Pointed fusion systems of blocks

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Abstract

The pointed fusion system of a block is a structure consisting of the fusions and relative multiplicities between the local pointed groups associated with a maximal Brauer pair. We show that the pointed fusion system is preserved by splendid Morita equivalences and part of the pointed fusion system is preserved by splendid stable equivalences of Morita type.

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1 Introduction

The pointed fusion system of a block, defined precisely in Section 2, is an invariant that can be viewed as a refinement of the fusion system of a block. Roughly, the pointed fusion system is a package of data consisting of some local pointed groups, the fusions between the local pointed groups and the relative multiplicities between the local pointed groups. The structure of the pointed fusion system includes the structure of a category and the structure of a poset, together with a labelling of the inclusions by the relative multiplicities.

We mention that a category with at least some resemblance to the pointed fusion system was introduced by Puig in [7] and was named the Puig category in Thévenaz [8, Section 47]. We do not know whether the pointed fusion system, as a category, coincides with the Puig category. We shall discuss the relationship between the two categories in Section 3.

One of our three main results, Theorem 3.3, says that, up to isomorphism, the pointed fusion system of block can be constructed from the source algebra of the block. A second main result, Theorem 5.6, describes how a splendid Morita equivalence between blocks gives rise to an isomorphism of pointed fusion systems. The third main result, Theorem 5.7, describes how a splendid stable equivalence of Morita type gives rise to an isomorphism between the stable parts of the pointed fusion systems.

In view of the first two of those theorems, it makes sense to view the pointed fusion system as a local invariant. A consideration of the 2-blocks of the non-trivial semidirect product of C_3 over Q_8 shows that the fusion system does not determine the number of isomorphism classes of simple modules of a block algebra. As we shall note in Remark 2.1, the pointed fusion system does determine that number.

We shall assume familiarity with block theory, the theory of Brauer pairs and the theory of pointed groups, all of which are discussed in Linckelmann [6].

After defining, in Section 2, the notion of a pointed fusion system and its stable part, we shall show, in Section 3, that the pointed fusion system of a block is determined by a source algebra.

We shall explain, in Section 4, how the pointed fusion system of a block yields a connection between the Weak Donovan Conjecture and a weak version of Puig's Conjecture.

In Section 5, we shall consider two block algebras of group algebras. We shall show, in Theorems 5.6 and 5.7, how two kinds of equivalences between the two block algebras each give rise to appropriate isomorphisms between the structures under consideration.

We shall give some examples. In Section 5, for blocks with Klein-four defect group, we shall decribe the three possible underlying multiposets of the pointed fusion system. We shall find that there is only one possibility for the underlying multiposet of the stable part of the pointed fusion system.

2 Pointed fusion systems and their stable parts

In the three theorems mentioned in Section 1, and also in our pair of equivalent definitions of the notion of a pointed fusion system, we shall be making use of suitable notions of isomorphism. Formulating those notions of isomorphism will require some abstraction.

We define a **poset category** to be a category $\mathcal{C}$ equipped with:

• a partial ordering $\leq$ on the set of $\mathcal{C}$-objects,

• a family of monomorphisms ${}_P\mathrm{inc}_Q : P \leftarrow Q$, called the **$\mathcal{C}$-inclusions**, defined for all $\mathcal{C}$-objects P and Q satisfying $P \geq Q$,

such that the following three conditions hold:

Strictness of inclusions: For all $\mathcal{C}$-objects $P \geq Q$, the inclusion ${}_P\mathrm{inc}_Q$ is an isomorphism if and only if $P = Q$, furthermore, ${}_P\mathrm{inc}_P = \mathrm{id}_P$.

Composability of inclusions: For all $\mathcal{C}$-objects $P \geq Q \geq R$, we have ${}_P\mathrm{inc}_Q . {}_Q\mathrm{inc}_R = {}_P\mathrm{inc}_R$.

Factorization of morphisms: For all $\mathcal{C}$-morphisms $\phi : P \leftarrow Q$ and $\mathcal{C}$-objects $R \leq Q$, there exists a unique $\mathcal{C}$-object $\phi(R)$ and a unique $\mathcal{C}$-isomorphism $\psi : \phi(R) \leftarrow R$ such that $\phi . {}_Q\mathrm{inc}_R = {}_P\mathrm{inc}_{\phi(R)} . \psi$. We call ψ the isomorphism with domain R **restricted** from ϕ.

We define a **multiposet** to be a poset $\mathcal{M}$ equipped with a function $m_\mathcal{M} : \mathbb{N} \leftarrow \mathcal{M} \times \mathcal{M}$, such that, given $x, y \in \mathcal{M}$, then $x \leq y$ if and only if $m_\mathcal{M}(x, y) \neq 0$. Thus, a multiposet is an enrichment of a poset where each inclusion is labelled with a positive integer.

Let $\mathcal{C}$ be a poset category. We define a **pointed refinement** of $\mathcal{C}$ to be a structure with the following three constituents satisfying the subsequent four conditions.

• For each $P \in \mathcal{C}$, there is a set $\mathcal{P}_P$ whose elements are called the **$\mathcal{P}$-points** of P. Given $x \in \mathcal{P}_P$, we write $P_x = (P, x)$ and we call P_x a **$\mathcal{P}$-object**.

• For any $\mathcal{C}$-morphism ϕ with domain P, there is a bijection $\mathcal{P}_{\phi(P)} \leftarrow \mathcal{P}_P$, written ${}^\phi x \leftarrow\!\shortmid x$, which depends only on the isomorphism with domain P restricted from ϕ. For $Q \leq P$ and $y \in \mathcal{P}_Q$, we define ${}^{\phi}y = {}^{\psi}y$ where ψ is the isomorphism with domain Q restricted from ϕ. We also write ${}^{\phi}(Q_y) = \phi(Q)_{{}^{\phi}y}$. Note that ${}^{\phi}(Q_y) = {}^{\psi}(Q_y)$.

• For any $\mathcal{P}$-objects Q_y and P_x, there is a natural number $m_{\mathcal{P}}(Q_y, P_x)$ called the **$\mathcal{P}$-multiplicity** of Q_y in P_x.

Bijection composition condition: Given $\mathcal{C}$-morphisms ψ and ϕ such that the composite $\psi\phi$ is defined, letting P be the domain of ϕ and $x \in \mathcal{P}_P$, then ${}^{\psi\phi}x = {}^{\psi}({}^{\phi}x)$.

Multiposet condition: The set of $\mathcal{P}$-objects, equipped with the function $m_{\mathcal{P}}$, is a multiposet.

Refinement condition: Given $\mathcal{P}$-objects Q_y and P_x such that $Q_y \leq P_x$, then $Q \leq P$, furthermore, if $Q = P$ then $y = x$

Compatibility condition: Given a $\mathcal{C}$-morphism ϕ with domain P, a $\mathcal{C}$-object $Q \leq P$ and points $x \in \mathcal{P}_P$ and $y \in \mathcal{P}_Q$, then $m({}^{\phi}(Q_y), {}^{\phi}(P_x)) = m(Q_y, P_x)$.

Let $\mathcal{C}$ be a poset category and let $\mathcal{P}'$ and $\mathcal{P}$ be pointed refinements of $\mathcal{C}$. We define a **$\mathcal{C}$-identical isomorphism** $\iota : \mathcal{P}' \leftarrow \mathcal{P}$ to be a family of bijections $\iota_P : \mathcal{P}'_P \leftarrow \mathcal{P}_P$ such that P runs over the $\mathcal{C}$-objects and the following two conditions hold:

Preservation of morphisms: For all $\mathcal{C}$-morphisms ϕ with domain P and $\mathcal{P}$-points x of $\mathcal{P}$, we have

$$ {}^{\phi}(\iota_P(x)) = \iota_{\phi(P)}({}^{\phi}x) \; . $$

Preservation of multiplicities: For all $\mathcal{C}$-objects Q_y and P_x, we have

$$ m(Q_{\iota_Q(y)}, P_{\iota_P(x)}) = m(Q_y, P_x) \; . $$

Throughout the rest of this paper, we let $\mathcal{O}$ be a complete local Noetherian ring with an algebraically closed residue field $\mathbb{F}$ of prime characteristic p. We let G be a finite group, we let b a block of $\mathcal{O}G$ with defect group D and we let B a source D-algebra of the block algebra $\mathcal{B} = \mathcal{O}Gb$. We let $\mathcal{F}$ be the fusion system on D associated with B. Recall, Linckelmann [Lin18, 8.7.1] tells us that $\mathcal{F}$ is determined by the interior D-algebra structure of B and, in fact, by the interior $\mathcal{O}D$-$\mathcal{O}D$-module structure of B.

