 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$.

Part (d). We have $A^n = PD^nP^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 2 & 5 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{bmatrix} \begin{bmatrix} 6 & 1 & -3 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 2 \cdot 2^n & 3 \cdot 3^n \\ 1 & 3 \cdot 2^n & 3 \cdot 3^n \\ 2 & 5 \cdot 2^n & 7 \cdot 3^n \end{bmatrix} \begin{bmatrix} 6 & 1 & -3 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

whose (2, 2) entry is $1 + 3 \cdot 2^n - 3 \cdot 3^n = 1 + 3(2^n - 3^n)$.

7.4: Part (a). We have $\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & 8 & -4 \\ 8 & \lambda - 1 & -4 \\ -4 & -4 & \lambda - 7 \end{vmatrix}$

$$= (\lambda - 1) \begin{vmatrix} \lambda - 1 & -4 \\ -4 & \lambda - 7 \end{vmatrix} - 8 \begin{vmatrix} 8 & -4 \\ -4 & \lambda - 7 \end{vmatrix} - 4 \begin{vmatrix} 8 & \lambda - 1 \\ -4 & -4 \end{vmatrix}$$

$$= (\lambda - 1)(\lambda^2 - 8\lambda - 9) - 8(8\lambda - 72) - 4(4\lambda - 36)$$

$$= \lambda^3 - 9\lambda - 81\lambda + 729 = \lambda^3 - 9\lambda^2 - 9^2\lambda + 9^3.$$

Part (b). Applying part (a), $\det(\lambda I - A) = (\lambda - 9)^2(\lambda + 9)$. So the eigenvalues can be enumerated as $\lambda_1 = \lambda_2 = 9$ and $\lambda_3 = -9$. Any 9-eigenvector (x, y, z) satisfies

$$0 = (A - 9I) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -8 & -8 & 4 \\ -8 & -8 & 4 \\ 4 & 4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

equivalently $2x + 2y - z = 0$. So two linearly independent 9-eigenvectors of A are $f_1 = (1, 0, 2)$ and $f_2 = (0, 1, 2)$. Meanwhile, any -9 -eigenvector (x, y, z) satisfies

$$0 = (A + 9I) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 & -8 & 4 \\ -8 & 10 & 4 \\ 4 & 4 & 16 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

A direct calculation shows that the equality holds when $x = y = -2$ and $z = 1$. So there is a -9 -eigenvector $f_3 = (-2, -2, 1)$. (To find f_3 , we observed that, by the symmetry $x \leftrightarrow y$, we must have $x = y$. It then becomes clear that $z \neq 0$. So we can put $z = 1$, whereupon $x = y = -2$.)

Taking the columns of P to be the eigenvectors f_1, f_2, f_3 , we arrive at

$$P = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 2 & 2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -9 \end{bmatrix}.$$

7.5: The characteristic equation is

$$0 = \begin{vmatrix} i - \lambda & i \\ i & i - \lambda \end{vmatrix} = (i - \lambda)^2 - i^2 = \lambda^2 - 2i\lambda = \lambda(\lambda - 2i).$$

So the eigenvalues are 0 and $2i$. The eigenvectors for eigenvalue 0 are the vectors $(s, -s)$ where s is a nonzero complex number. The eigenvectors for eigenvalue $2i$ are the vectors (t, t) where t is a nonzero complex number.

8: Miscellaneous exercises

Question 8.1: Let $H = \begin{bmatrix} 99 & 1 & 1 \\ 1 & 99 & 1 \\ 1 & 1 & 99 \end{bmatrix}$.

- (a) Which vectors (x, y, z) are eigenvectors of H ?
- (b) Find an orthonormal basis of eigenvectors of H .

Solutions

8.1: Part (a). The eigenvectors of H are the same as the eigenvectors of

$$H_\mu = \begin{bmatrix} \mu & 1 & 1 \\ 1 & \mu & 1 \\ 1 & 1 & \mu \end{bmatrix}$$

where μ is any real number. We have

$$\ker(H_1) = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}, \quad \ker(H_{-2}) = \{(x, y, z) \in \mathbb{R}^3 : x = y = z\}$$

and $\ker(H_\mu) = \{0, 0, 0\}$ for $\mu \notin \{1, -2\}$. Therefore (x, y, z) is an eigenvector of H if and only if (x, y, z) is nonzero and either $x + y + z = 0$ or else $x = y = z$.

Part (b). By part (a), one orthonormal basis of eigenvectors of H is

$$\{(1, -1, 0)/\sqrt{2}, (1, 1, -2)/\sqrt{6}, (1, 1, 1)/\sqrt{3}\}.$$

Comments: In the context of an introductory course focussed on routines of calculation, this question is hard only because, in the standard drill for diagonalization, the student habitually finds only a basis of eigenvectors. Nevertheless, granted a theoretical grasp of the notion of an eigenvector, it is obvious that any matrix has infinitely many eigenvectors.

Part (b) of the question can be done in more systematic way by taking an obvious basis and applying the Gram–Schmidt process. Nevertheless, it is easier to simply contrive an orthonormal basis straight away.

If use of arbitrary complex numbers is admitted, a more elegant answer to part (b) is

$$\{(1, \omega^r, \omega^{2r}) : r \in \mathbb{Z}\}$$

where $\omega = e^{2\pi i/3}$. This orthonormal basis is of fundamental importance in the theory of the discrete Fourier transform.