The pointed p-groups on a block algebra

Laurence Barker

Department of Mathematics
Bilkent University
06800 Bilkent, Ankara
Turkey

16 March 2023

Abstract

A pointed p-group is a pointed group P_γ such that P is a p-group. We parameterize the pointed p-groups on a group algebra or on a block algebra of a group algebra. The parameterization involves p-subgroups and irreducible characters of centralizers of p-subgroups.

2020 *Mathematics Subject Classification:* 20C20.

Keywords: Puig category; relative multiplicity; pointed p-group; generalized piece.

1 Introduction

For a block of a group algebra, Puig [Pui86] introduced a refinement of the fusion system which Thévenaz [The95, Section 47] called the Puig category of the block. Let $\mathbb{F}$ be an algebraically closed field of prime characteristic p, let G be a finite group and let b be a block of the group algebra $\mathbb{F}G$. Let D_λ be a maximal local pointed group on the block algebra $\mathbb{F}Gb$. Recall that the **Puig category** $\mathcal{L}$ of $\mathbb{F}Gb$ associated with D_λ is defined as follows. The objects of $\mathcal{L}$ are the local pointed subgroups of D_λ. For local pointed subgroups P_γ and Q_δ of D_λ, the $\mathcal{L}$-morphisms $P_\gamma \leftarrow Q_\delta$ are the conjugation monomorphisms $P \ni {}^g y \leftarrow\!\shortmid y \in Q$ where $g \in G$ satisfies $P_\gamma \geq {}^g(Q_\delta)$. The composition is the usual composition of group monomorphisms. Since D_λ is determined by $\mathbb{F}Gb$ up to G-conjugacy, the category $\mathcal{L}$ is determined by $\mathbb{F}Gb$ up to isomorphism of categories.

To explicitly specify a local pointed group P_γ on $\mathbb{F}Gb$, or more generally on $\mathbb{F}G$, we make use of Brauer characters. We understand an $\mathbb{F}G$-character to be a Brauer character of $\mathbb{F}G$, defined by means of a fixed embedding of the torsion unit group of $\mathbb{F}$ in the torsion unit group of some algebraically closed field of characteristic 0. A well-known result of Puig, recorded as Theorem 2.1 below, expresses, for any p-subgroup P of G, a bijective correspondence between the local points of P on $\mathbb{F}G$ and the irreducible $\mathbb{F}C_G(P)$-characters. However, the Puig category is hard to determine explicitly because the inclusion relation between the local pointed groups on $\mathbb{F}G$ is difficult to describe in terms of the corresponding irreducible characters.

We shall assume familiarity with the theory of G-algebras. For accounts of the topic, see Linckelmann [Lin18], Thévenaz [The95]. For any pointed groups W_ω and U_μ on $\mathbb{F}G$ with $W \leq U$, write $m(W_\omega, U_\mu)$ to denote the relative multiplicity of W_ω in U_μ. Recall that $W_\omega \leq U_\mu$ if and only if $m(W_\omega, U_\mu) \neq 0$. It is easy to see that, given $W \leq V \leq U$, then we have a matrix relation

$$m(W_\omega, U_\mu) = \sum_\nu m(W_\omega, V_\nu)\, m(V_\nu, U_\mu)$$

where ν runs over the points of V on $\mathbb{F}G$. That matrix relation cannot, in general, be confined to local pointed groups. Indeed, supposing that W_ω and U_μ are local pointed groups on $\mathbb{F}G$, whereupon W, V, U must be p-groups, evaluation of the sum still requires us to consider all the points ν of V on $\mathbb{F}G$, not just the local points of V on $\mathbb{F}G$. We call a pointed group U_μ on $\mathbb{F}G$ a **pointed p-group** when U is a p-group. In order to make use of the above matrix relation, as was done in the proof of [BG22, 5.2] for instance, it may be necessary to consider all the pointed p-groups on $\mathbb{F}G$ or on $\mathbb{F}Gb$.

The main aim of this paper is to parameterize the pointed p-groups on $\mathbb{F}G$ and, in particular, on $\mathbb{F}Gb$. That is to say, we shall put the pointed p-groups in a bijective correspondence with the elements of a set that can be explicitly determined.

In Section 2, we shall recall a result of Puig which establishes a bijective correspondence between the local pointed groups on $\mathbb{F}G$ and some pairs which we call the pieces of $\mathbb{F}G$. We shall define the notion of a piece in Section 2. For now, let us just say that, if one is armed with an explicit description of the poset of p-subgroups of G and the modular character tables of the centralizers of the p-subgroups of G, then one knows explicitly what the pieces of $\mathbb{F}G$ are. Towards a parameterization of the pointed p-subgroups on $\mathbb{F}G$, we shall introduce the notion of a generalized piece of $\mathbb{F}G$. Again, granted an explicit description of the poset of p-subgroups and the modular character tables, one knows explicitly what the generalized pieces of $\mathbb{F}G$ are. Each pointed p-group on $\mathbb{F}G$ is associated with a unique generalized piece of $\mathbb{F}G$. The question to be answered, then, is as to which of the generalized pieces are associated with local pointed groups. We shall introduce the notion of a substantive generalized piece. The defining condition for substantivity involves involves a simple module constructed for a semidirect product $C_G(Q) \rtimes (N_P(Q)/Q)$, where $Q \leq P$ are p-subgroups of G. Proposition 2.3 allows the condition to be reformulated in terms of a Clifford-theoretic construction.

Our main result, Theorem 2.8, stated in Section 2, describes a bijective correspondence between the pointed p-groups on $\mathbb{F}G$ and the substantive generalized pieces of $\mathbb{F}G$. Corollary 2.9 describes how the bijective correspondence is compatible with blocks.

The proof of Theorem 2.8 will be given in Section 3, where we shall introduce the notion of the absolute multiplicity of a generalised piece. The substantive generalized pieces are those generalized pieces whose absolute multiplicity is nonzero. Proposition 3.12 says that the bijective correspondence between the local pointed groups and the substantive generalized pieces preserves absolute multiplicity.

To illustrate the theory, we shall present two examples in Section 4. For the principal 2-blocks of the symmetric groups S_4 and S_5, we shall calculate the relative multiplicities between the substantive generalized pieces. That will yield, in particular, the relative multiplicities between the pieces of the blocks, in other words, the relative multiplicities between the local pointed groups. The method is to first determine which generalized pieces are substantive (in effect, classifying the pointed p-groups on the block algebras), then determining the absolute multiplicities of the substantive generalized pieces (in effect, determining the absolute multiplicities of the pointed p-groups).

2 Qualitative results

In this section, we shall define the notions of a piece of $\mathbb{F}G$, a generalized piece of $\mathbb{F}G$ and a substantive generalized piece of $\mathbb{F}G$. In Theorem 2.8, we shall describe a bijective correspondence between the pointed p-groups on $\mathbb{F}G$ and the substantive generalized pieces of $\mathbb{F}G$. Three of the results in this section, Propositions 2.3, 2.7 and Theorem 2.8, are expressed in a qualitative way, and their proofs, which rely on some formulas for multiplicities, will be deferred to the next section.

We let $\mathrm{Irr}(\mathbb{F}G)$ denote the set of irreducible $\mathbb{F}G$-characters. Given $\xi \in \mathrm{Irr}(\mathbb{F}G)$, we write $V(\xi)$ to denote a simple $\mathbb{F}G$-module with modular character ξ. We write $E(\xi)$ to denote an indecomposable projective $\mathbb{F}G$-module with a quotient isomorphic to $V(\xi)$. Of course, $V(\xi)$ and $E(\xi)$ are well-defined up to isomorphism.

For a p-subgroup P of G, we write the P-relative Brauer map as $\mathrm{br}_P : \mathbb{F}C_G(P) \leftarrow (\mathbb{F}G)^P$. We define a **piece** of $\mathbb{F}G$ to be a pair (P, θ), usually written as P_θ, where P is a p-subgroup of G and $\theta \in \mathrm{Irr}(\mathbb{F}C_G(P))$. We allow G to permute the pieces of $\mathbb{F}G$ by defining ${}^g(P_\theta) = ({}^gP)_{{}^g\theta}$ for $g \in G$. In Theorem 2.8 below, we shall be extending the following fundamental theorem, which can be found in Thévenaz [The95, 37.6].

