The Puig category of a block and conjectural isomorphism invariance of the multiplicities of the objects

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Joint work with Matthew Gelvin

The cast

Let p be a prime.

We shall associate a given p-block — that is, a given part of the p-modular representation theory of a given finite group — with:

- A finite p-group D called the **defect group**.
- A finite-dimensional algebra A, called an **almost-source algebra**, which comes equipped with a group homomorphism $A^\times \leftarrow D$. In particular, D acts as automorphisms of A.
- A finite category $\mathcal{F}$, called the **fusion system**, whose objects of $\mathcal{F}$ are the subgroups of D and whose morphisms are certain group monomorphisms.
- A finite category $\mathcal{LP}$, called the **Puig category**, whose objects are pairs (P, γ), where $P \leq D$, and the morphisms, again, are certain group monomorphisms.

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Uniformity Conjecture for Ω

The almost-source algebra A has a basis Ω that is stable under the left and right multiplicative action of the defect group D. As a D-D-biset (as a $D{\times}D$-set), Ω is well-defined up to isomorphism.

Uniformity Conjecture for Ω

The unit group $A^\times$ contains a D-D-stable basis for A. That is, we can choose $\Omega \subset A^\times$.

Given $P \leq D \geq Q$ and a group monomorphism $\phi : P \leftarrow Q$, which we call an **isomorphism under** D, we define the diagonal subgroups

$$\Delta(\phi) = \Delta(P, \phi, Q) = \{(\phi(y), y) : y \in Q\} \leq P{\times}Q \,, \qquad \Delta(P) = \Delta(P, \mathrm{id}, P).$$

An implication of that conjecture

The D-D-biset Ω is $\mathcal{F}$-**semicharacteristic**, we mean, given an isomorphism $\phi : P \leftarrow Q$ under D, then

$$|\Omega^{\Delta(P)}| = |\Omega^{\Delta(\phi)}| = |\Omega^{\Delta(Q)}|$$

and moreover, this is nonzero if and only if ϕ is a morphism in the fusion system $\mathcal{F}$

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Uniformity conjecture for $\mathcal{LP}$

The objects of the Puig category $\mathcal{LP}$ have the form (P, γ) where $P \leq D$ and $\gamma \in \mathcal{LP}_P(A)$. For now, let us just understand $\mathcal{LP}_A(P)$ to be a set of decorations associated with P and determined also by A.

Each object (P, γ) of $\mathcal{LP}$ is associated with a postive integer $m_A(P, \gamma)$, called the **multiplicity**.

Uniformity conjecture for $\mathcal{LP}$

Let $P \leq D \geq Q$ and $\delta \in \mathcal{LP}_A(Q)$. Let $\phi \in \mathrm{Iso}_{\mathcal{F}}(P \leftarrow Q)$. Then there exists a unique $\gamma \in \mathcal{LP}_A(P)$ such that $\phi \in \mathrm{Iso}_{\mathcal{LP}}((P, \gamma) \leftarrow (Q, \delta))$. Furthermore,

$$m_A(P, \gamma) = m_A(Q, \delta).$$

Implications of this conjecture

(1) It yields a classification of the objects of $\mathcal{LP}$ (which are currently known only up to isomorphism).
(2) It also tells us exactly what the morphisms in $\mathcal{LP}$ are (which are currently known only when at least one of the involved subgroups of D is fully $\mathcal{F}$-centralised).

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Statement of the theorems

Theorem 1

For fixed A, the Uniformity Conjecture for Ω is equivalent to the Uniformity Conjecture for $\mathcal{LP}$.

When the conjecture holds for A, we call A a **uniform almost-source algebra**. It can be shown that A is uniform when:

- the defect group D is abelian,
- the defect group D is normal in the given ambient finite group,
- generally, when $\mathcal{F} = N_{\mathcal{F}}(D)$, in other words, every $\mathcal{F}$-morphism is a restriction of an $\mathcal{F}$-automorphism of D.

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When the ambient group is p-solvable, the ambient p-block has a uniform almost-source algebra.

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The fusion system of a finite group

An ideal of p-local finite group theory:

Given p and just some p-local data about a finite group, proceed to examine the p-local properties of the group.

In practice, we let G be a finite group (but fear that only the p-local data will be given).

What strongly p-local data about G ought to be:

Some p-subgroups of G and a finite amount of further information.

Let $\mathcal{F}(G)$ denote the category with:

- Objects: the p-subgroups of G.
- Morphisms $P \leftarrow Q$: the conjugation maps ${}^g v \hookleftarrow v$ where $P \geq {}^g Q$.