Given any p-subgroup P of G, we write $\mathrm{br}_P : \mathbb{F}C_G(P) \leftarrow (\mathcal{O}G)^P$ for the P-relative Brauer map. Let γ be a local point of P on $\mathcal{B}$ and let e be a block of $\mathbb{F}C_G(P)\mathrm{br}_P(b)$. Consider the local pointed group P_γ and the Brauer pair (P, e). We write $P_\gamma \in (P, e)$ when $\mathrm{br}_P(\gamma) \subset \mathbb{F}C_G(P)e$. Given $Q \leq D$, we write e_Q for the unique block of $\mathbb{F}C_G(Q)$ such that $\mathrm{br}_Q(1_B) \in \mathbb{F}C_G(Q)e_Q$. For any local pointed group P_γ on $\mathbb{F}Gb$, we say that P_γ is **overshadowed** by B provided $P \leq D$ and $P_\gamma \in (P, e_P)$.

Given a group R and a monomorphism θ with domain R, we define

$$ \Delta(\theta) = \{(\theta(z), z) : z \in R\} $$

as a subgroup of $\phi(R) \times R$. We define the **pointed fusion system** $\mathcal{LP}(B)$ of $\mathcal{B}$ associated with B to be the pointed refinement of $\mathcal{F}$ characterized as follows. Given $P \leq D$, then the $\mathcal{LP}(B)$-points of P are those local points γ of P on $\mathcal{B}$ such that P_γ is overshadowed by B. Thus, the $\mathcal{LP}(B)$-objects are the pointed groups on $\mathcal{B}$ overshadowed by B. Given an $\mathcal{LP}(B)$-object P_γ and an $\mathcal{F}$-morphism ϕ with domain P, we define ${}^{\phi}\gamma$ to be the point of $\phi(P)$ on $\mathcal{B}$ such that, letting $i \in \gamma$ and letting u be a unit in $\mathcal{B}^{\Delta(\phi)}$, then ${}^{u}i \in {}^{\phi}\gamma$. Given another $\mathcal{LP}(B)$-object Q_δ, then the $\mathcal{LP}(B)$-multiplicity $m(Q_\delta, P_\gamma)$ is 0 unless $Q \leq P$, in which case $m(Q_\delta, P_\gamma)$ is the relative multiplicity of Q_δ in P_γ, we mean, the number of elements of δ that appear when an element of γ is expressed as a sum of mutually orthogonal primitive idempotents of A^Q.

We understand algebras and modules to be finitely generated over their coefficient rings. For an algebra Λ over $\mathcal{O}$, we let $\ell(\Lambda)$ denote the number of isomorphism classes of simple Λ-modules.

Remark 2.1. *The number $\ell(\mathcal{B})$ is equal to the number of minimal objects of $\mathcal{LP}(B)$ as a poset.*

Proof. Since B and $\mathcal{B}$ are Morita equivalent, the condition $\tau' \subseteq \tau$ characterizes a bijective correspondence between the points τ' on B and the points τ on the algebra $C_G(1)e_1 = \mathcal{B}$. So every pointed group on $\mathcal{B}$ having the form 1_τ is overshadowed by B. □

To construct pointed fusion systems explicitly in particular cases, it is convenient to work with an isomorphic copy $\mathcal{P}(B)$ of $\mathcal{LP}(B)$ defined as follows. We make use of Brauer characters of $\mathbb{F}G$-modules, which we understand to be $\mathbb{K}$-valued, where $\mathbb{K}$ is a sufficiently large field of characteristic 0 whose group of p'-roots of unity is indentified with the group of torsion units of $\mathbb{F}$. Some notation will be needed. Given an algebra Λ over $\mathcal{O}$ and Λ-modules L and M with L indecomposable, we write $m(L, M)$ to denote the multiplicity of L as a direct summand of M. Given a subalgebra $\Gamma \leq \Lambda$, we write ${}_\Gamma\mathrm{Res}_\Lambda$ to denote the restriction functor to Γ-modules from Λ-modules.

For $P \leq D$, we define the $\mathcal{P}(B)$-points of P to be the irreducible Brauer characters of $\mathbb{F}C_G(P)e_P$. Given a $\mathcal{P}(B)$-point ξ of P and an $\mathcal{F}$-morphism ϕ with domain P, we define a $\phi(P)$-point ${}^\phi\xi = {}^g\xi$ where $g \in G$ and ϕ is conjugation by g. Let $V(\xi)$ be simple $\mathbb{F}C_G(P)e_P$-module with Brauer character ξ. We define the **diagonal module**

$$\mathrm{Dia}(P_\xi) = \mathcal{O}Gi$$

as an $\mathbb{F}(G{\times}P)$-module, where i is a primitive idempotent of $(\mathcal{O}G)^P$ such that $\mathrm{br}_P(i)$ does not annihilate $V(\xi)$, and the actions of G and P on $\mathcal{O}Gi$ are by left and right translation, respectively. Plainly, $\mathrm{Dia}(P_\xi)$ is well-defined up to isomorphism, independently of the choice of i. The primitivity of i ensures that $\mathrm{Dia}(P_\xi)$ is indecomposable. Given $Q \leq P$ and a $\mathcal{P}(B)$-point η of Q, we define the $\mathcal{P}(B)$-multiplicity of Q_η in P_ξ to be

$$m(Q_\eta, P_\xi) = m(\mathrm{Dia}(Q_\eta), {}_{G\times Q}\mathrm{Res}_{G\times P}(\mathrm{Dia}(P_\eta)))\ .$$

Thus, we have defined $\mathcal{P}(B)$ as a pointed refinement of $\mathcal{F}$. The following remark is clear.

Remark 2.2. *There is an $\mathcal{F}$-identical isomorphism $\iota : \mathcal{LP}(B) \leftarrow \mathcal{P}(B)$ such that, given a $\mathcal{P}(B)$-point P_ξ, then $\iota_P(\xi) = \gamma$, where γ is the local point of P on $\mathcal{B}$ such that* $\mathrm{br}_P(\gamma)$ *does not annihilate $V(\xi)$.*

Although the pointed fusion system $\mathcal{LP}(B) \cong \mathcal{P}(B)$ of $\mathcal{B}$ does depend on B, the uniqueness of D and B up to G-conjugacy implies that, as a category and as a poset, $\mathcal{LP}(B)$ is well-defined up to isomorphism as an invariant of $\mathcal{B}$. To avoid clutter, we have refrained from writing out the evident definitions of isomorphism for poset categories and multiposets. We have also refrained from writing out the evident general definition of isomorphism for pointed refinements of poset categories. But it is easy to see that, with those notions of isomorphism understood, $\mathcal{LP}(B)$ is well-defined up to isomorphism of pointed refinements of fusion systems, independently of the choices of D and B.

As an aside, we make some brief comments about another pointed refinement of $\mathcal{F}$ which, again, can be described, up to $\mathcal{F}$-identical isomorphism, in two ways. Let $\mathcal{LP}^|(B)$ be the pointed refinement of $\mathcal{F}$ defined as follows. For $P \leq D$, we define the $\mathcal{LP}^|(B)$-points of P to be those points α of P on $\mathcal{B}$ such that a maximal local pointed subgroup of P_α is overshadowed by B. Given an $\mathcal{F}$-morphism ϕ with domain P, we define ${}^\phi\alpha$ as above. The $\mathcal{LP}^|(B)$-multiplicities are, again, the usual relative multiplicities between pointed groups. In analogy with $\mathcal{P}(B)$, we let $\mathcal{P}^|(B)$ be the following pointed refinement of $\mathcal{F}$. Employing some notation and terminology from [1], we take the $\mathcal{P}^|(B)$-points of P to be the substantive generalized pieces having the form $P{\uparrow}Q_\eta$ where Q_η is a $\mathcal{P}(B)$-point of P. The action of an $\mathcal{F}$-morphism ϕ with domain P is given by ${}^\phi(P{\uparrow}Q_\eta) = ({}^gP){\uparrow}{}^g(Q_\eta)$ where g is as before. Each $\mathcal{P}^|(B)$-point $P{\uparrow}Q_\eta$ is associated with an indecomposable $G{\times}P$-bimodule $\mathrm{Dia}(P{\uparrow}Q_\eta)$ as defined in [1] and, given a $\mathcal{P}^|(B)$-point $S{\uparrow}T_\tau$ with $S \leq P$, the $\mathcal{P}^|(B)$-multiplicity of $S{\uparrow}T_\tau$ in $P{\uparrow}Q_\eta$ is defined to be

$$m(S{\uparrow}T_\tau, P{\uparrow}Q_\eta) = m(\mathrm{Dia}(S{\uparrow}T_\tau), {}_{G\times S}\mathrm{Res}_{G\times P}(\mathrm{Dia}(P{\uparrow}Q_\eta))) \ .$$

It is not hard to show that there is an $\mathcal{F}$-identical isomorphism

$$\mathcal{LP}^|(B) \cong \mathcal{P}^|(B) \ .$$

An advantage of $\mathcal{LP}^|(B)$ over $\mathcal{LP}(B)$, useful when calculating multiplicties for particular cases, is that, as explained in [1], the $\mathcal{LP}^|(B)$-multiplicities satisfy a matrix relation, and it follows that all the multiplicties are determined by those multiplicities $m(S{\uparrow}T_\tau, P{\uparrow}Q_\eta)$ for which $|P : S| = p$. We omit fuller details because we shall not be discussing $\mathcal{LP}^|(B)$ any further in this paper.