Theorem 2.1. (Puig.) *There is a G-equivariant bijective correspondence between:*

- *the local pointed groups P_γ on $\mathbb{F}G$,*
- *the pieces P'_θ of $\mathbb{F}G$,*

such that $P_\gamma \leftrightarrow P'_\theta$ if and only if $P = P'$ and $E(\theta) \cong \mathbb{F}C_G(P)\mathrm{br}_P(i)$ where $i \in \gamma$. The condition is independent of the choice of i.

To generalize that theorem, we shall need to generalize the notion of a piece. Consider the pairs (P, Q_ϕ) where P is a p-subgroup of G and Q_ϕ is a piece of $\mathbb{F}G$ such that $P \geq Q$. Two such pairs (P, Q_ϕ) and $(P', Q'_{\phi'})$ are to be deemed equivalent provided $P = P'$ and the pieces Q_ϕ and $Q'_{\phi'}$ are P-conjugate. We write $P{\uparrow}Q_\phi$ to denote the equivalence class of (P, Q_ϕ). We call $P{\uparrow}Q_\phi$ a **generalized piece** of $\mathbb{F}G$. We allow G to permute the generalized pieces of $\mathbb{F}G$ by defining ${}^g(P{\uparrow}Q_\phi) = ({}^gP){\uparrow}{}^g(Q_\phi)$ for $g \in G$. We identify any piece P_θ with the generalized piece $P{\uparrow}P_\theta$.

To define the substantivity condition on generalized pieces, we shall be needing the following abstract lemma.

Lemma 2.2. *Let $K \trianglelefteq G$ such that G/K is a p-group. Given $\xi \in \mathrm{Irr}(\mathbb{F}G)$ and $\eta \in \mathrm{Irr}(\mathbb{F}K)$, then $E(\xi) \cong {}_G\mathrm{Ind}_K(E(\eta))$ if and only if ${}_K\mathrm{Res}_G(V(\xi))$ is a direct sum of mutually non-isomorphic G-conjugates of $V(\eta)$. Furthermore, those equivalent conditions characterize a bijective correspondence $\xi \leftrightarrow [\eta]_G$ between the irreducible $\mathbb{F}G$-characters ξ and the G-conjugacy classes $[\eta]_G$ of irreducible $\mathbb{F}K$-characters η.*

Proof. A special case of Linckelmann [Lin18, 5.12.10] asserts that any primitive idempotent of $\mathbb{F}K$ remains primitive in $\mathbb{F}G$. So there is a function $\mathrm{Irr}(\mathbb{F}K) \ni \eta \mapsto \xi \in \mathrm{Irr}(\mathbb{F}G)$ such that, letting i be a primitive idempotent of $\mathbb{F}K$ satisfying $E(\eta) \cong \mathbb{F}Ki$, then $E(\xi) \cong \mathbb{F}Gi \cong {}_G\mathrm{Ind}_K(E(\eta))$. By Mackey decomposition, the restriction ${}_K\mathrm{Res}_G(E(\xi))$ is a sum of G-conjugates of $E(\eta)$. So the condition $E(\xi) \cong {}_G\mathrm{Ind}_K(E(\eta))$ characterizes a bijection $[\eta]_G \leftrightarrow \xi$.

Suppose $[\eta]_G \leftrightarrow \xi$. Since $V(\eta)$ and $V(\xi)$, respectively, are the unique simple modules of $\mathbb{F}K$ and $\mathbb{F}G$ not annihilated by i, Clifford's Theorem implies that ${}_K\mathrm{Res}_G(V(\xi))$ is a direct sum of G-conjugates of $V(\eta)$. Since i is primitive in $\mathbb{F}G$, we have $\dim_{\mathbb{F}}(iV(\xi)) = 1$, so each G-conjugate of $V(\xi)$ occurs in ${}_K\mathrm{Res}_G(V(\eta))$ with multiplicity 1. $\square$

We shall be applying the lemma in the following special case. Let S be a finite p-group acting as automorphisms on a finite group K. Via the canonical isomorphism $K \cong K{\rtimes}1$, we embed K in the semidirect product $K{\rtimes}S$. Given $\eta \in \mathrm{Irr}(\mathbb{F}K)$, we write $\eta_{\rtimes S}$ to denote the irreducible $\mathbb{F}(K{\rtimes}S)$-character such that $\eta_{\rtimes S}$ corresponds to the S-orbit of η.

Let $P{\uparrow}Q_\phi$ be a generalized piece of $\mathbb{F}G$. Now $P{\uparrow}Q_\phi$ determines Q_ϕ only up to P-conjugacy, but let us make a choice of Q_ϕ and write $\overline{P} = N_P(Q)/Q$. Via the conjugation action of $N_P(Q)$ on $C_G(Q)$, we allow $\overline{P}$ to act as automorphisms on $C_G(Q)$ and we form the semidirect product $C_G(Q){\rtimes}\overline{P}$. Via the canonical isomorphism $\overline{P} \cong 1{\rtimes}\overline{P}$, we embed $\overline{P}$ in $C_G(Q){\rtimes}\overline{P}$. We call $P{\uparrow}Q_\phi$ **substantive** when the simple $\mathbb{F}(C_G(Q){\rtimes}\overline{P})$-module $V(\phi_{\rtimes\overline{P}})$ restricts to an $\mathbb{F}\overline{P}$-module with a nonzero free direct summand. That condition is clearly independent of the choice of Q_ϕ. Observe that any piece of $\mathbb{F}G$ is a substantive generalized piece of $\mathbb{F}G$.

The following criterion for substantivity will be proved, in a stronger quantitative form, in Proposition 3.2 below.

Proposition 2.3. *Let $P{\uparrow}Q_\phi$ be a generalized piece of $\mathbb{F}G$. Write $\overline{N}_P(Q) = N_P(Q)/Q$. Write $N_P(Q_\phi)$ for the stabilizer of Q_ϕ in P and write $\overline{N}_P(Q_\phi) = N_P(Q_\phi)/Q$. Let $\overline{N}_P(Q_\phi) \leq S \leq \overline{N}_P(Q)$. Then $P{\uparrow}Q_\phi$ is substantive if and only if the simple $\mathbb{F}(C_G(Q){\rtimes}S)$-module $V(\phi_{\rtimes\overline{S}})$ restricts to an $\mathbb{F}S$-module with a nonzero free direct summand.*

The proposition gives the following means of determining whether a given generalized piece $P{\uparrow}Q_\phi$ is substantive. Let $T = \overline{N}_P(Q_\phi)$. In the evident way, $\mathbb{F}C_G(Q)$ becomes a T-algebra. Via the representation of $V(\phi)$, we regard $\mathrm{End}_\mathbb{F}(V(\phi))$ as a T-algebra. Since the cohomology groups $H^1(S, \mathbb{F}^\times)$ and $H^2(S, \mathbb{F}^\times)$ are trivial, the T-algebra structure of $\mathrm{End}_\mathbb{F}(V(\phi))$ enriches, in a unique way, to an interior T-algebra structure. Thus, $V(\phi)$ becomes an $\mathbb{F}T$-module. The generalized piece $P{\uparrow}Q_\phi$ is substantive if and only if, regarding $V(\phi)$ as an $\mathbb{F}T$-module, the regular $\mathbb{F}T$-module occurs as a direct summand of $V(\phi)$.

We note an immediate corollary of the proposition.