A strongly p-local construction

The **fusion system** of G is the full subcategory $\mathcal{F}_D(G)$ of $\mathcal{F}(G)$ whose objects are the subgroups of a Sylow p-subgroup D of G.

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blocks

Let $\mathbb{F}$ be an algebraically closed field of characteristic p. The group algebra of G over $\mathbb{F}$ decomposes as

$$\mathbb{F}G = \bigoplus_b \mathbb{F}Gb$$

where b runs over the **blocks** of $\mathbb{F}G$, we mean, the primitive idempotents of $Z(\mathbb{F}G)$.

Let us fix $\mathbb{F}$ and G.

Every indecomposable $\mathbb{F}G$-module is an $\mathbb{F}Gb$-module for some block b of $\mathbb{F}G$. In view of that observation, let us also fix b.

Implication of Alperin's Weight Conjecture

The number $\ell(\mathbb{F}Gb)$ of isomorphism classes of simple $\mathbb{F}Gb$-modules is determined by (a satisfying notion of) the p-local data for $\mathbb{F}$ and G and b.

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The defect groups of a p-block

A **Brauer pair** of $\mathbb{F}G$ is defined to be a pair (P, e) where P is a p-subgroup of G and e is a block of $\mathbb{F}C_G(P)$.

The inclusion relation on Brauer pairs (definition omitted) has the property that, if $(P, e) \geq (Q, f)$, then $P \geq Q$.

When $(1, b) \leq (P, e)$, we call (P, e) a **Brauer pair** on the block algebra $\mathbb{F}Gb$. In block theory, the Brauer pairs on $\mathbb{F}Gb$ take the role which, in pure group theory, are played by the p-subgroups of G.

The maximal Brauer pairs on $\mathbb{F}Gb$ are permuted transitively by G. When (D, e) is a maximal Brauer pair on $\mathbb{F}Gb$, we call D a **defect group** of $\mathbb{F}Gb$. Thus, the defect group D of $\mathbb{F}Gb$ is well-defined up to G-conjugation.

Alas, D is not enough to determine $\ell(\mathbb{F}Gb)$. We have only:

Donovan's Conjecture

There are only finitely many Morita equivalence classes of blocks with a given defect group.

So let us try to incorporate, along with D, a finite amount of further data.

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Definition of the fusion system

We still fix $\mathbb{F}$ and G and b. Let $\mathcal{F}(\mathbb{F}Gb)$ be the category with:

- Objects: the Brauer pairs on $\mathbb{F}Gb$.
- Morphisms $(P, e) \leftarrow (Q, d)$: the conjugation maps $P \ni {}^g v \leftarrow v \in Q$ where $(P, e) \geq {}^g(Q, d)$.

We let (D, e_D) be a maximal Brauer pair on $\mathbb{F}Gb$. In particular, D is a defect group of b. It can be shown that, for each $P \leq D$, there exists a unique block e_P of $\mathbb{F}C_G(P)$ such that $(P, e_P) \leq (D, e_D)$.

A strongly p-local construction

The fusion system of b can be viewed as the full subcategory $\mathcal{F} = \mathcal{F}_{(D,e_D)}(\mathbb{F}Gb)$ of $\mathcal{F}(\mathbb{F}Gb)$ whose objects are the Brauer pairs having the form (P, e_P) with $P \leq D$. Usually, we understand the objects of $\mathcal{F}$ to be the subgroups P rather that the corresponding Brauer pairs (P, e_P).

Alas, D and $\mathcal{F}$ are not enough to determine $\ell(\mathbb{F}Gb)$. An example is provided by the two 3-blocks of a semidirect product $Q_8 \ltimes (C_3 \times C_3)$.

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Definition of the almost-source algebra

Observe that G acts as automorphisms on $\mathbb{F}Gb$ via the group homomorphism $(\mathbb{F}Gb)^\times \leftarrow G$ given by $gb \leftarrow\!\shortmid g$. We now replace $\mathbb{F}Gb$ with a smaller algebra and replace G with a smaller group.

For any p-subgroup P of G, the projection

$$\mathrm{br}_P : \mathbb{F}C_G(P) \leftarrow (\mathbb{F}G)^P$$

is an algebra map. We choose an idempotent $1_A \in (\mathbb{F}G)^D$ such that

$$\mathrm{br}_P(1_A) \leq e_P$$

for all $P \leq D$. Such 1_A does indeed exist. We call

$$A = 1_A \mathbb{F}G 1_A$$

an **almost-source algebra** for $\mathbb{F}Gb$.