We define the stable part of $\mathcal{F}$, denoted $\overline{\mathcal{F}}$, to be the poset category obtained from $\mathcal{F}$ by removing the trivial subgroup of D from the set of objects. Of course, $\mathcal{F}$ and $\overline{\mathcal{F}}$ determine each other. The reason for considering $\overline{\mathcal{F}}$ is that it allows us to make the following definition. We define the **stable part** of $\mathcal{LP}(B)$, denoted $\overline{\mathcal{LP}}(B)$, to be the pointed refinement of $\overline{\mathcal{F}}$ obtained from $\mathcal{LP}(B)$ by removing the minimal objects. Thus, the $\overline{\mathcal{LP}}(B)$-objects are the local pointed groups P_γ on $\mathcal{B}$ such that $1 < P \leq D$ and P_γ is overshadowed by B.

3 Determination by the source algebra

We continue to work with the block algebra $\mathcal{B} = \mathcal{O}Gb$, the source D-algebra B and the fusion system $\mathcal{F}$ introduced in the previous section. We consider the pointed fusion system $\mathcal{LP} = \mathcal{LP}(B)$. Theorem 3.3, below, says that $\mathcal{LP}$ is determined by B.

The **Puig category** of $\mathcal{B}$ associated with B, which we write as $\mathcal{L} = \mathcal{L}(B)$, was introduced by Puig [7]. See also Thévenaz [8, Section 47]. We define $\mathcal{L}$ as follows. The $\mathcal{L}$-objects are the local pointed groups on the D-algebra B. Given $\mathcal{L}$-objects P_γ and Q_δ, then the $\mathcal{L}$-morphisms $P_\gamma \leftarrow Q_\delta$ are those group monomorphisms $\phi : P \leftarrow Q$ such that, choosing $i \in \gamma$ and $j \in \delta$, then there exists a unit $u \in B^{\Delta(\phi)}$ satisfying ${}^uj \leq i$, where $\leq$ denotes the usual partial ordering of idempotents. It is easy to check that the existence condition is independent of the choices of i and j.

We shall be making use of an isomorphic copy $\mathcal{L}'$ of $\mathcal{L}$ defined as follows. Let λ be the point of D on $\mathcal{B}$ such that $1_B \in \lambda$. In other words, λ is the unique point of D on B such that $D_\lambda \in (D, e_D)$. The $\mathcal{L}'$-objects are the local pointed subgroups of D_λ. Given $\mathcal{L}'$-objects $P_{\gamma'}$ and $Q_{\delta'}$, then the $\mathcal{L}$-morphisms $P_{\gamma'} \leftarrow Q_{\delta'}$ are the conjugation maps ${}^gy \hookleftarrow y$ where $g \in G$ satisfying $P_{\gamma'} \geq {}^g(Q_{\delta'})$. A theorem of Puig [7, 3.6], also in Thévenaz [8, 47.10], asserts that there is an isomorphism of categories $\mathcal{L}' \leftarrow \mathcal{L}$ acting as the identity on morphisms and sending each $\mathcal{L}$-object P_γ to the $\mathcal{L}'$-object $P_{\gamma'}$ such that $\gamma' \supseteq \gamma$.

Observe that $\mathcal{L}'$ is a full subcategory of $\mathcal{LP}$. We do not know whether $\mathcal{L}' = \mathcal{LP}$ as categories. We mention that the conjecture [2, 1.5] implies that $\mathcal{L}' = \mathcal{LP}$. We do not have an explicit parameterization of the $\mathcal{L}'$-objects or, equivalently, the $\mathcal{L}$-objects. The next result, part of Linckelmann [6, 8.7.3], implies that the inclusion of $\mathcal{L}'$ in $\mathcal{LP}$ is an equivalence of categories. So we do have an explicit parameterization of the isomorphism classes of $\mathcal{L}$-objects in terms of irreducible Brauer characters, indeed, writing $\mathcal{P} = \mathcal{P}(B)$ then, via the chain of functors $\mathcal{P} \cong \mathcal{LP} \hookleftarrow \mathcal{L}' \cong \mathcal{L}$, the isomorphism classes of $\mathcal{L}$-objects are in a bijective correspondence with the $\mathcal{P}$-objects.

Theorem 3.1. (Linckelmann.) *Let P_γ be a local pointed group on $\mathcal{B}$ overshadowed by B. Suppose P is fully $\mathcal{F}$-centralized. Then $P_\gamma \leq D_\lambda$, in other words, $B \cap \gamma$ is a local point of P on B.*

In view of the isomorphism between $\mathcal{L}$ and the full subcategory $\mathcal{L}'$ of $\mathcal{LP}$, the latest theorem has the following corollary.

Corollary 3.2. *Let P_γ be a local pointed group on B such that P is fully $\mathcal{F}$-centralized. Let ϕ be an $\mathcal{F}$-automorphism of P. Then there exists a local point ${}^\phi\gamma$ of P on B such that ϕ is an $\mathcal{L}$-isomorphism $P_{{}^\phi\gamma} \leftarrow P_\gamma$.*

Theorem 3.3. *The pointed fusion system $\mathcal{LP}$ is determined up to $\mathcal{F}$-identical isomorphism by the interior D-algebra structure of B.*

Proof. From B, we shall construct a pointed refinement $\widehat{\mathcal{P}}$ of $\mathcal{F}$ and an $\mathcal{F}$-identical isomorphism $\iota : \widehat{\mathcal{P}} \leftarrow \mathcal{LP}$. Let $\widetilde{\mathcal{F}}$ be a set of fully $\mathcal{F}$-centralized subgroups of D such that $\widetilde{\mathcal{F}}$ is a set of representatives of the $\mathcal{F}$-isomorphism classes. For each $R \in \widetilde{\mathcal{F}}$, let $\mathcal{L}_R$ denote the set of points of R on B. For each $P \leq D$, let $\widetilde{P}$ be the unique element of $\widetilde{\mathcal{F}}$ such that $P \cong_{\mathcal{F}} \widetilde{P}$. We choose and fix an $\mathcal{F}$-isomorphism $\theta_P : P \leftarrow \widetilde{P}$ and a set $\widehat{\mathcal{P}}_P$ together with a bijection $\Theta_P : \widehat{\mathcal{P}}_P \leftarrow \mathcal{L}_{\widetilde{P}}$. For each $\widehat{\gamma} \in \widehat{\mathcal{P}}_P$, we define

$$\widetilde{\gamma} = \Theta_P^{-1}(\widehat{\gamma}) \ .$$

For each $\mathcal{F}$-morphism ϕ with domain P, we define

$$\widetilde{\phi} = \theta_{\phi(P)}^{-1} \phi \theta_P \ .$$

Since $\widetilde{\phi}$ is an $\mathcal{F}$-automorphism of the fully $\mathcal{F}$-centralized subgroup $\widetilde{P}$, Corollary 3.2 ensures that we can form the local point ${}^{\widetilde{\phi}}\widetilde{\gamma}$ of $\widehat{P}$ on B. We define

$${}^\phi\widehat{\gamma} = \Theta_{\phi(P)}({}^{\widetilde{\phi}}\widetilde{\gamma}) \ .$$

Let $Q \leq D$ and $\widehat{\delta} \in \widehat{\mathcal{P}}_Q$. When $Q \not\leq P$, we let $m_{\widehat{\mathcal{P}}}(Q_{\widehat{\delta}}, P_{\widehat{\gamma}}) = 0$. Now suppose $Q \leq P$. Put $\theta = \theta_P^{-1}\theta_Q$ as an isomorphism with domain $\widetilde{Q}$. We define

$$m_{\widehat{\mathcal{P}}}(Q_{\widehat{\delta}}, P_{\widehat{\gamma}}) = m({}^\theta(\widetilde{Q}_{\widetilde{\delta}}), \widetilde{P}_{\widetilde{\gamma}})$$

when θ appears as an $\mathcal{L}$-isomorphism with domain $\widetilde{Q}_{\widetilde{\delta}}$, otherwise $m_{\widehat{\mathcal{P}}}(Q_{\widehat{\delta}}, P_{\widehat{\gamma}}) = 0$. Thus far, we have specified all the data determining $\widehat{\mathcal{P}}$, but we have not yet shown that $\widehat{\mathcal{P}}$ is a pointed refinement of $\mathcal{F}$.