Corollary 2.4. *Let $P{\uparrow}Q_\phi$ be a generalized piece of $\mathbb{F}G$ and let $Q \leq P' \leq P$.*

(1) *If $P{\uparrow}Q_\phi$ is substantive, then $P'{\uparrow}Q_\phi$ is substantive.*

(2) *If $N_P(Q_\phi) \leq P'$ and $P'{\uparrow}Q_\phi$ is substantive, then $P{\uparrow}Q_\phi$ is substantive.*

For any piece $P{\uparrow}Q_\phi$ of $\mathbb{F}G$, we now construct an $\mathbb{F}(G{\times}P)$-module $\mathrm{Dia}_G(P{\uparrow}Q_\phi)$, called the **diagonal module** of $P{\uparrow}Q_\phi$. Still working with a choice of Q_ϕ, let

$$N = N_{G\times P}(\Delta(Q)) = (C_G(Q){\times}1)\Delta(N_P(Q)) \;, \qquad \overline{N} = \overline{N}_{G\times P}(\Delta(Q)) \cong C_G(Q){\rtimes}\overline{P} \;.$$

Via the canonical isomorphism $C_G(Q) \cong (C_G(Q){\times}1)\Delta(Q)/\Delta(Q)$, we embed $C_G(Q)$ in $\overline{N}$ and form the induced $\mathbb{F}\overline{N}$-module

$$\overline{\mathrm{Dia}}^0_G(P{\uparrow}Q_\phi) = {}_{\overline{N}}\mathrm{Ind}_{C_G(Q)}(E(\phi)) \cong {}_{\overline{N}}\mathrm{Iso}_{C_G(Q)\rtimes\overline{P}}(E(\phi_{\rtimes\overline{P}}))$$

which is indecompoable and projective. We define the inflated $\mathbb{F}N$-module

$$\mathrm{Dia}^0_G(P{\uparrow}Q_\phi) = {}_N\mathrm{Inf}_{\overline{N}}(\overline{\mathrm{Dia}}^0_G(P{\uparrow}Q_\phi))$$

which is indecomposable with vertex $\Delta(Q)$. We define $\mathrm{Dia}_G(P{\uparrow}Q_\phi)$ to be the indecomposable $\mathbb{F}(G{\times}P)$-module with vertex $\Delta(Q)$ in Green correspondence with $\mathrm{Dia}^0_G(P{\uparrow}Q_\phi)$. It is easy to check that, given a P-conjugate $Q'_{\phi'}$ of Q_ϕ, then $\mathrm{Dia}_G(P{\uparrow}Q_\phi) \cong \mathrm{Dia}_G(P{\uparrow}Q'_{\phi'})$. Thus, $\mathrm{Dia}_G(P{\uparrow}Q_\phi)$ is determined by $P{\uparrow}Q_\phi$ up to isomorphism, independently of the choice of Q_ϕ. The next result tells us that, in fact, $\mathrm{Dia}_G(P{\uparrow}Q_\phi)$ is uniquely determined by $P{\uparrow}Q_\phi$.

Lemma 2.5. *Let P be a p-subgroup of G and let Q_ϕ and $Q'_{\phi'}$ be pieces of $\mathbb{F}G$ such that $P \geq Q$ and $P \geq Q'$. Then $\mathrm{Dia}_G(P{\uparrow}Q_\phi) \cong \mathrm{Dia}_G(P{\uparrow}Q'_{\phi'})$ if and only if $P{\uparrow}Q_\phi = P{\uparrow}Q'_{\phi'}$.*

Proof. The conclusion in one direction has been observed already. For the converse, suppose $\mathrm{Dia}_G(P{\uparrow}Q_\phi) \cong \mathrm{Dia}_G(P{\uparrow}Q'_{\phi'})$. By considering vertices, $\Delta(Q)$ and $\Delta(Q')$ are $G{\times}P$-conjugate, hence Q and Q' are P-conjugate and we may assume that $Q = Q'$. By considering Green correspondents, $E(\phi_{\rtimes \overline{P}}) \cong E(\phi'_{\rtimes \overline{P}})$. By Lemma 2.2, ϕ and ϕ' lie in the same $\overline{P}$-orbit of $\mathrm{Irr}(\mathbb{F}C_G(Q))$, hence Q_ϕ and $Q_{\phi'}$ are P-conjugate. □

We shall be making use of the following characterization of the diagonal module $\mathrm{Dia}(P{\uparrow}Q_\phi)$.

Lemma 2.6. *Given a generalized piece $P{\uparrow}Q_\phi$ on $\mathbb{F}G$, then*

$$\mathrm{Dia}_G(P{\uparrow}Q_\phi) \cong {}_{G\times P}\mathrm{Ind}_{G\times Q}(\mathrm{Dia}_G(Q_\phi)) \; .$$

Proof. By its definition, $\mathrm{Dia}_G(P{\uparrow}Q_\phi)$ is the isomorphically unique indecomposable $\mathbb{F}(G{\times}P)$-module with vertex $\Delta(Q)$ that appears as a direct summand of the $\mathbb{F}(G{\times}P)$-module

$$L = {}_{G\times P}\mathrm{Ind}_{N_{G\times P}(\Delta(Q))}\mathrm{Inf}_{\overline{N}_{G\times P}(\Delta(Q))}\mathrm{Ind}_{C_G(Q)}(E(\phi)) \; .$$

As a special case, $\mathrm{Dia}_G(Q_\phi)$ is the isomorphically unique indecomposable $\mathbb{F}(G{\times}Q)$-module with vertex $\Delta(Q)$ that appears as a direct summand of

$$M = {}_{G\times Q}\mathrm{Ind}_{N_{G\times Q}(\Delta(Q))}\mathrm{Inf}_{C_G(Q)}(E(Q))$$

where the inflation is via the canonical epimorphism $N_{G\times Q}(\Delta(Q)) \to \overline{N}_{G\times Q}(\Delta(Q)) \cong C_G(Q)$. Using the Mackey formula for bisets in Bouc [Bou10, 2.3.24], we obtain an equality of bisets

$${}_{N_{G\times P}(\Delta(Q))}\mathrm{Inf}_{\overline{N}_{G\times P}(\Delta(Q))}\mathrm{Ind}_{C_G(Q)} \cong {}_{N_{G\times P}(\Delta(Q))}\mathrm{Ind}_{N_{G\times Q}(\Delta(Q))}\mathrm{Inf}_{C_G(Q)} \; .$$

Hence, $L \cong {}_{G\times P}\mathrm{Ind}_{G\times Q}(M)$. Therefore, writing $D = {}_{G\times P}\mathrm{Ind}_{G\times Q}(\mathrm{Dia}_G(Q_\phi))$, then D is a direct summand of L. By Green's Indecomposability Criterion, D is indecomposable with vertex $\Delta(Q)$. By the uniqueness of $\mathrm{Dia}_G(P{\uparrow}Q_\phi)$, we have $\mathrm{Dia}_G(P{\uparrow}Q_\phi) \cong D$. □

When we write an $\mathbb{F}(G{\times}P)$-module in the form ${}_GM_P$, we are indicating that the actions of $G{\times}1$ and $1{\times}P$ are by left translation and right translation, respectively. Given $\mathbb{F}G$-modules L and M, we write $L \,|\, M$ when L is isomorphic to a direct summand of M. A stronger quantitative version of the next result will appear below as Proposition 3.5

Proposition 2.7. *A generalized piece $P{\uparrow}Q_\phi$ on $\mathbb{F}G$ is substantive if and only if*

$$\mathrm{Dia}_G(P{\uparrow}Q_\phi) \,|\, {}_G\mathbb{F}G_P \; .$$

Our main result is the following classification of the pointed p-groups on $\mathbb{F}G$. We shall prove it in the next section.

Theorem 2.8. *There is a G-equivariant bijective correspondence between:*

- *the pointed p-groups P_α on $\mathbb{F}G$,*
- *the substantive generalized pieces $P'{\uparrow}Q_\phi$,*

such that $P_\alpha \leftrightarrow P'{\uparrow}Q_\phi$ if and only if $P = P'$ and, letting Q_δ be the local pointed group on $\mathbb{F}G$ corresponding to the piece Q_ϕ, also letting $i \in \alpha$, the following two equivalent conditions hold:

(a) *Q_δ is a maximal local pointed subgroup of P_α,*

(b) *we have $\mathrm{Dia}_G(P{\uparrow}Q_\phi) \cong {}_G(\mathbb{F}Gi)_P$.*

The bijective correspondences in Theorems 2.1 and 2.8 are compatible with blocks in the following ways. Let b be a block of $\mathbb{F}G$. A piece P_θ of $\mathbb{F}G$ is called a **piece** of $\mathbb{F}Gb$ provided $\mathrm{br}_P(b)$ acts as the identity on the $\mathbb{F}C_G(P)$-module $E(\theta)$. Obviously, letting P_γ be the local pointed group on $\mathbb{F}G$ corresponding to P_θ, then P_θ is a piece of $\mathbb{F}Gb$ if and only if P_γ is a local pointed group on $\mathbb{F}Gb$. A generalized piece $P{\uparrow}Q_\phi$ of $\mathbb{F}G$ is called a **generalized piece** of $\mathbb{F}Gb$ provided Q_ϕ is a piece of $\mathbb{F}Gb$. Theorem 2.8 has the following immediate corollary.

