On why $(-3)^2 = 9$

Laurence Barker, 8 March 2023

This note is in response to a query, at a secondary school level, about how to evaluate x^2 when $x = -3$.

The natural numbers are the numbers 0, 1, 2, 3 and so on. The integers are the numbers 0, 1, -1, 2, -2, 3, -3 and so on. Let us examine how some familiar patterns pertaining to the natural numbers seem to extend to the integers in general.

We have: 2 times 2 equals 4 and 1 times 2 equals 2 and 0 times 2 equals 0. Extending the pattern, -1 times 2 equals -2. Generalizing, -1 times a equals $-a$ for any natural number a.

Now, 2 times -1 equals -1 times 2 which, as we decided above, is equal to -2. We have 1 times -1 equals -1 and 0 times -1 equals 0. Extending the pattern, -1 times -1 equals 1.

The equality $(xy)^2 = x^2y^2$ holds for all natural numbers x and y. Let us assume that it also holds for all integers x and y. Then

$$(-3)^2 = \big((-1)3\big)^2 = (-1)^2 3^3$$

which is equal to 1 times 9, which is equal to 9.

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A weakness in the above argument is that it assumes that the negative numbers exist and that they behave according to the same rules as the natural numbers.

Historically, mathematicians have often resisted fantastic constructions, such as fractions, negative numbers, square roots of negative numbers. Those qualms have been reasonable and productive. A major stimulus to the great advances in mathematics made during the 4th century BC was the realization that there is some rather fascinating trouble with the notion of a square root of 2.

Suppose there exist natural numbers a and b such that $(a/b)^2 = 2$. Then we can take a to be as small as possible. Since $a^2 = 2b^2$, the number a is even, and we can write $a = 2\alpha$ for some natural number α. Then $2\alpha^2 = b^2$, so b is even and we can write $b = 2\beta$ for some natural number β. But now $(\alpha/\beta)^2 = 2$, which is impossible, indeed, α is smaller than a, and we have contradicted the condition that a is as small as possible. What we have shown is that, whatever a square root of 2 might be, if it exists, it cannot be expressed as one natural number divided by another natural number.

Granted that the notion of a square root of 2 may be unclear, then what, if anything, is -1 or, for that matter, -3? Maybe all the manipulations we did above, arriving at $(-3)^2 = 9$, was just sophistry about things that do not genuinely exist.

To make sense of the patterns that we used to arrive at $(-3)^2 = 9$, we must define the integers and we must define their addition and multiplication operations. We shall do so in a direct way, without going through the set theory that is a prerequisite for a standard textbook account.

Let us take the natural numbers 0, 1, 2, ... for granted. Actually, the construction of the natural numbers is a story in itself, and a great saga if one wishes to follow the historical development.

We define an integer to be a thing represented by an expression $[a, b]$ where a and b are natural numbers. Two integers $[a, b]$ and $[a', b']$ are understood to be equal to each other if and only if $a + b' = a' + b$. We define addition to be such that

$$[a, b] + [c, d] = [a + c, b + d] .$$

We define multiplication to be such that

$$[a, b][c, d] = [ac + bd, ad + bc] .$$

Some work is needed to confirm that the formulas make sense. Suppose $[a, b] = [a', b']$ and $[c, d] = [c', d']$. Since $c + d' = c' + d$, we have

$$a(c + d') + b(c' + d) = a(c' + d) + b(c + d') .$$

That equality can be rewritten as

$$(ac + bd) + (ad' + bc') = (ac' + bd') + (ad + bc) .$$

So $[ac+bd, ad+bc] = [ac'+bd', ad'+bc']$. Similarly, $[ac'+bd', ad'+bc'] = [a'c'+b'd', a'd'+b'c']$. Therefore,

$$[ac + bd, ad + bc] = [a'c' + b'd', a'd' + b'c'] .$$

We have shown that, given two integers x and y then, no matter what integers a, b, c, d one chooses such that $x = [a, b]$ and $y = [c, d]$, the formula for the multiplication operation will always give the same integer xy. In the jargon, we say that the multiplication of integers is well-defined. A similar but easier argument shows that the addition of integers is well-defined.

We identify each natural number a with the integer $[a, 0]$. Thus, each natural number can now be regarded as an integer, and the addition and multiplication of natural numbers has now been extended to operations, still called addition and multiplication, on the integers.

For a natural number a, we define $-a = [0, a]$. More generally, we define $-[a, b] = [b, a]$. It is easy to see that the operation sending an integer x to $-x$ is well-defined. Note that -1 times x equals $-x$. We define subtraction to be such that, given integers x and y, then $x - y = x + (-y)$. Finally,

$$(-3)^2 = [0, 3][0, 3] = [9, 0] = 9 .$$

Having completed the edifice, we can now get rid of the scaffolding. We have

$$[a, b] = a - b$$

for any natural numbers a and b. Our formula for multiplication of integers can now be seen as

$$(a - b)(c - d) = ac + bd - (ad + bc) .$$

We leave it, as an exercise for the reader, to prove that the integers obey many of the fundamental rules that are familiar from work with the natural numbers, such as $x + y = y + x$ and $x(y + z) = xy + xz$.

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To complete the discussion, one further point ought to be made. A novice might very reasonably feel that the above theory is useless, and that a grasp of the arithmetic of the integers is best obtained simply through practice, just presuming that all will be consistent.

But the idea behind the theory is adapted very frequently in many areas of mathematics. When the objects involved are more sophisticated, experts can become confused unless everything is defined precisely, in a matter that can be checked step by step. For instance, there is an intuitively digestible sense in which a plane can be combined with a line to form a three-dimensional space. In my area, we work with objects called modules, which have a kind of spacial structure. A 4-dimensional module can be added to a 7-dimensional module to give an 11-dimensional module. But sometimes, it is convenient to subtract a bigger module from a smaller one. The algebra becomes elegant if one understands a 4-dimensional module minus a 7-dimensional module to be what is called a virtual module, with dimension -3. Virtual modules can be useful, just as negative numbers are. So a critically minded novice, reacting very naturally against the theory above, ought to suspend judgement about whether the theory is valuable.