For any $P \leq D$, when γ denotes an element of $\mathcal{LP}_P$, it is to be understood that

$$\widetilde{\gamma} = B \cap {}^{\theta_P^{-1}}\gamma$$

which, by Theorem 3.1, is a point of $\widetilde{P}$ on B. For such γ, it is also to be understood that

$$\widehat{\gamma} = \Theta_P(\widetilde{\gamma}) \ .$$

We let ι be the family of bijections $\iota_P : \widehat{P}_P \leftarrow \mathcal{LP}_P$ such that $\iota_P(\gamma) = \widehat{\gamma}$.

Simultaneously, we shall show that $\widehat{\mathcal{P}}$ is a pointed refinement of $\mathcal{F}$ and that ι is an $\mathcal{F}$-identical isomorphism $\widehat{\mathcal{P}} \leftarrow \mathcal{LP}$. We are to show preservation of morphisms and multiplicities. That is to say, we are to show that, for all $\mathcal{F}$-morphisms ϕ with domain P and $\gamma \in \mathcal{LP}_P$, we have ${}^{\phi}\widehat{\gamma} = \widehat{{}^{\phi}\gamma}$, and we are also to show that, for all $\mathcal{LP}$-objects Q_δ and P_γ with $Q \leq P$, we have $m_{\widehat{\mathcal{P}}}(Q_{\widehat{\delta}}, P_{\widehat{\gamma}}) = m(Q_\delta, P_\gamma)$.

Given ϕ and P_γ as specified, then

$$\widetilde{{}^{\phi}\gamma} = B \cap {}^{\theta_{\phi(P)}^{-1}\phi}\gamma = B \cap {}^{\widetilde{\phi}\theta_P^{-1}}\gamma = {}^{\widetilde{\phi}}\widetilde{\gamma} = {}^{\widetilde{\phi}}\Theta_P^{-1}(\widehat{\gamma}) \ .$$

So $\widehat{{}^{\phi}\gamma} = \Theta_{\phi(P)}(\widetilde{{}^{\phi}\gamma}) = {}^{\phi}\widehat{\gamma}$. We have established preservation of morphisms.

Given an $\mathcal{LP}$-object Q_δ with $Q \leq P$, then

$$m(Q_\delta, P_\gamma) = m({}^{\theta_P^{-1}}(Q_\delta), {}^{\theta_P^{-1}}(P_\gamma)) \ .$$

Let θ be as above. If θ appears as an $\mathcal{L}$-isomorphism with domain $\widetilde{Q}_{\widetilde{\delta}}$, then

$$m({}^{\theta_P^{-1}}(Q_\delta), {}^{\theta_P^{-1}}(P_\gamma)) = m({}^{\theta}(\widetilde{Q}_{\widetilde{\delta}}), \widetilde{P}_{\widetilde{\gamma}}) = m_{\widehat{\mathcal{P}}}(Q_{\widehat{\delta}}, P_{\widehat{\gamma}}) \ .$$

Now suppose θ does not appear as an $\mathcal{L}$-isomorphism with domain $\widetilde{Q}_{\widetilde{\delta}}$. Then $m_{\widehat{\mathcal{P}}}(Q_{\widehat{\delta}}, P_{\widehat{\gamma}}) = 0$. Since $\widetilde{Q}$ is fully $\mathcal{F}$-centralized, Theorem 3.1 implies that ${}^{\theta_Q}(Q_\delta) \leq D_\lambda$. Hence, by the hypothesis on θ, we have ${}^{\theta_P^{-1}}(Q_\delta) \not\leq D_\lambda$. Perforce, ${}^{\theta_P^{-1}}(Q_\delta) \not\leq {}^{\theta_P^{-1}}(P_\gamma)$. So $Q_\delta \not\leq P_\gamma$, in other words, $m(Q_\delta, P_\gamma) = 0$. We have established preservation of multiplicities. □

4 Conjectures on bounds

Let G, D, B, $\mathcal{B}$ be as introduced in Section 2. We continue to work with the pointed fusion system $\mathcal{LP} = \mathcal{LP}(B)$. We shall discuss three related conjectures concerning bounds in terms of the defect group D.

A statement of Puig's Conjecture can be found in Thévenaz [8, 38.5]. Confirmation of an assertion stated without proof in [8, 38.6] would imply that the following conjecture is equivalent to Puig's Conjecture. For a finitely generated algebra Λ over $\mathcal{O}$, we define $\mathbb{F}\Lambda = \mathbb{F} \otimes_{\mathcal{O}} \Lambda$ as an algebra over $\mathbb{F}$.

Conjecture 4.1. (Weak Puig Conjecture.) *Fixing D, there is a bound on the dimension of the source D-algebra $\mathbb{F}B$ of the block algebra $\mathbb{F}\mathcal{B}$.*

The next conjecture was raised in Eaton–Kessar–Külshammer–Sambale [5, 9.1].

Conjecture 4.2. (Weak Donovan Conjecture.) *Fixing D, then there is a bound on the Cartan invariants of the block algebra $\mathbb{F}\mathcal{B}$.*

Conjecture 4.3. (Bounded Multiplicities Conjecture.) *Fixing D, then there is a bound on the multiplicities of the pointed fusion system $\mathcal{LP}$ of $\mathcal{B}$.*

Proposition 4.4. *Fixing D, then the Weak Puig Conjecture holds for D if and only if the Weak Donovan Conjecture and the Bounded Multiplicities Conjecture hold for D.*

Proof. Fix $\mathcal{B}$ and B. Let c be the maximum of the Cartan invariants of $\mathcal{B}$. Let m be the maximum of the $\mathcal{LP}$-multiplicities. We shall show that $c \leq \dim_{\mathbb{F}}(\mathbb{F}B) \geq m$ and

$$\dim_{\mathbb{F}}(\mathbb{F}B) \leq cm^2|D|^4 \ .$$

Since $\mathbb{F}\mathcal{B}$ and $\mathbb{F}B$ are Morita equivalent, they have the same Cartan invariants and $c \leq \dim_{\mathbb{F}}(\mathbb{F}B)$. Given $\mathcal{LP}$-objects Q_δ and P_γ with $Q \leq P$ then, for all $Q \leq R \leq P$, we have

$$m(Q_\delta, P_\gamma) \geq \sum_{\epsilon \in \mathcal{LP}_R} m(Q_\delta, R_\epsilon)\, m(R_\epsilon, P_\gamma) \ .$$

Therefore, $m = m(1_\tau, D_\lambda)$ for some point τ on $\mathcal{B}$. Letting W be a simple B-module not annihilated by the point $B \cap \tau$ on B, then $m = \dim_{\mathbb{F}}(W) \leq \dim_{\mathbb{F}}(\mathbb{F}B)$.

By the above Morita equivalence, the number ℓ of isomorphism classes of simple B-modules is equal to the number of isomorphism classes of simple $\mathcal{B}$-modules. A theorem of Brauer and Feit in Linckelmann [6, 6.12.1] asserts that $\ell \leq |D|^2/4 + 1$. If D is trivial, then $\ell = 1$. So $\ell \leq |D|^2$. We have

$$\dim_{\mathbb{F}}(\mathbb{F}B) = \sum_V \dim_{\mathbb{F}}(V) \dim_{\mathbb{F}}(L_V)$$

where V runs over the simple $\mathbb{F}B$-modules up to isomorphism and L_V denotes the projective cover of V. Each $\dim_{\mathbb{F}}(V) \leq m$ and $\dim_{\mathbb{F}}(L_V) \leq cm\ell$. Therefore, $\dim_{\mathbb{F}}(\mathbb{F}B) \leq cm^2\ell^2$. □

In the case where $p = 2$ and D is abelian, the Weak Donovan Conjecture was proved in [5, 9.2]. Hence we obtain the following corollary.

Corollary 4.5. *Fixing D and supposing $p = 2$ and D is abelian, then the Weak Puig Conjecture holds for D if and only if the Bounded Multiplicities Conjecture holds for D.*

5 Isomorphisms induced by equivalences

Let G, b, D, B, $\mathcal{B}$ be as in Section 2. In Theorem 5.6, we shall describe how a splendid Morita equivalence from $\mathcal{B}$ gives rise to an isomorphism of pointed fusion systems. In Theorem 5.7, we shall describe how a splendid stable equivalence of Morita type from $\mathcal{B}$ gives rise to isomorphisms between stable parts of pointed fusion systems.