Corollary 2.9. *Let b be a block of $\mathbb{F}G$. Let P_α be a pointed p-group on $\mathbb{F}G$. Let $P{\uparrow}Q_\phi$ be the substantive generalized piece of $\mathbb{F}G$ corresponding to P_α. Then the following three conditions are equivalent:*

(a) *P_α is a pointed p-group on $\mathbb{F}Gb$,*

(b) *$P{\uparrow}Q_\phi$ is a piece of $\mathbb{F}Gb$,*

(c) *we have $\mathrm{Dia}_G(P{\uparrow}Q_\phi) \,|\, {}_G(\mathbb{F}Gb)_P$.*

3 Stronger quantitative results

We shall define the absolute multiplicity of a generalized piece. That will enable us to prove stronger quantitative versions of Propositions 2.3, 2.7 and Theorem 2.8.

For any pointed group U_μ, we write $m(U_\mu)$ to denote the absolute multiplicity of U_μ, we mean to say, the maximal size of a set of mutually orthogonal elements of μ. Given a piece P_θ of $\mathbb{F}G$, we define the **absolute multiplicity** of P_θ to be

$$m(P_\theta) = \theta(1) = \dim_{\mathbb{F}}(V(\theta)) \,.$$

The next remark says that the bijective correspondence in Theorem 2.1 preserves absolute multiplicities.

Remark 3.1. *Given a local pointed group P_γ on $\mathbb{F}G$ with corresponding piece P_θ on $\mathbb{F}G$, then $m(P_\gamma) = m(P_\theta)$.*

Proof. Letting $i \in \gamma$, then $m(P_\gamma)$ and $m(P_\theta)$ are both equal to the multiplicity of the projective indecomposable $\mathbb{F}C_G(P)$-module $\mathbb{F}C_G(P)\mathrm{br}_P(i)$ as a direct summand of the regular $\mathbb{F}C_G(P)$-module. □

To prove the results in the previous section, we shall need to extend the notion of absolute multiplicity to generalized pieces. Given $\mathbb{F}G$-modules L and M with L indecomposable, we write $m(L, M)$ to denote the multiplicity of L as a direct summand of M.

Let $P{\uparrow}Q_\phi$ be a generalized piece on $\mathbb{F}G$. As before, we make a choice of Q_ϕ. Again, we write $\overline{P} = N_P(Q)$ and we consider the simple $\mathbb{F}(C_G(Q) \rtimes \overline{P})$-module $V(\phi_{\rtimes \overline{P}})$. We define the **absolute multiplicity** of $P{\uparrow}Q_\phi$ to be the natural number

$$m(P{\uparrow}Q_\phi) = m(\mathbb{F}\overline{P}, {}_{\overline{P}}\mathrm{Res}_{C_G(Q) \rtimes \overline{P}}(V(\phi_{\rtimes \overline{P}})))$$

where $\mathbb{F}\overline{P}$ denotes the regular $\mathbb{F}\overline{P}$-module. Plainly, $m(P{\uparrow}Q_\phi)$ is well-defined, independently of the choice of Q_ϕ. Observe that, for a piece P_θ of $\mathbb{F}G$, the absolute multiplicity $m(P_\theta) = m(P{\uparrow}P_\theta)$ is unambiguous. The generalized piece $P{\uparrow}Q_\phi$ is substantive if and only if $m(P_\theta) \neq 0$.

The next result is a stronger quantitative version of Proposition 2.3.

Proposition 3.2. *Let $P{\uparrow}Q_\phi$ be a generalized piece on $\mathbb{F}G$. Let $\overline{N}_P(Q_\phi) \leq S \leq \overline{N}_P(Q)$. Then*

$$m(P{\uparrow}Q_\phi) = m(\mathbb{F}S, {}_S\mathrm{Res}_{C_G(Q)\rtimes S}(V(\phi_{\rtimes S}))) \ .$$

Proof. Write $C = C_G(Q)$ and $T = \overline{N}_P(Q_\phi)$. Define $m_S = m(\mathbb{F}S, {}_S\mathrm{Res}_{C_G(Q)\rtimes S}(V(\phi_{\rtimes S})))$. By considering the case where $S = \overline{N}_P(Q)$, we see that it suffices to show that $m_T = m_S$. We have ${}_C\mathrm{Res}_{C\rtimes T}(V(\phi_{\rtimes T})) \cong V(\phi)$ and

$${}_C\mathrm{Res}_{C\rtimes S}(V(\phi_{\rtimes S})) \cong \bigoplus_{sT \subseteq S} {}^sV(\phi) \ .$$

So $\dim_{\mathbb{F}}(V(\phi_{\rtimes S})) = |S : T| \dim_{\mathbb{F}}(V(\phi_{\rtimes T}))$.

Since $C\rtimes T$ is subnormal in $C\rtimes S$, Clifford's Theorem implies that ${}_{C\rtimes T}\mathrm{Res}_{C\rtimes S}(V(\phi_{\rtimes S}))$ is semisimple. But $V(\phi)$ occurs in the semisimple $\mathbb{F}C$-module ${}_C\mathrm{Res}_{C\rtimes S}(V(\phi_{\rtimes S}))$ and $V(\phi_{\rtimes T})$ is the isomorphically unique simple $\mathbb{F}(C\rtimes T)$-module such that $V(\phi)$ occurs in the semisimple $\mathbb{F}(C\rtimes T)$-module ${}_C\mathrm{Res}_{C\rtimes T}(V(\phi_{\rtimes T}))$. Therefore, $V(\phi_{\rtimes T})$ occurs in the semisimple $\mathbb{F}(C\rtimes T)$-module ${}_{C\rtimes T}\mathrm{Res}_{C\rtimes S}(V(\phi_{\rtimes S}))$. By Frobenius reciprocity, $V(\phi_{\rtimes S})$ is isomorphic to a submodule of ${}_{C\rtimes S}\mathrm{Ind}_{C\rtimes T}(V(\phi_{\rtimes T}))$. A consideration of dimensions yields

$$V(\phi_{\rtimes S}) \cong {}_{C\rtimes S}\mathrm{Ind}_{C\rtimes T}(V(\phi_{\rtimes T})) \ .$$

By Mackey decomposition,

$${}_S\mathrm{Res}_{C\rtimes S}(V(\phi_{\rtimes S})) \cong {}_S\mathrm{Ind}_T\mathrm{Res}_{C\rtimes T}(V(\phi_{\rtimes T})) \ .$$

The required equality $m_T = m_S$ follows because, given any indecomposable direct summand M of ${}_T\mathrm{Res}_{C\rtimes T}(V(\phi_{\rtimes T}))$, then M is free if and only if ${}_S\mathrm{Ind}_T(M)$ is free. □

We shall be needing the following result of Broué [Bro85, 3.2]. Given a p-subgroup S of G and an $\mathbb{F}G$-module M, we define the $\mathbb{F}\overline{N}_G(S)$-module $M(S)$ to be the quotient of M^S by the sum of the images of the trace maps $\mathrm{tr}_T^S : M^T \to M^S$, running over the strict subgroups $T < S$.

Proposition 3.3. (Broué.) *Let S be a p-subgroup of G, let E be an indecomposable projective $\mathbb{F}\overline{N}_G(S)$-module, and let F be the indecomposable $\mathbb{F}G$-module with vertex S in Green correspondence with the inflated $\mathbb{F}N_G(S)$-module ${}_{N_G(S)}\mathrm{Inf}_{\overline{N}_G(S)}(E)$. Let M be a p-permutation $\mathbb{F}G$-module. Then $m(F, M) = m(E, M(S))$.*

Another necessary ingredient is the following result of Robinson [Rob89, Proposition 1].