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Features of the almost-source algebra

We view A as coming equipped with the group homomorphism $A \leftarrow D$ given by $x1_A \leftarrow\!\dashv x$.

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It can be shown that A determines $\mathcal{F}$. In fact, the D-D-set Ω determines $\mathcal{F}$.

Theorem: (Puig)

The algebra A is Morita equivalent to $\mathbb{F}Gb$. In particular, A determines $\ell(\mathbb{F}Gb)$.

However, even if we impose the further condition that 1_A is a primitive idempotent of $(\mathbb{F}G)^P$, whereupon A is said to be a **source algebra**, then we have only:

Puig's Conjecture

Given a defect group D then, up to isomorphism, there are only finitely many possibilities for the source algebra, as an algebra equipped with the structure involving D.

Thus, A seems too big, $\mathcal{F}$ seems too small, and we seek something intermediate, determined by D and only a finite amout of further data about $\mathbb{F}Gb$.

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A precursor to the Puig category

For any finite-dimensional algebra B over $\mathbb{F}$, we define a **point** of B to be a conjugacy class of primitive idempotents of B.

There is an evident bijective correspondence between the points of B and the isomorphism classes of simple B-modules.

For $P \leq D$, we define $\mathcal{P}_A(P)$ to be the set of points of A^P. We call (P, α) a **pointed group** on A.

The set of pointed groups on A admits a partial ordering where $(P, \alpha) \geq (Q, \beta)$ provided, for all $i \in \alpha$, there exists $j \in \beta$ satisfying $i \geq j$.

Let $\mathcal{P}$ denote the category with:

- Objects: the pointed groups on A.
- Morphisms $(P, \alpha) \leftarrow (Q, \beta)$: the conjugations $P \ni {}^g v \leftarrowtail v \in Q$ where $(P, \alpha) \geq {}^g(Q, \beta)$.

It can be shown that each $\mathcal{P}_A(P)$ is finite. So $\mathcal{P}$ is a strongly p-local construction.

Definition of the Puig category

Consider a pointed group (P, γ) on A. We call (P, γ) a **local pointed group** and we call γ a **local point** provided

$$\mathrm{br}_P(\gamma) \neq \{0\} \ .$$

That is equivalent to the condition that $\mathrm{br}_P(\gamma)$ is a point of $\mathbb{F}C_G(P)$. Thus, each local point of A^P is associated with an isomorphism class of simple $\mathbb{F}C_G(P)$-modules. We let $\mathcal{LP}_A(P)$ denote the set of local points of P on A.

Another strongly p-local construction

The **Puig category** of A is defined to be the full subcategory $\mathcal{LP}$ of $\mathcal{P}$ whose objects are the local pointed groups on A.

We regard $\mathcal{LP}$ as an enrichment of $\mathcal{F}$. The set of objects of this enriched structure $\mathcal{LP}$ can still be regarded as a poset.

Remark

The minimal objects of $\mathcal{LP}$ are in a bijective correspondence with the isomorphism classes of simple $\mathbb{F}Gb$-modules. In particular, $\mathcal{LP}$ determines $\ell(\mathbb{F}Gb)$.

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The theorems

Given a pointed group (P, α) on A, we define the **multiplicity** $m_A(P, \alpha)$ to be the multiplicity of α as a point of A^P.

Theorem 1

Given $\mathbb{F}$, G, b, A, then the following three conditions are equivalent:
(1) Let $\phi : P \leftarrow Q$ be an $\mathcal{F}$-isomorphism. Then there is a bijection

$$\mathcal{LP}_A(P) \ni \gamma \leftrightarrow \delta \in \mathcal{LP}_A(Q)$$

characterized by the condition that $\phi : (P, \gamma) \leftarrow (Q, \delta)$ is a $\mathcal{LP}$-isomorphism.
(2) The analogue of (1) holds with $\mathcal{P}$ in place of $\mathcal{LP}$.
(3) The unit group $A^\times$ contains a D-D-stable basis for A.

We call A a **uniform almost-source algebra** when those equivalent conditions hold.

Theorem 2

When G is p-solvable, $\mathbb{F}Gb$ has a uniform almost-source algebra.

The theorems

Given a pointed group (P, α) on A, we define the **multiplicity** $m_A(P, \alpha)$ to be the multiplicity of α as a point of A^P.