Given a group R, we define $\Delta(R) = \{(z, z) : z \in R\}$ as a subgroup of $R \times R$. Recall, given a p-subgroup P of G and an $\mathcal{O}G$-module M, the **Brauer construction** of M at P is defined to be the $\mathbb{F}N_G(P)/P$-module

$$M(P) = M^P / \sum_{Q \leq P} \mathrm{tr}_Q^P(M^Q)$$

where tr_Q^P denotes the transfer map $M^P \leftarrow M^Q$. A theorem of Broué [3, 3.2] implies that if M has vertex P, then $M(P)$ is projective and indecomposable.

Lemma 5.1. *Let E be a finite p-group and let M be a permutation $\mathcal{O}E$-$\mathcal{O}E$-bimodule that is free as a left $\mathcal{O}E$-module and as a right $\mathcal{O}E$-module. Let $P \leq E$ and let N be a permutation $\mathcal{O}E$-$\mathcal{O}P$-bimodule that is free as a left $\mathcal{O}E$-module and as a right $\mathcal{O}P$-module. Suppose*

$$(M \otimes_{\mathcal{O}E} N)(\Delta(P)) \neq 0 \ .$$

Then there exists a monomorphism $\phi : E \leftarrow P$ such that $N(\Delta(\phi)) \neq 0$.

Proof. Let S be an E-E-stable basis for M. Let T be an E-P-stable basis for N. The hypothesis implies that there exists $(s,t) \in S{\times}T$ such that $xs \otimes tx^{-1} = s \otimes t$ for all $x \in P$. There is a monomorphism $\phi : E \leftarrow P$ given by $(xs\phi(x)^{-1}, \phi(x)tx^{-1}) = (s,t)$. □

Again, let P be a p-subgroup of G. In view of the equality

$$N_{G\times P}(\Delta(P)) = (C_G(P){\times}1)\Delta(P)$$

we have an evident isomorphism

$$N_{G\times P}(\Delta(P))/\Delta(P) \cong C_G(P) \ .$$

Via that isomorphism, given an $\mathcal{O}(G{\times}P)$-module M, we can regard $M(\Delta(P))$ as an $\mathbb{F}C_G(P)$-module.

Adapting a definition in Section 2, for any pointed group P_μ on $\mathcal{O}G$, choosing $i \in \mu$, we define the **diagonal module**

$$\mathrm{Dia}(P_\mu) = \mathcal{O}Gi$$

as an $\mathcal{O}(G{\times}P)$-module. Again, it is clear that $\mathrm{Dia}(P_\mu)$ is well-defined independently of the choice of i. Again, the primitivity of i implies that $\mathrm{Dia}(P_\mu)$ is indecomposable. We claim that the point μ of P is local if and only if $\mathrm{Dia}(P_\mu)$ has vertex $\Delta(P)$. Since $\mathrm{Dia}(P_\mu)$ is a direct summand of the permutation $\mathcal{O}(G{\times}P)$-module $\mathcal{O}G \cong \mathcal{O}(G{\times}P)/\Delta(P)$, some vertex of $\mathrm{Dia}(P_\mu)$ is contained in $\Delta(P)$. There is an $\mathcal{O}$-linear isomorphism

$$\mathrm{End}_{\mathcal{O}(G\times 1)}(\mathcal{O}Gi) \cong \mathcal{O}Gi$$

given by $\sigma \leftrightarrow \sigma(i)$ for an $\mathcal{O}(G{\times}1)$-endomorphism σ of $\mathcal{O}Gi$. Given $Q \leq P$, then $\mathrm{tr}_{\Delta(Q)}^{\Delta(P)}(\sigma) = \mathrm{id}_{\mathcal{O}Gi}$ if and only if $\mathrm{tr}_{\Delta(Q)}^{\Delta(P)}(\sigma(i)) = i$. The claim follows. We mention that diagonal modules are discussed more systematically in [1], but our present use of them is independent of the material there.

Let P_γ be a local pointed group on $\mathcal{B}$. Then P_γ is a pointed group on $\mathcal{O}G$ and we can form the diagonal module $M = \mathrm{Dia}(P_\gamma)$, which is indecomposable with vertex $\Delta(P)$. Since $bM = M$, we can regard M as a $\mathcal{B}$-$\mathcal{P}$-bimodule. We have $\mathrm{br}_P(b)M(\Delta(P)) = M(\Delta(P))$. So $M(\Delta(P))$ is a projective indecomposable $\mathbb{F}C_G(P)\mathrm{br}_P(b)$-module.

Proposition 5.2. *Let $P \leq D$. Then the condition* $\mathrm{Dia}(P_\gamma) \cong M$ *characterizes a bijective correspondence $\gamma \leftrightarrow [M]$ between:*

(a) *the local points γ of P on $\mathcal{B}$ such that P_γ is overshadowed by B,*

(b) *the isomorphism classes $[M]$ of indecomposable $\mathcal{B}$-$\mathcal{O}P$-bimodules M with vertex $\Delta(P)$ such that $M \,|\, \mathcal{B}$ and $e_P M(\Delta(P)) = M(\Delta(P))$.*

Proof. By comments above, the specified condition characterizes a bijective correspondence between the local points γ of P on $\mathcal{B}$ and the isomorphism classes $[M]$ of indecomposable $\mathcal{B}$-$\mathcal{O}P$-bimodules M with vertex $\Delta(P)$ such that $M \,|\, \mathcal{B}$. For such γ and M, supposing $\gamma \leftrightarrow [M]$, then P_γ is overshadowed by B if and only if $e_P M(\Delta(P)) = M(\Delta(P))$. □

We shall be needing two lemmas describing how isomorphisms between $\mathcal{LP}(B)$-objects induce isomorphisms between diagonal modules and how multiplicities between $\mathcal{LP}(B)$-objects can be expressed in terms of diagonal modules.

Lemma 5.3. *Given an $\mathcal{LP}(B)$-object P_γ and an $\mathcal{F}$-morphism ϕ with domain P, then*

$$\mathrm{Dia}({}^{\phi}(P_\gamma)) \cong \mathrm{Dia}(P_\gamma) \otimes_{\mathcal{O}P} (P{\times}\phi(P))/\Delta(\phi^{-1}) \, .$$

Proof. Let $i \in \gamma$ and let u be a unit in $\mathcal{B}^{\Delta(\phi)}$. Then ${}^u i \in {}^\phi\gamma$. We have $\mathcal{O}G.{}^u i = \mathcal{O}Giu^{-1}$ and $u^{-1} \in \mathcal{B}^{\Delta(\phi^{-1})}$. □

Lemma 5.4. *Given $\mathcal{LP}(B)$-objects P_γ and Q_δ with $Q \leq P$, then*

$$m(Q_\delta, P_\gamma) = m(\mathrm{Dia}(Q_\delta), {}_{G\times Q}\mathrm{Res}_{G\times P}(P_\gamma)) = m(\mathrm{Dia}(Q_\delta), \mathrm{Dia}(P_\gamma) \otimes_{\mathcal{O}P} (P{\times}Q)/\Delta(Q)) \, .$$

Proof. The first equality is clear. The functor $- \otimes_{\mathcal{O}P} (P{\times}Q)/\Delta(Q)$ is the restriction functor to right $\mathcal{O}Q$-modules from right $\mathcal{O}P$-modules. □

We shall also be needing the following part of Linckelmann [6, 8.7.1].

Theorem 5.5. (Linckelmann.) *Given $P, Q \leq D$, then every indecomposable direct summand of B, as an $\mathcal{O}P$-$\mathcal{O}Q$-bimodule, is isomorphic to $\mathcal{O}(P{\times}Q)/\Delta(\psi)$ for some $\mathcal{F}$-isomorphism ψ to a subgroup of P from a subgroup of Q.*

We now introduce another block with the same defect group D. Let F be a finite group, let a be a block of $\mathcal{O}F$ with defect group D and let A be a source D-algebra of the block algebra $\mathcal{A} = \mathcal{O}Fa$.

An $\mathcal{A}$-$\mathcal{B}$-bimodule M is said to induce a **splendid Morita equivalence** to $\mathcal{A}$ from $\mathcal{B}$ with respect to A and B provided the following two conditions hold:

- M and the dual M^* induce a Morita equivalence between $\mathcal{A}$ and $\mathcal{B}$,
- M is an indecomposable direct summand of $\mathcal{O}F1_A \otimes_{\mathcal{O}D} 1_B\mathcal{O}G$.