Proposition 3.4. (Robinson.) *Given $\xi \in \mathrm{Irr}(\mathbb{F}G)$ and a p-subgroup S of G, then*

$$m(E(\xi), \mathbb{F}G/S) = m(\mathbb{F}S, {}_S\mathrm{Res}_G(V(\xi))) \ .$$

The next result implies Proposition 2.7 and, more precisely, it characterises the multiplicity of a generalized piece in terms of the associated diagonal module.

Proposition 3.5. *Given a generalized piece $P{\uparrow}Q_\phi$ on $\mathbb{F}G$, then*

$$m(P{\uparrow}Q_\phi) = m(\mathrm{Dia}_G(P{\uparrow}Q), {}_G\mathbb{F}G_P) \ .$$

Proof. Let $m = m(\mathrm{Dia}_G(P{\uparrow}Q), {}_G\mathbb{F}G_P)$. By Proposition 3.3,

$$m(\mathrm{Dia}_G(P{\uparrow}Q_\phi), M) = m(\overline{\mathrm{Dia}}^0_G(P{\uparrow}Q_\phi), M(\Delta(Q)))$$

for any p-permutation $\mathbb{F}(G{\times}P)$-module M. Put $M = {}_G\mathbb{F}G_P$. Let N and $\overline{N}$ be as in Section 2. Define $L = {}_N\mathrm{Res}_{G\times P}(M)$. Then $M(\Delta(Q)) = L(\Delta(Q))$ and

$$m = m(\overline{\mathrm{Dia}}^0_G(P{\uparrow}Q_\phi), L(\Delta(Q)))\ .$$

Let $\Gamma \subseteq G$ such that $1 \in \Gamma$ and $\{(g,1) : g \in \Gamma\}$ is a set of representatives of the double cosets $N\backslash(G{\times}P)/\Delta(P)$. For each $g \in \Gamma$, we define a permutation $\mathbb{F}N$-module $L_g = \mathbb{F}N/H_g$ where $H_g = N \cap {}^g\Delta(P)$. Since $M \cong \mathbb{F}(G \times P)/\Delta(P)$, Mackey decomposition yields

$$L(\Delta(Q)) \cong \bigoplus_{g\in\Gamma} L_g(\Delta(Q))\ .$$

Fix $g \in \Gamma$ such that $L_g(\Delta(Q)) \neq 0$. Then $\Delta(Q) \leq H_g \leq {}^g\Delta(P)$, hence $g \in C_G(Q)$ and $(g,1) \in N$. The condition $1 \in \Gamma$ implies that $g = 1$. Therefore, $L(\Delta(Q)) \cong L_1(\Delta(Q))$. Since $H_1 = \Delta(N_P(Q))$, we have $L(\Delta(Q)) \cong \mathbb{F}\overline{N}/\overline{P}$. We have shown that

$$m = m({}_{\overline{N}}\mathrm{Ind}_{C_G(Q)}(E(\phi)), \mathbb{F}\overline{N}/\overline{P})\ .$$

In view of the isomorphism $\overline{N} \cong C_G(Q){\rtimes}\overline{P}$, Lemma 2.2 yields

$$m = m(E(\phi_{\rtimes\overline{P}}), \mathbb{F}(C_G(Q){\rtimes}\overline{P})/\overline{P})\ .$$

Lemma 3.4 now implies that $m = m(P{\uparrow}Q_\phi)$. □

We shall be needing two abstract lemmas. We write $\leq$ to denote the usual partial ordering on the idempotents of a ring.

Lemma 3.6. *For any local pointed group P_γ on $\mathbb{F}G$, there exists $i \in \gamma$ such that $i \leq \mathrm{br}_P(i)$.*

Proof. Let $i_0 \in \gamma$. Then $\mathrm{br}_P(i_0\mathrm{br}_P(i_0)) = \mathrm{br}_P(i_0) \notin J(\mathbb{F}C_G(P))$. So $i_0\mathrm{br}_P(i_0) \notin J((\mathbb{F}G)^P)$. It follows that $i \leq \mathrm{br}_P(i_0)$ for some $i \in \gamma$. We have $\mathrm{br}_P(i) \leq \mathrm{br}_P(\mathrm{br}_P(i_0)) = \mathrm{br}_P(i_0)$. But $\mathrm{br}_P(i_0)$ is a primitive idempotent of $\mathbb{F}C_G(P)$. Therefore, $\mathrm{br}_P(i) = \mathrm{br}_P(i_0)$. □

We point out that the proof of the lemma yields a stronger result, namely, that for all $i_0 \in \gamma$, there exists $i \in \gamma$ satisfying $i \leq \mathrm{br}_P(i) = \mathrm{br}_P(i_0)$.

Lemma 3.7. *Let P_α be a pointed p-group on $\mathbb{F}G$. Given $i \in \alpha$, then the $\mathbb{F}(G{\times}P)$-module ${}_G(\mathbb{F}Gi)_P$ is indecomposable. Given $Q \leq P$, then Q is a defect group of P_α if and only if $\Delta(Q)$ is a vertex of ${}_G(\mathbb{F}Gi)_P$.*

Proof. Writing $\circ$ to indicate an opposite algebra, there is an interior P-algebra isomorphism

$$\mathrm{End}_{\mathbb{F}(G\times 1)}(\mathbb{F}Gi) \cong (i\mathbb{F}Gi)^\circ$$

such that, given $r \in \mathrm{End}_{\mathbb{F}(G\times 1)}(\mathbb{F}Gi)$ and $a \in i\mathbb{F}Gi$, then $r \leftrightarrow a^\circ$ provided $r(x) = xa$ for all $x \in \mathbb{F}Gi$. Hence $\mathrm{End}_{\mathbb{F}(G\times P)}(\mathbb{F}Gi) \cong ((i\mathbb{F}Gi)^P)^\circ$, which is a local algebra. Therefore, ${}_G(\mathbb{F}Gi)_P$ is indecomposable. Since ${}_G(\mathbb{F}Gi)_P$ is a direct summand of the $\mathbb{F}(G{\times}P)$-module ${}_G\mathbb{F}G_P \cong \mathbb{F}(G{\times}P)/\Delta(P)$, some vertex of ${}_G(\mathbb{F}Gi)_P$ is contained in $\Delta(P)$.

Suppose Q is a defect group of P_α. Let $a \in (i\mathbb{F}Gi)^Q$ such that $i = \mathrm{tr}_Q^P(a)$. Let $r \in \mathrm{End}_{\mathbb{F}(G\times Q)}(\mathbb{F}Gi)$ such that $r \leftrightarrow a^\circ$. Then $\mathrm{tr}_{G\times Q}^{G\times P}(r) = \mathrm{tr}_{\Delta(Q)}^{\Delta(P)}(r) = \mathrm{id}_{\mathbb{F}Gi}$. So $G\times Q$ contains a vertex S of $\mathbb{F}Gi$. But $S \leq {}^{(g,u)}\Delta(P)$ for some $(g,u) \in G\times P$. So the vertex ${}^{(ug^{-1},1)}S$ of $\mathbb{F}Gi$ is contained in the subgroup $(G\times Q) \cap \Delta(P) = \Delta(Q)$.

For the reverse inclusion, suppose $\Delta(Q)$ is a vertex of $\mathbb{F}Gi$. Let $r \in \mathrm{End}_{\mathbb{F}(G\times Q)}(\mathbb{F}Gi)$ such that $\mathrm{tr}_{G\times Q}^{G\times P}(r) = \mathrm{id}_{\mathbb{F}Gi}$. Let $a \in (i\mathbb{F}Gi)^Q$ such that $r \leftrightarrow a^\circ$. Then $i = \mathrm{tr}_Q^P(a)$. We deduce that Q contains a defect group of P_α. $\square$

Proposition 3.8. *Let P_γ be a local pointed group on $\mathbb{F}G$. Let P_θ be the piece of $\mathbb{F}G$ corresponding to P_γ. Let $i \in \gamma$. Then* $\mathrm{Dia}_G(P_\theta) \cong {}_G(\mathbb{F}Gi)_P$.