Theorem 1

Given $\mathbb{F}$, G, b, A, then the following three conditions are equivalent:
(1) Let $\phi : P \leftarrow Q$ be an $\mathcal{F}$-isomorphism. Then there is a bijection

$$\mathcal{LP}_A(P) \ni \gamma \leftrightarrow \delta \in \mathcal{LP}_A(Q)$$

characterized by the condition that $\phi : (P, \gamma) \leftarrow (Q, \delta)$ is a $\mathcal{LP}$-isomorphism.
(2) The analogue of (1) holds with $\mathcal{P}$ in place of $\mathcal{LP}$.
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Theorem 2

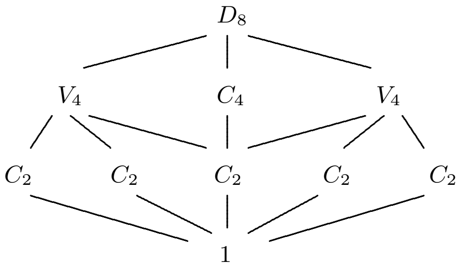
When G is p-solvable, $\mathbb{F}Gb$ has a uniform almost-source algebra.

The poset $\mathcal{F}$ for the principal 2-block of S_5

As a concrete example, we put $p = 2$ and $G = S_5$. We let b be the principal block of $\mathbb{F}G$ and we take A to be a source algebra.

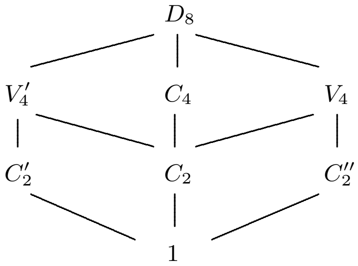
To illustrate the finite categories $\mathcal{F}$ and $\mathcal{LP}$ and $\mathcal{LP}$, we shall draw Hasse diagrams of the associated posets of objects. The morphisms, recall, are the G-conjugations.

As a poset, $\mathcal{F}$ is simply the poset of subgroups of the defect group D_8, thus:



the poset of D-conjugacy classes of $\mathcal{F}$

Before turning to the posets $\mathcal{LP}$ and $\mathcal{P}$, let us simplify the picture by merging each D-conjugacy class into a single vertex.

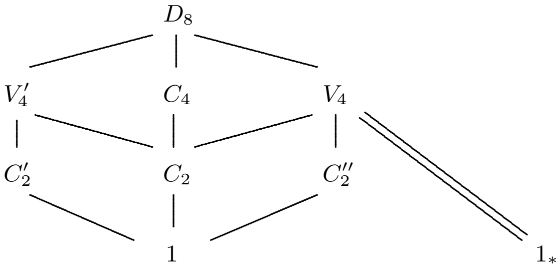


Resolving an ambiguity by taking $V_4 = O_2(S_4)$, we have an $\mathcal{F}$-isomorphism $C_2 \cong C_2''$.

the poset of D-conjugacy classes of $\mathcal{LP}$ for the principal 2-block of S_5

Again merging D-conjugacy classes, the poset for $\mathcal{LP}$ is as shown, where P indicates the local pointed group (P, γ_0) with the point γ_0 corresponding to the trivial $\mathbb{F}C_G(P)$-module, while 1_* indicates the local pointed group $(1, \gamma_1)$ with the point γ_1 corresponding to the 4-dimensional simple $\mathbb{F}G$-module.

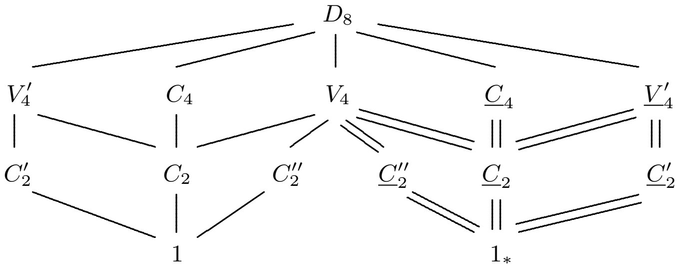
The double line indicates a relative multiplicity of 2. All the other relative multiplicities are 1.



Note that we can read off $\ell(\mathbb{F}Gb)$ from the diagram. It is the number of minimal vertices, $\ell(\mathbb{F}Gb) = 2$.

The poset of D-conjugacy classes of $\mathcal{P}$ for the principal 2-block of S_5

And the D-orbits of the poset $\mathcal{P}$ is as follows, where $\underline{P}$ indicates a non-local pointed group having the form (P, α).



We have $\mathcal{P}$-isomorphisms $C_2 \cong C_2''$ and $\underline{C}_2 \cong \underline{C}_2''$.