A theorem of Puig and Scott in [6, 9.7.4] asserts that there is a splendid Morita equivalence to $\mathcal{A}$ from $\mathcal{B}$ with respect to A and B if and only if there is an interior D-algebra isomorphism $A \cong B$. We mention that the hypothesis on the coefficient ring in [6] is slightly different, but the proof in [6] carries over, without change, to the case of arbitrary $\mathcal{O}$. Note that, when the two equivalent conditions hold, the fusion system $\mathcal{F}$ associated with B is also the fusion system associated with A.

Theorem 5.6. *Suppose there is an $\mathcal{A}$-$\mathcal{B}$-bimodule M inducing a splendid Morita equivalence to $\mathcal{A}$ from $\mathcal{B}$ with respect to A and B. Then an $\mathcal{F}$-identical isomorphism $\iota : \mathcal{LP}(A) \leftarrow \mathcal{LP}(B)$ is given by* $\mathrm{Dia}(P_{\iota_P(\gamma)}) \cong M \otimes_{\mathcal{B}} \mathrm{Dia}(P_\gamma)$ *for any $\mathcal{LP}(B)$-object P_γ.*

Proof. Fix an $\mathcal{LP}(B)$-object P_γ and let

$$L = M \otimes_{\mathcal{B}} \mathrm{Dia}(P_\gamma)$$

as an $\mathcal{A}$-$\mathcal{O}P$-bimodule. We shall show that there exists a point α of P on $\mathcal{A}$ such that P_α is a $\mathcal{LP}(A)$-object and $L \cong \mathrm{Dia}(P_\alpha)$. Since $\mathrm{Dia}(P_\gamma) \cong M^* \otimes_{\mathcal{A}} L$, it will then follow, by Proposition 5.2, that the function $P_\alpha \leftarrow\!\shortmid P_\gamma$ is a bijection to the set of $\mathcal{LP}(A)$-objects from the set of $\mathcal{LP}(B)$-objects.

The functor $M \otimes_{\mathcal{B}} -$ is a Morita equivalence to $\mathcal{A} \otimes_{\mathcal{O}} \mathcal{O}P$ from $\mathcal{B} \otimes_{\mathcal{O}} \mathcal{O}P$, so L is indecomposable. Letting $i \in \gamma$, we have $\mathrm{Dia}(P_\gamma) \cong \mathcal{O}Gi$, hence $L \cong Mi$. So $L \,|\, \mathcal{O}F1_A \otimes_{\mathcal{O}D} W$ for some indecomposable direct summand W of the $\mathcal{O}D$-$\mathcal{O}P$-bimodule $1_B\mathcal{O}Gi$. Therefore,

$$\mathrm{Dia}(P_\gamma) \cong M^* \otimes_{\mathcal{A}} L \,|\, M^* 1_A \otimes_{\mathcal{O}D} W \; .$$

Since $\mathrm{Dia}(P_\gamma)$ has vertex $\Delta(P)$, we have $\mathrm{Dia}(P_\gamma)(\Delta(P)) \neq 0$, so $(M^*1_A \otimes_{\mathcal{O}D} W)(\Delta(P)) \neq 0$. By Lemma 5.1, $W(\Delta(\psi)) \neq 0$ for some monomorphism $\psi : D \leftarrow P$. Since W is indecomposable, $W \cong \mathcal{O}(D \times P)/\Delta(\psi)$. By Theorem 5.5, ψ is an $\mathcal{F}$-morphism. Let $I = \mathcal{O}(\psi(P) \times P)/\Delta(\psi)$ as an $\mathcal{O}\psi(P)$-$\mathcal{O}P$-bimodule. Writing I° for the opposite bimodule of I, we have

$$W \otimes_{\mathcal{O}P} I^\circ \cong \mathcal{O}(D \times \psi(P))/\Delta(\psi(P)) \; .$$

So $- \otimes_{\mathcal{O}D} W \otimes_{\mathcal{O}P} I^\circ$ is the restriction functor to right $\mathcal{O}\psi(P)$-modules from right $\mathcal{O}D$-modules. Therefore,

$$L \otimes_{\mathcal{O}P} I^\circ \,|\, \mathcal{O}F1_A \otimes_{\mathcal{O}D} W \otimes_{\mathcal{O}P} I^\circ \cong \mathcal{O}F1_A$$

as $\mathcal{A}$-$\mathcal{O}\psi(P)$-bimodules. Since $L \otimes_{\mathcal{O}P} I^\circ$ is indecomposable, there exists a primitive idempotent h' of $A^{\psi(P)}$ such that

$$L \otimes_{\mathcal{O}P} I^\circ \cong \mathcal{O}Fh'$$

as $\mathcal{A}$-$\mathcal{O}\psi(P)$-bimodules. Let $f \in F$ such that ψ is conjugation by f. Let h be the primitive idempotent of A^P such that $h' = {}^f h$. Let α be the point of P on $\mathcal{A}$ owning h. We have

$$L \cong L \otimes_{\mathcal{O}P} I^\circ \otimes_{\mathcal{O}\psi(P)} I \cong \mathcal{O}Fh' \otimes_{\mathcal{O}\psi(P)} \mathcal{O}\psi(P)fP \cong \mathcal{O}Fh \cong \mathrm{Dia}(P_\alpha) \; .$$

To show that α is local, let Q be a defect group of α. As an $\mathcal{O}(F \times P)$-module, L is isomorphic to a direct summand of a module induced from $F \times Q$. By Green's indecomposability Criterion,

$$L \cong {}_{F \times P}\mathrm{Ind}_{F \times Q}(K) \cong K \otimes_{\mathcal{O}Q} \mathcal{O}P$$

for some $\mathcal{A}$-$\mathcal{O}Q$-bimodule K. We have

$$\mathrm{Dia}(P_\gamma) \cong M^* \otimes_{\mathcal{A}} L \cong M^* \otimes_{\mathcal{A}} K \otimes_{\mathcal{O}Q} \mathcal{O}P \cong {}_{G \times P}\mathrm{Ind}_{G \times Q}(M^* \otimes_{\mathcal{A}} K) \; .$$

But $\mathrm{Dia}(P_\gamma)$ has vertex $\Delta(P)$, so $Q = P$ and α is local.

To show that P_α is overshadowed by A, let α' be the point of $\psi(P)$ on $\mathcal{A}$ owning h'. Since $h' \in A^{\psi(P)}$, we have $\psi(P)_{\alpha'} \in (\psi(P), e_{\psi(P)})$. But $\psi(P)_{\alpha'} = {}^f(P_\alpha)$ and $(\psi(P), e_{\psi(P)}) = {}^f(P, e_P)$, so $P_\alpha \in (P, e_P)$, in other words, P_α is overshadowed by A.

We have now established that P_α is an $\mathcal{LP}(A)$-object. It remains to show that ι_P preserves fusions and multiplicities. Let ϕ be an $\mathcal{F}$-morphism with domain P. Lemma 5.3 tells us that, applying the functor $- \otimes_{\mathcal{O}P} (P \times \phi(P))/\Delta(\phi^{-1})$, we have

$$\mathrm{Dia}({}^\phi(P_\alpha)) \cong M \otimes \mathrm{Dia}({}^\phi(P_\gamma)) \; .$$

So ${}^\phi\alpha = \iota_{\phi(P)}({}^\phi\gamma)$ and ι preserves fusions. Let $Q \leq P$ and let δ be an $\mathcal{LP}(B)$-point of Q. Let β be the $\mathcal{LP}(A)$-point of Q such that $\mathrm{Dia}(Q_\beta) \cong M \otimes_{\mathcal{B}} \mathrm{Dia}(Q_\delta)$. Since $M \otimes_{\mathcal{B}} -$ is a Morita equivalence to $\mathcal{A} \otimes_{\mathcal{O}} \mathcal{O}Q$ from $\mathcal{B} \otimes_{\mathcal{O}} \mathcal{O}Q$, Lemma 5.4 yields $m(Q_\beta, P_\alpha) = m(Q_\delta, P_\gamma)$. So ι preserves multiplicities. $\square$

An $\mathcal{A}$-$\mathcal{B}$-bimodule M is said to induce a **splendid stable equivalence of Morita type** to $\mathcal{A}$ from $\mathcal{B}$ with respect to A and B provided the following two conditions hold:

- M and the dual M^* induce a stable equivalence of Morita type between $\mathcal{A}$ and $\mathcal{B}$,
- M is an indecomposable direct summand of $\mathcal{O}F1_A \otimes_{\mathcal{O}D} 1_B\mathcal{O}G$.

By [6, 9.8.2], when such M exists, the fusion system associated with A is $\mathcal{F}$.