Proof. It is easy to check that the isomorphism class of the $\mathbb{F}(G\times P)$-module ${}_G(\mathbb{F}Gi)_P$ is determined by P_γ, independently of i. So, in view of Lemma 3.6, we may assume that $i \leq \mathrm{br}_P(i)$. Let $\mathrm{br}_P(i) = i + \sum_k k$ be a primitive idempotent decomposition in $(\mathbb{F}G)^P$. By Lemma 3.7,

$$
{}_G(\mathbb{F}G\mathrm{br}_P(i))_P = {}_G(\mathbb{F}Gi)_P \oplus \bigoplus_k {}_G(\mathbb{F}Gk)_P
$$

as a direct sum of indecomposable $\mathbb{F}(G\times P)$-modules. Each k has a defect group Q_k strictly contained in P. By the same lemma, ${}_G(\mathbb{F}Gi)_P$ has vertex $\Delta(P)$ while each ${}_G(\mathbb{F}Gk)_P$ has vertex $\Delta(Q_k)$. Inflating via the canonical epimorphism $N_{G\times P}(\Delta(P)) \to C_G(P)$, we have

$$
\mathbb{F}C_G(P)\mathrm{br}_P(i) \cong {}_{N_{G\times P}(\Delta(P))}\mathrm{Inf}_{C_G(P)}(E(\theta)) \cong \mathrm{Dia}_G^0(P_\theta) \; .
$$

As a direct sum of $\mathbb{F}N_{G\times P}(\Delta(P))$-modules,

$$
\mathbb{F}G\mathrm{br}_P(i) = \mathbb{F}C_G(P)\mathrm{br}_P(i) \oplus \mathbb{F}(G - C_G(P))\mathrm{br}_P(i) \; .
$$

The conjugation action of P on $G - C_G(P)$ has no fixed points, so each indecomposable direct summand of $\mathbb{F}(G - C_G(P))\mathrm{br}_P(i)$ has a vertex strictly contained in $\Delta(P)$. The required isomorphism now follows from the Green Correspondence Theorem. $\square$

The next result is a theorem of Puig that can be found in Linckelmann [Lin18, 5.12.20] or Thévenaz [The95, 18.3].

Theorem 3.9. (Puig.) *Let Q_δ be a local pointed group on a G-algebra A over $\mathbb{F}$. Let P_α be be a pointed group on A such that $Q \leq P$. Let $i \in \alpha$. Then Q_δ is a maximal local pointed subgroup of P_α if and only if there exists $j \in \delta$ such that $i = \mathrm{tr}_Q^P(j)$ and $\{{}^x j : xQ \subseteq P\}$ is a set of mutually orthogonal idempotents.*

We shall also be using the following result of [BG22, 3.1].

Theorem 3.10. *Let Q_δ be a local pointed group on a G-algebra A over $\mathbb{F}$. Let P be a p-subgroup of G such that $Q \leq P$. Then there exists at most one point α of P on A such that Q_δ is a maximal local pointed subgroup of P_α.*

Lemma 3.11. *Let P be a p-subgroup of G, let α and α' be points of P on $\mathbb{F}G$ and let $i \in \alpha$ and $i' \in \alpha'$. Then ${}_G(\mathbb{F}Gi)_P \cong {}_G(\mathbb{F}Gi')_P$ if and only if $\alpha = \alpha'$.*

Proof. If $\alpha = \alpha'$, then the isomorphism is clear. Conversely, assume the isomorphism. Lemma 3.7 implies that P_α and $P_{\alpha'}$ have a common defect group Q. Let δ and δ' be local points of Q on $\mathbb{F}G$ such that Q_δ and $Q_{\delta'}$ are maximal local pointed subgroups of P_α and $P_{\alpha'}$, respectively. By Theorem 3.9, there exist $j \in \delta$ and $j' \in \delta'$ such that $i = \mathrm{tr}_Q(j)$ as a sum of $|P : Q|$ mutually orthogonal idempotents and similarly for i' and j'. We have

$$ {}_G(\mathbb{F}Gi)_P \cong {}_{G\times P}\mathrm{Ind}_{G\times Q}(\mathbb{F}Gj) \; . $$

Restricting to $G{\times}Q$ and then applying Mackey decomposition,

$$ (\mathbb{F}Gi)(\Delta(Q)) \cong ({}_{G\times Q}\mathrm{Res}_{G\times P}\mathrm{Ind}_{G\times Q}(\mathbb{F}Gj))(\Delta(Q)) \cong \bigoplus_{xQ \subseteq N_P(Q)} \mathbb{F}C_G(Q)\,\mathrm{br}_Q({}^x j) $$

and similarly for i' and j'. Therefore, $\mathbb{F}C_G(Q)\,\mathrm{br}_Q(j') \cong \mathbb{F}C_G(Q)\,\mathrm{br}_Q({}^x j)$ for some $x \in N_P(Q)$. Hence, $\delta' = {}^x\delta$. So we may assume that $\delta' = \delta$. By Theorem 3.10, $\alpha = \alpha'$. □

We now prove Theorem 2.8. For each p-subgroup P of G, Lemmas 2.5 and 3.11 imply that the isomorphism classes of $\mathrm{Dia}_G(P{\uparrow}Q_\phi)$ and ${}_G(\mathbb{F}Gi)_P$ determine $P{\uparrow}Q_\phi$ and P_α, respectively. So Propositions 3.5 and 3.8 imply that condition (b) characterizes a bijective correspondence $P_\alpha \leftrightarrow P{\uparrow}Q_\phi$.

It remains only to show that the conditions (a) and (b) are equivalent. Let Q_δ be the local pointed group corresponding to Q_ϕ. Let $j \in \delta$. By Proposition 3.8, $\mathrm{Dia}_G(Q_\phi) \cong {}_G(\mathbb{F}Gj)_Q$. So by Lemma 2.6,

$$ \mathrm{Dia}_G(P{\uparrow}Q_\phi) \cong {}_{G\times P}\mathrm{Ind}_{G\times Q}(\mathbb{F}Gj) \; . $$

If (a) holds then, by Theorem 3.9, we can take the choice of j to be such that $i = \mathrm{tr}_Q^P(j)$ as a sum of $|P : Q|$ mutually orthogonal idempotents. We deduce (b).

Conversely, assume (b). By Lemma 3.7, P_γ has defect group Q. Let δ' be a point of Q on $\mathbb{F}G$ such that P_γ has maximal local pointed subgroup $Q_{\delta'}$. By what we have already shown, $\mathrm{Dia}_G(P{\uparrow}Q_{\delta'}) \cong \mathrm{Dia}_G(P{\uparrow}Q_\delta)$. By Lemma 2.5, $P{\uparrow}Q_{\delta'} = P{\uparrow}Q_\delta$. So $Q_{\delta'}$ and Q_δ are P-conjugate, and (a) holds. The proof of Theorem 2.8 is complete.

Let us note that the bijective correspondence in that theorem preserves absolute multiplicities in the following sense.

Proposition 3.12. *Let P_α be a pointed p-group on $\mathbb{F}G$. Let $P{\uparrow}Q_\phi$ be the substantive generalized piece on $\mathbb{F}G$ corresponding to P_α. Then $m(P_\alpha) = m(P{\uparrow}Q_\phi)$.*

Proof. This follows immediately from Proposition 3.5 □

4 The principal 2-blocks of S_4 and S_5

By a method that involves calculation of the absolute multiplicities, we shall determine the relative multiplicities between the substantive generalized pieces for the principal 2-blocks of S_4 and S_5. The method is based on the following immediate implication of Theorem 2.8, Corollary 2.9, Proposition 3.12.

Corollary 4.1. *Given a p-subgroup P of G, then we have an $\mathbb{F}(G{\times}P)$-isomorphism*

$$ {}_G\mathbb{F}G_P \cong \bigoplus_{Q_\phi} m(P{\uparrow}Q_\phi)\, {}_G(\mathbb{F}Gi_{Q_\phi})_P $$

where Q_ϕ runs over the P-conjugacy classes of pieces of $\mathbb{F}G$ such that the generalized piece $P{\uparrow}Q_\phi$ is substantive, and i_{Q_ϕ} is an element of the point α of P on $\mathbb{F}G$ such that P_α is the local pointed group on $\mathbb{F}G$ corresponding to $P{\uparrow}Q_\phi$. Furthermore, given a block b of $\mathbb{F}G$, then

$$_G(\mathbb{F}Gb)_P \cong \bigoplus_{Q_\phi} m(P{\uparrow}Q_\phi)\, _G(\mathbb{F}Gi_{Q_\phi})_P$$

where Q_ϕ now runs over the P-conjugacy classes of pieces of $\mathbb{F}Gb$ such that $P{\uparrow}Q_\phi$ is substantive.