The multiplicities of the pointed groups

Again reading off from the diagrams, the multiplicities of the objects of $\mathcal{LP}$, that is, the multiplicities of the local pointed groups on A, are:

(P,γ)	1	C_2	C_2'	C_2''	V_4'	C_4	V_4	D	1_*
$m_A(P,\gamma)$	1	1	1	1	1	1	1	1	4

The multiplicities of the other objects of $\mathcal{P}$, the non-local pointed groups, are:

(P,α)	$\underline{C}_2$	$\underline{C}_2'$	$\underline{C}_2''$	$\underline{V}_4'$	$\underline{C}_4$
$m_A(P,\gamma)$	2	2	2	1	1

The Uniformity Conjecture, in this case, is the prediction of the equalities

$$m_A(C_2) = m_A(C_2'') \,, \qquad m_A(\underline{C}_2) = m_A(\underline{C}_2'') \,.$$

Why we have trouble classifying the objects and morphisms of $\mathcal{LP}$

For all three categories $\mathcal{F}$ and $\mathcal{LP}$ and $\mathcal{P}$, the problem of explicitly describing the morphisms is equivalent to the problem of describing the isomorphisms, since an arbitrary morphism is the composite of an inclusion and an isomorphism.

The above example seems to suggest that each $\mathcal{F}$-isomorphism $\phi : P \leftarrow Q$ gives rise to several $\mathcal{LP}$-isomorphisms $\phi_{\gamma,\delta} : (P, \gamma) \leftarrow (Q, \delta)$ in $\mathcal{LP}$.

Such $\phi_{\gamma,\delta}$ does always make sense when we extend to local pointed groups (P, γ) and (Q, δ) on the larger algebra $\mathbb{F}Gb$.

Unfortunately, when we assume that one of (P, γ) or (Q, δ) lies in $\mathcal{LP}$, it is not clear that both must belong to $\mathcal{LP}$.

Alperin's Fusion Theorem tells us that any $\mathcal{F}$-isomorphism can be expressed as a composite $\phi = \phi_n ... \phi_1$ of restrictions $\phi_i : P_i \leftarrow P_{i-1}$ of automorphisms of $\mathcal{F}$-centric subgroups of D. It is not hard to see that each ϕ_i gives rise to an $\mathcal{LP}$-isomorphism $(P_i, \gamma_i') \leftarrow (P_{i-1}, \gamma_{i-1})$. But to hence form a composite, we would need to ensure that each $\gamma_i = \gamma_i'$, and I see no way of doing that.

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Proof of Theorem 1, the three equivalent criteria for uniformity of A

The proof is long, but one of the key elements is as follows.

Given any $\mathbb{F}G$-module M and a p-subgroup P of G, we define the **Brauer quotient**

$$M(P) = M^P / M^P_<$$

where $M^P_<$ is the span of the images of the transfer maps $A^P \leftarrow A^Q$ for strict subgroups $Q < P$. When M has a P-stable basis Γ, the projection $\mathrm{br}^M_P : M(P) \leftarrow M^P$ restricts to a linear isomorphism

$$M(P) \cong \mathbb{F}(\Gamma^P) \; .$$

Thus, for an $\mathcal{F}$-isomorphism ϕ, we obtain a connection between $A(\Delta(\phi))$ and $\Omega^{\Delta(\phi)}$.

The three conditions in question are all equivalent to the condition that, given $\mathcal{F}$-isomorphisms ϕ and ψ such that the composite $\phi\psi$ is defined, then the multiplication operation on A gives rise to a linear epimorphism

$$A(\Delta(\phi\psi)) \leftarrow A(\Delta(\phi)) \times A(\Delta(\psi)) \; .$$

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Proof of Theorem 2, uniformity in the p-solvable case

Our first proof was another long one, by the kind of Clifford theory routinely used in the p-solvable case: an inductive argument dealing separately with the cases $O_p(G) \neq 1$ and $O_{p'}(G) \neq 1$. But there is a much quicker argument, as follows.

The block b is said to be of **principal type** when $\mathbb{F}Gb$ is an almost-source algebra of $\mathbb{F}Gb$. In that case, it is easy to show that, in fact, $\mathbb{F}Gb$ is a uniform almost-source algebra. The following theorem reduces to that case.

Theorem, Harris–Linckelmann, 2000

When G is p-solvable, there exists $H \leq G$ and a block c of $\mathbb{F}H$ such that c is of principal type and the induction and restriction functors between H and G yield a Morita equivalence between $\mathbb{F}Hc$ and $\mathbb{F}Gb$.

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References

Theorems 1 and 2, and references for the background, can be found in:

Laurence Barker, Matthew Gelvin, *Conjectural invariance with respect to the fusion system of an almost-source algebra*, Journal of Group Theory, **25**, 974-995 (2022).

Thank you!