Theorem 5.7. *Suppose there is an $\mathcal{A}$-$\mathcal{B}$-bimodule M inducing a splendid stable equivalence of Morita type to $\mathcal{A}$ from $\mathcal{B}$ with respect to A and B. Then there is an $\mathcal{F}$-identical isomorphism $\bar{\iota} : \overline{\mathcal{LP}}(A) \leftarrow \overline{\mathcal{LP}}(B)$ such that $\mathrm{Dia}(P_{\bar{\iota}_P(\gamma)})$ is the non-projective part of $M \otimes_{\mathcal{B}} \mathrm{Dia}(P_\gamma)$ for any $\overline{\mathcal{LP}}(B)$-object P_γ.*

Proof. Write $M^* \otimes_{\mathcal{A}} M \cong \mathcal{B} \oplus Y$ where Y is a projective $\mathcal{B}$-$\mathcal{B}$-module. Fix an $\overline{\mathcal{LP}}(B)$-object P_γ. We have a direct sum of $\mathcal{A}$-$\mathcal{O}P$-bimodules

$$L \oplus L' \cong M \otimes_{\mathcal{B}} \mathrm{Dia}(P_\gamma)$$

where L is indecomposable and non-projective while L' is projective. Letting $i \in \gamma$, then $\mathrm{Dia}(P_\gamma) \cong \mathcal{O}Gi$ and

$$M^* \otimes_{\mathcal{A}} L \oplus M^* \otimes_{\mathcal{A}} L' \cong (\mathcal{B} \oplus Y) \otimes_{\mathcal{B}} \mathrm{Dia}(P_\gamma) \cong \mathrm{Dia}(P_\gamma) \oplus Yi\ .$$

Since $M^* \otimes_{\mathcal{A}} L'$ and Yi are projective, $\mathrm{Dia}(P_\gamma)$ is the non-projective part of $M^* \otimes_{\mathcal{A}} L$. We shall show that there exists an $\overline{\mathcal{LP}}(A)$-object P_α such that $L \cong \mathrm{Dia}(P_\alpha)$. It will then follow, by Proposition 5.2, that there is a bijective correspondence $P_\alpha \leftrightarrow P_\gamma$ between the $\overline{\mathcal{LP}}(A)$-objects and the $\overline{\mathcal{LP}}(B)$-objects.

We have $L \oplus L' \cong Mi$. So $L \,|\, \mathcal{O}F1_A \otimes_{\mathcal{O}D} W$ for some indecomposable $\mathcal{O}D$-$\mathcal{O}P$-bimodule W such that $W \,|\, 1_B\mathcal{O}Gi$. Since $\mathrm{Dia}(P_\gamma)$ is the non-projective part of $M^* \otimes_{\mathcal{A}} L$, we have

$$(M^* \otimes_{\mathcal{A}} L)(\Delta(P)) \cong (\mathrm{Dia}(P_\gamma))(\Delta(P)) \neq 0\ .$$

But $M^* \otimes_{\mathcal{A}} L \,|\, M^* \otimes_{\mathcal{A}} \mathcal{O}F1_A \otimes_{\mathcal{O}D} W \cong M^*1_A \otimes_{\mathcal{O}D} W$. So

$$(M^*1_A \otimes_{\mathcal{O}D} W)(\Delta(P)) \neq 0\ .$$

The next stage of the argument proceeds much as in the proof of Theorem 5.6. Let us summarize it. Again, we find that $W \cong \mathcal{O}(D{\times}P)/\Delta(\psi)$ for some $\mathcal{F}$-morphism ψ. Letting I be as before, we find that there exists a primitive idempotent h' of $A^{\psi(P)}$ such that $L \otimes_{\mathcal{O}P} I^\circ \cong \mathcal{O}Fh'$. By considering f, h, α as before, we deduce that $L \cong \mathrm{Dia}(P_\alpha)$.

To show that α is local, adaptation of the analogous argument in the proof of Theorem 5.6 requires some care. We again use Green's Indecomposability Criterion to show that $L \cong K \otimes_{\mathcal{O}Q} \mathcal{O}P$ where K is an $\mathcal{A}$-$\mathcal{O}Q$-bimodule and Q is a defect group of α. Again,

$$M^* \otimes_{\mathcal{A}} L \cong {}_{G\times P}\mathrm{Ind}_{G\times Q}(M^* \otimes_{\mathcal{A}} K)\ .$$

But we saw above that $(M^* \otimes_{\mathcal{A}} L)(\Delta(P)) \neq 0$. So $Q = P$ and α is local. To show that P_α is overshadowed by A the argument is very similar to what we did before.

Thus, we have established that P_α is an $\overline{\mathcal{LP}}(A)$-object, and it remains only to check preservation of fusions and multiplicities. Let ϕ be as before. Write $J = \mathcal{O}(P{\times}\phi(P))/\Delta(\phi^{-1})$.

Applying the functor $-\otimes_{\mathcal{O}P} J$ to the isomorphism $\mathrm{Dia}(P_\alpha) \oplus L' \cong M \otimes_{\mathcal{B}} \mathrm{Dia}(P_\gamma)$ and using Lemma 5.3, we obtain

$$\mathrm{Dia}({}^\phi(P_\alpha)) \oplus L' \otimes_{\mathcal{O}P} J \cong M \otimes_{\mathcal{B}} \mathrm{Dia}({}^\phi(P_\gamma)) \ .$$

Since $L' \otimes_{\mathcal{O}P} J$ is a projective $\mathcal{A}$-$\mathcal{O}\phi(P)$-bimodule, ${}^\phi\alpha = \bar{\iota}_{\phi(P)}({}^\phi\gamma)$ and $\bar{\iota}$ preserves fusions.

To show that $\bar{\iota}$ preserves multiplicities, let Q be a non-trivial subgroup of P, let δ be an $\overline{\mathcal{LP}}(B)$-point of Q and let β be the $\overline{\mathcal{LP}}(A)$-point of Q such that

$$\mathrm{Dia}(Q_\beta) \oplus N \cong M \otimes_{\mathcal{B}} \mathrm{Dia}(Q_\delta)$$

for some projective $\mathcal{B}$-$\mathcal{O}Q$-bimodule N. Since $L' \otimes_{\mathcal{O}P} \mathcal{O}(P{\times}Q)/\Delta(Q)$ is a projective $\mathcal{A}$-$\mathcal{O}Q$-bimodule, Lemma 5.4 yields

$$m(Q_\beta, P_\alpha) = m(\mathrm{Dia}(Q_\beta), M \otimes_{\mathcal{B}} \mathrm{Dia}(P_\alpha) \otimes_{\mathcal{O}P} \mathcal{O}(P{\times}Q)/\Delta(Q)) \ .$$

The non-projective part of $M^* \otimes_{\mathcal{A}} \mathrm{Dia}(Q_\beta)$ is $\mathrm{Dia}(Q_\delta)$ and the non-projective part of $M^* \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \mathrm{Dia}(P_\gamma) \otimes_{\mathcal{O}P} \mathcal{O}(P{\times}Q)/\Delta(Q)$ is isomorphic to the non-projective part of $\mathrm{Dia}(P_\gamma) \otimes_{\mathcal{O}P} (P{\times}Q)/\Delta(Q)$, so

$$m(Q_\beta, P_\alpha) = m(\mathrm{Dia}(Q_\delta), \mathrm{Dia}(P_\gamma) \otimes_{\mathcal{O}P} \mathcal{O}(P{\times}Q)/\Delta(Q)) \ .$$

By Lemma 5.4 again, $m(Q_\beta, P_\alpha) = m(Q_\delta, P_\gamma)$. So $\bar{\iota}$ preserves multiplicities. □

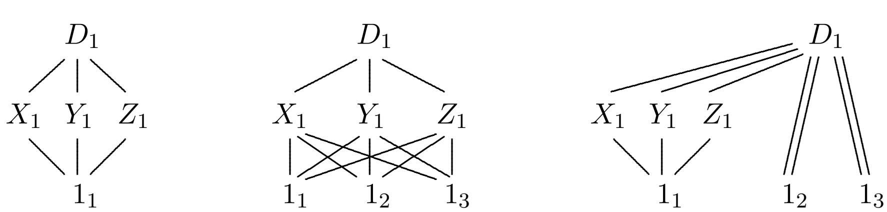
6 Klein-four defect groups

Once again, let G, b, D, B, $\mathcal{B}$ be as in Section 2. In the case where $p = 2$ and $D \cong V_4$, we shall describe all the possibilities for the underlying multiposet of the pointed fusion system $\mathcal{LP} = \mathcal{LP}(B)$. This will be an application of the following theorem of Craven–Eaton–Kessar–Linckelmann [4, 1.1]. Their proof of the theorem relies on the classification of simple finite groups.