Consider a block b of $\mathbb{F}G$. Let us make some comments on some combinatorial structures possessed by the set $\mathcal{P}(\mathbb{F}Gb)$ of pieces of $\mathbb{F}Gb$ and the set $\mathcal{P}^{|}(\mathbb{F}Gb)$ of substantive generalized pieces of $\mathbb{F}Gb$. We understand a **multiposet** to be a poset such that each inclusion $x' \leq x$ is associated with a natural number $m(x', x)$ called the **multiplicity** of x' in x. We regard $\mathcal{P}^{|}(\mathbb{F}Gb)$ as a poset such that, given $P'{\uparrow}Q'_{\phi'}, P{\uparrow}Q_\phi \in \mathcal{P}^{|}(\mathbb{F}Gb)$, then $P'{\uparrow}Q'_{\phi'} \leq P{\uparrow}Q_\phi$ provided $P' \leq P$ and the relative multiplicity $m(P'{\uparrow}Q'_{\phi'}, P{\uparrow}Q_\phi)$ is nonzero. We regard $\mathcal{P}^{|}(\mathbb{F}Gb)$ as a multiposet whose multiplicities are the relative multiplicities. In an evident sense, $\mathcal{P}(\mathbb{F}Gb)$ is a submultiposet of $\mathcal{P}^{|}(\mathbb{F}Gb)$.

By the matrix relation for relative multiplicities in Section 1, the whole family of relative multiplicities $m(P'{\uparrow}Q'_{\phi'}, P{\uparrow}Q_\phi)$ is determined by those relative multiplicities such that $P' \leq P$ and $|P : P'| = p$. The Hasse diagram for $\mathcal{P}^{|}(\mathbb{F}Gb)$ as a poset has an upwards line from $P'{\uparrow}Q'_{\phi'}$ to $P{\uparrow}Q_\phi$ if and only if that condition on P' and P holds. So the structure of $\mathcal{P}^{|}(\mathbb{F}Gb)$ is determined by an enriched Hasse diagram where any upwards line from an element $P'{\uparrow}Q'_{\phi'}$ to an element $P{\uparrow}Q_\phi$ is labled with $m(P'{\uparrow}Q'_{\phi'}, P{\uparrow}Q_\phi)$.

Put $p = 2$. Let $H = S_4$. The principal block of $\mathbb{F}H$ is the unique block of $\mathbb{F}H$. Let D be a Sylow 2-subgroup of H. We have $D \cong D_8$, the dihedral group of order 8. Let

$$\mathbb{S} = \{1, C_2, C_2', Z, C_4, V_4, V_4', D\}$$

be a set of representatives of the D-conjugacy classes of subgroups of D, named according to the isomorphism classes and such that $Z =_G C_2 < V_4$. Let $\mathbb{P}_H$ be the submultiposet of $\mathcal{P}^{|}(\mathbb{F}H)$ consisting of those substantive generalized pieces on $\mathbb{F}H$ that have the form $P{\uparrow}Q_\phi$ where $P, Q \in \mathbb{S}$. The elements of $\mathbb{P}_H$ comprise a set of representatives for the D-conjugacy classes of substantive generalized pieces of $\mathbb{F}H$ having the form $P{\uparrow}Q_\phi$ where $P \leq D$. So a Hasse diagram for $\mathbb{P}_H$, labelled with the relative multiplicities, will supply a complete description of the multiposet $\mathcal{P}^{|}(\mathbb{F}H)$.

Determining the pieces in $\mathbb{P}_H$ will be straightforward. To find the other elements of $\mathbb{P}_H$, we shall use the Clifford-theoretic criterion for substantivity of a generalized piece. That criterion was presented above immediately following the statement of Proposition 2.3.

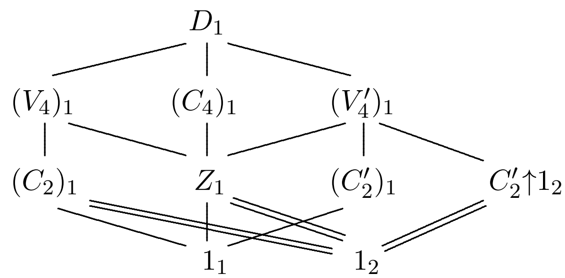
Let us set up some conventions of notation for expressing pieces of $\mathbb{F}H$ concisely. For any subgroup $L \leq H$, we write $\mathrm{Irr}(\mathbb{F}L) = \{\theta_1^L, \theta_2^L, ...\}$, enumerated such that θ_1^L is the trivial $\mathbb{F}L$-character. For any p-subgroup $P \leq H$, we write $P_i = P_{\theta_i^C}$ where $C = C_H(P)$. We have $\mathrm{Irr}(\mathbb{F}H) = \{\theta_1^H, \theta_2^H\}$. For all $P \in \mathbb{S} - \{1\}$, the centralizer $C_H(P)$ is a p-group, so $\mathrm{Irr}(\mathbb{F}C_G(P)) = \{\theta_1^{C_H(P)}\}$. Therefore, the pieces in $\mathbb{P}_H$ are

$$1_1, 1_2, (C_2)_1, (C_2')_1, Z_1, (C_4)_1, (V_4)_1, (V_4')_1, D_1.$$

We claim that the only other element of $\mathbb{P}_H$ is $C_2'{\uparrow}1_2$. To demonstrate the claim, we first note that, for all $Q \in \mathbb{S}$, we have $\theta_1^{C_H(Q)}(1) = 1$, so there is no element $P{\uparrow}Q_1 \in \mathbb{P}_H$ with

$P > Q$. The kernel of $V(\theta_2^H)$ is V_4. So, given $P \in \mathbb{S}$, then ${}_P\mathrm{Res}_H(V(\theta_2^H))$ has a nonzero free direct summand if and only if $P \in \{\{1, C_2'\}$. The claim is established.

Recall, the absolute multiplicity of a piece is the degree of the associated irreducible character. For the sole element of $\mathbb{P}_H$ that is not a piece, we have ${}_{C_2'}\mathrm{Res}_H(V(\theta_2^H)) \cong \mathbb{F}C_2'$ and the absolute multiplicity is $m(C_2'{\uparrow}1_2) = 1$. Applying Corollary 4.1, ${}_H\mathbb{F}G_1 \cong \mathrm{Dia}_H(1_1) \oplus 2\,\mathrm{Dia}_H(1_2)$. Also, ${}_H\mathbb{F}H_{C_2'} \cong \mathrm{Dia}_H((C_2')_1) \oplus \mathrm{Dia}_H(C_2'{\uparrow}1_2)$ and ${}_H\mathbb{F}H_R \cong \mathrm{Dia}_H(R_1)$ for $R \in \mathbb{S} - \{1, C_2'\}$. By Lemma 2.6 and Mackey decomposition, ${}_{H\times 1}\mathrm{Res}_{H\times C_2'}(\mathrm{Dia}_H(C_2'{\uparrow}1_2)) \cong 2\,\mathrm{Dia}_H(1_2)$. So ${}_{H\times 1}\mathrm{Res}_{H\times C_2'}(\mathrm{Dia}_H((C_2')_1) \cong \mathrm{Dia}_H(1_1)$. Therefore, the multiposet $\mathbb{P}_H$ has the following Hasse diagram, where single and double lines indicate relative multiplicities 1 and 2, respectively.



Still putting $p = 2$, now put $G = S_5$. Let b be the principal block of $\mathbb{F}G$. We embed H in G and take D and $\mathbb{S}$ to be the same as before. Let $\mathbb{P}_b$ be the multiposet of substantive generalized pieces of $\mathbb{F}Gb$ having the form $P{\uparrow}Q_\phi$ where $P, Q \in \mathbb{S}$. The elements of $\mathbb{P}_b$ comprise a set of representatives for the D-conjugacy classes of substantive generalized pieces of $\mathbb{F}Gb$ having the form $P{\uparrow}Q_\phi$ where $P \leq D$. So, as before, to specify the poset $\mathcal{P}^{|}(\mathbb{F}Gb)$, it will be enough to display a Hasse diagram for $\mathbb{P}_b$, labelled with the relative multiplicities.