Theorem 6.1. (Craven–Eaton–Kessar–Linckelmann.) *Supposing $p = 2$ and $D \cong V_4$ then, as an interior D-algebra, B is isomorphic to $\mathcal{O}D$ or $\mathcal{O}A_4$ or the principal block algebra of $\mathcal{O}A_5$. In the latter two cases, D is identified with a Sylow 2-subgroup of A_4 or A_5.*

Suppose $p = 2$ and $D \cong V_4$. By Theorem 6.1, together with Theorem 5.6, we shall have described all the possible multiposet structures for $\mathcal{LP}$ when we have done so in the three cases where $G \in \{D, A_4, A_5\}$ and b is the principal block of $\mathcal{O}G$.

Let X, Y, Z be the proper subgroups of D. For any $R \in \{X, Y, Z, D\}$, we have $C_G(R) = D$. So there exists a unique local point γ^R of R on $\mathcal{B}$. We write $R_1 = R_{\gamma^R}$. Let V_1, ... be representatives of the isomorphism classes of simple $\mathcal{B}$-modules, enumerated such that V_1 is trivial. Let γ_i be the point on $\mathcal{B}$ that does not annihilate V_i. We write $1_i = 1_{\gamma_i}$. Thus, every $\mathcal{LP}$-object has the form R_1 or 1_i. We shall show that, in the three cases where G is D, A_4, A_5, respectively, the multiposet structure of $\mathcal{LP}$ is as shown, where the double lines indicate multiplicity 2 and all the other multiplicities are 1.



For the rest of this paper, we regard $\mathcal{LP}$ as a multiposet. Plainly, if $G = D$, then $\mathcal{LP}$ is as depicted in the left-hand diagram. The remaining two cases share some common features. Henceforth, suppose that $G = A_4$ or $G = A_5$ and let b be the principal block of $\mathcal{O}G$. By Theorem 6.1 (or an easy direct argument which we omit), $B = \mathcal{B}$. The points on $\mathcal{B}$ are the γ_i with $i \in \{1, 2, 3\}$. Let E_i be the indecomposable projective $\mathcal{B}$-module, well-defined up to isomorphism, such that $V_i \cong E_i/J(E_i)$. Transporting via the isomorphism $G{\times}1 \cong G$, we have

$$\mathrm{Dia}(1_i) \cong {}_{G\times 1}\mathrm{Iso}_G(E_i) \ .$$

To proceed further, we consider the two cases separately.

Suppose $G = A_4$. Then $\mathcal{O}G \cong E_1 \oplus E_2 \oplus E_3$ as $\mathcal{O}G$-modules. So

$$\mathcal{O}G \cong \mathrm{Dia}(1_1) \oplus \mathrm{Dia}(1_2) \oplus \mathrm{Dia}(1_3)$$

as $\mathcal{O}(G{\times}1)$-modules. We also have $\mathcal{O}G \cong \mathrm{Dia}(X_1) \oplus E$ as $\mathcal{O}(G{\times}X)$-modules, where E is projective. Restricting via the embedding $G \cong G{\times}1 \hookrightarrow G{\times}X$, we have ${}_G\mathrm{Res}_{G\times X}(E) \,|\, E_1 \oplus E_2 \oplus E_3$. But every projective $\mathcal{O}(G{\times}X)$-module restricts to a direct sum of 2 isomorphic copies of a projective $\mathcal{O}G$-module. Therefore, $E = 0$, that is, $\mathrm{Dia}(X_1) \cong \mathcal{O}G$ as $\mathcal{O}(G{\times}X)$-modules and

$${}_{G\times 1}\mathrm{Res}_{G\times X}(\mathrm{Dia}(X_1)) \cong \mathrm{Dia}(1_1) \oplus \mathrm{Dia}(1_2) \oplus \mathrm{Dia}(1_3) \ .$$

It follows that $\mathrm{Dia}(D_1) \cong \mathcal{O}G$ as $\mathcal{O}(G{\times}D)$-modules and

$${}_{G\times X}\mathrm{Res}_{G\times D}(\mathrm{Dia}(D_1)) \cong \mathrm{Dia}(X_1) \ .$$

Bearing in mind that the subgroups X, Y, Z are G-conjugate, we deduce, using Lemma 5.4, that $\mathcal{LP}$ is as depicted in the middle diagram above.

Now suppose $G = A_5$. Let $X < H < G$ with $H \cong D_{10}$, the dihedral group of order 10. We have $\mathcal{O}H \cong L \oplus L_0$ as $\mathcal{O}(H{\times}X)$-modules, where L_0 is projective and L is indecomposable with vertex $\Delta(X)$ and $\mathcal{O}$-rank $\mathrm{rk}_{\mathcal{O}}(L) = 2$. Since $\mathrm{Dia}(X_1)$ has vertex $\Delta(X)$ and

$$\mathrm{Dia}(X_1) \,|\, \mathcal{O}G \cong {}_{G\times X}\mathrm{Ind}_{H\times X}(\mathcal{O}H)$$

we have $\mathrm{Dia}(X_1) \,|\, {}_{G\times X}\mathrm{Ind}_{H\times X}(L)$. Therefore, $\mathrm{rk}_{\mathcal{O}}(\mathrm{Dia}(X_1)) \leq 12$. For $i \in \{1, 2, 3\}$, inducing via the embedding $G{\times}X \hookleftarrow G{\times}1 \cong G$, let $E_i^X = {}_{G\times X}\mathrm{Ind}_G(E_i)$. Then E_1^X, E_2^X, E_3^X comprise a set of representatives of the isomorphism classes of indecomposable projective $\mathcal{O}(G{\times}X)$-modules. Since $\mathrm{rk}_{\mathcal{O}}(E_1) = 12$ and $\mathrm{rk}_{\mathcal{O}}(E_2) = \mathrm{rk}_{\mathcal{O}}(E_3) = 8$, we have $\mathrm{rk}_{\mathcal{O}}(E_1^X) = 24$ and $\mathrm{rk}_{\mathcal{O}}(E_2^X) = \mathrm{rk}_{\mathcal{O}}(E_3^X) = 16$. Now

$$\mathcal{B} \cong \mathrm{Dia}(X_1) \oplus E$$

as $\mathcal{O}(G{\times}X)$-modules, where E is projective. We have $\mathrm{rk}_{\mathcal{O}}(\mathcal{B}) = 44$, so $32 \leq \mathrm{rk}_{\mathcal{O}}(E) < 44$. By considering an outer automorphism of G, we see that E_2^X and E_3^X have the same multiplicity as direct summands of E. The constraints we have obtained on the $\mathcal{O}$-ranks imply that $E \cong E_2^X \oplus E_3^X$. Therefore, $\mathrm{rk}_{\mathcal{O}}(\mathrm{Dia}(X_1)) = 12$.

We have $\mathcal{B} \cong E_1 \oplus 2E_2 \oplus 2E_3$ as $\mathcal{O}G$-modules, so

$$\mathcal{B} \cong \mathrm{Dia}(1_1) \oplus 2\mathrm{Dia}(1_2) \oplus 2\mathrm{Dia}(1_3)$$

as $\mathcal{O}(G{\times}1)$-modules. By the above isomorphism for E, we have

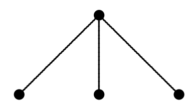
$${}_{G\times 1}\mathrm{Res}_{G\times X}(\mathrm{Dia}(X_1)) \cong \mathrm{Dia}(1_1) \ , \qquad {}_{G\times 1}\mathrm{Res}_{G\times X}(E) \cong 2\mathrm{Dia}(1_2) \oplus 2\mathrm{Dia}(1_3) \ .$$

We have $\mathrm{Dia}(D_1) \cong \mathcal{O}G1_B$ as $\mathcal{O}(G{\times}D)$-modules. But $B = \mathcal{B}$, so $\mathrm{Dia}(D_1) \cong \mathcal{B}$ as $\mathcal{O}(G{\times}D)$-modules and

$$_{G\times X}\mathrm{Res}_{G\times D}(\mathrm{Dia}(D_1)) \cong \mathrm{Dia}(X_1) \oplus E \; .$$

Again bearing in mind the G-conjugacy of X, Y, Z, an application of Lemma 5.4 yields the conclusion that $\mathcal{LP}$ is as depicted in the right-hand diagram above.

Our analysis of the three cases is now complete. It follows, in particular, that whenever the defect group of a 2-block is V_4, the underlying multiposet of the stable part $\overline{\mathcal{LP}}$ of the pointed fusion system is such that all the $\overline{\mathcal{LP}}$-multiplicities are unity and, as a poset, $\overline{\mathcal{LP}}$ has the following Hasse diagram.



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