Replacing H with G, we retain the above convention for expressing pieces and generalized pieces concisely. We can choose the enumeration $\mathrm{Irr}(\mathbb{F}G) = \{\theta_1^G, \theta_2^G, \theta_3^G\}$ such that θ_3^G is not in b. The irreducible $\mathbb{F}G$-characters θ_2^G and θ_3^G both have degree 4. The local pointed group corresponding to the piece 1_3 is not on $\mathbb{F}Gb$, so 1_3 is not a piece of $\mathbb{F}Gb$. Writing $C = C_G(C_2')$, then $C \cong C_2 \times S_3$. We have $\mathrm{Irr}(\mathbb{F}Cb_C) = \{\theta_1^C, \theta_2^C\}$. As before, the local pointed group corresponding to $(C_2')_2$ is not on the principal block algebra of $\mathbb{F}C$, so $(C_2')_2$ is not a piece of $\mathbb{F}Gb$. Given $R \in \mathbb{S} - \{1, C_2'\}$, then $C_G(R)$ is a p-group. So the pieces in $\mathbb{P}_b$ are 1_2 and P_1 with $P \in \mathbb{S}$.

We claim that the other elements of $\mathbb{P}_b$ are $P{\uparrow}1_2$, where $P \in \{C_2, C_2', Z, C_4, V_4'\}$. To prove the claim, we shall consider, for each $P \in \mathbb{S}$, the restriction to P of the simple $\mathbb{F}G$-module $V = V(\theta_2^G)$. Let Ω be the G-set of Sylow 5-subgroups of G. Observe that, as a D-set by restriction,

$$\Omega \cong D/C_4 \sqcup D/C_2 \, .$$

Enumerate $\Omega = \{L_1, ..., L_6\}$. Let $A = \{\sum_i \lambda_i L_i \in \mathbb{F}\Omega : \sum_i \lambda_i = 0\}$ and $B = \mathbb{F}\sum_i L_i$. The $\mathbb{F}G$-modules $\mathbb{F}\Omega/A$ and B are trivial and the Brauer character of $\mathbb{F}\Omega$ is easily shown to be $2\theta_1^G + \theta_2^G$. Therefore, $A/B \cong V$. We have

$$m(P{\uparrow}1_2) = \dim_{\mathbb{F}}(P^+.V) = \dim_{\mathbb{F}}((P^+.A + B)/B)$$

where P^+ denotes the sum of the elements of P.

If $P = C_2'$, then we can choose the enumeration L_1, ... such that the P-orbits are $\{L_1, L_2\}$, $\{L_3, L_4\}$, $\{L_5, L_6\}$, whereupon

$$P^+.A = \mathrm{span}_{\mathbb{F}}\{L_1 + L_2 + L_5 + L_6, L_3 + L_4 + L_5 + L_6\}$$

while $B \cap P^+.A = \{0\}$, hence $m(P{\uparrow}1_2) = 2$. If $P \in \{C_2, W\}$, then we can choose the enumeration such that the P-orbits are $\{L_1, L_2\}$, $\{L_3, L_4\}$, $\{L_5\}$, $\{L_6\}$, whereupon

$$P^+.A = \mathrm{span}_{\mathbb{F}}\{L_1 + L_2, L_3 + L_4\}$$

and again $B \cap P^+.A = \{0\}$, hence $m(P{\uparrow}1_2) = 2$. If $P \in \{C_4, V_4'\}$, then we can choose the enumeration such that $\{L_1, L_2, L_3, L_4\}$ is a P-orbit, whereupon

$$P^+.A = \mathrm{span}_{\mathbb{F}}\{L_1 + L_2 + L_3 + L_4\}$$

and yet again $B \cap P^+.A = \{0\}$, hence $m(P{\uparrow}1_2) = 1$. Finally, when $P \in \{V_4, D\}$, the whole of $\mathbb{F}\Omega$ is annihilated by P^+, hence $m(P{\uparrow}1_2) = 0$. The claim is established and moreover, the absolute multiplicities of the pieces in $\mathbb{P}_b$ having been clear already, we have now determined the absolute multiplicities of all the elements of $\mathbb{P}_b$.

By Corollary 4.1,

$$_G(\mathbb{F}Gb)_P = \begin{cases} \mathrm{Dia}_G(1_1) \oplus 4\,\mathrm{Dia}_G(1_2) & \text{if } P = 1, \\ \mathrm{Dia}_G(P_1) \oplus 2\,\mathrm{Dia}(P{\uparrow}1_2) & \text{if } |P| = 2, \\ \mathrm{Dia}_G(P_1) \oplus \mathrm{Dia}(P{\uparrow}1_2) & \text{if } P \in \{C_4, V_4'\}, \\ \mathrm{Dia}_G(P_1) & \text{if } P \in \{V_4, D\}. \end{cases}$$

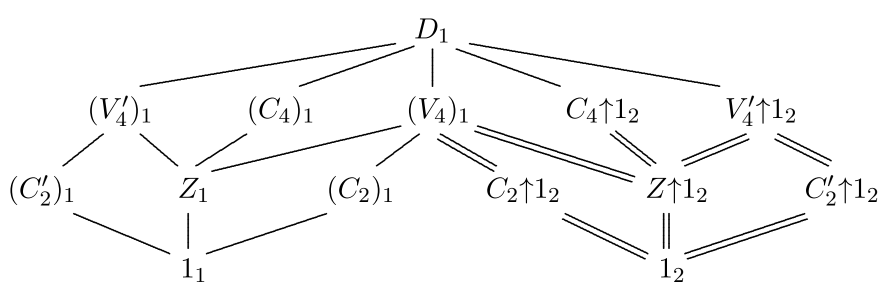
Given any element of $\mathbb{P}_b$ having the form $P{\uparrow}1_2$, Mackey decomposition yields

$$_{G\times 1}\mathrm{Res}_{G\times P}(\mathrm{Dia}_G(P{\uparrow}1_2)) \cong |P|\,\mathrm{Dia}_G(1_2)\ .$$

It follows that, for any element of $\mathbb{P}_b$ having the form P_1, we have

$$_{G\times 1}\mathrm{Res}_{G\times P}(\mathrm{Dia}_G(P_1)) = \begin{cases} \mathrm{Dia}_G(1_1) \oplus 4\,\mathrm{Dia}_G(1_2) & \text{if } P \in \{V_4, D\}, \\ \mathrm{Dia}_G(1_1) & \text{otherwise.} \end{cases}$$

Therefore, the multiposet $\mathbb{P}_b$ has the following Hasse diagram, the single and double lines again indicating relative multiplicities 1 and 2, respectively.



Comparing the two Hasse diagrams that we have produced, it may be of interest to note that, confining attention to the pieces, we see that the poset of local pointed groups on a source algebra of $\mathbb{F}H$ is not isomorphic to the poset of local pointed groups on a source algebra of $\mathbb{F}Gb$.

References

[BG22] L. Barker, M. Gelvin, *Conjectural invariance with respect to the fusion system of an almost-source algebra*, J. Group Theory **25**, 973-995 (2022).

[Bou10] S. Bouc, "Biset Functors for Finite Groups", Springer Lecture Notes in Math. 1990, (Springer, Heidelberg, 2010).

[Bro85] M. Broué, *On Scott modules and p-permutation modules: an approach through the Brauer morphism*, Proc. American Math. Soc. **93**, 401-408 (1985).

[Lin18] M. Linckelmann, "The Block Theory of Finite Group Algebras", Vol. 1 (Cambridge University Press, Cambridge, 2018).

[Pui86] L. Puig, *Local fusions in block source algebras*, J. Algebra **104**, 358-369 (1986).

[Rob89] G. R. Robinson, *On projective summands of induced modules*, J. Algebra **122**, 106-111 (1989).

[The95] J. Thévenaz, "G-algebras and Modular Representation Theory", (Clarendon Press, Oxford, 1995).