Exercises and Solutions, MATH 220, Spring 2024

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This file can be found on my homepage in the file: Exercises200spr24.pdf.

These are casual notes, just something extra, not a basis for study.

These notes are merely a supplement to the course. If you just focus on the material here, you will not do well. Most or all of the sample exercises here are very easy, very routine. In the exams, you will also be tested on comprehension of the material and ability to explain your reasoning. To do that, you need to grasp the logic behind the methods, and you also need skill at presenting deductive mathematical arguments.

These notes doubtless have mistakes. They will be subject to change.

1: Gaussian elimination and Gauss–Jordan elimination

Question 1.1: Find all the solutions to

$$2x + y + z = 7\,, \qquad 4x + 3y + z = 13\,, \qquad 6x - y - 2z = 7\,,$$

(a) Using Gaussian elimination,
(b) Using Gauss–Jordan elimination.

Question 1.2: Find all the solutions to

$$2x + y + z = 7\,, \qquad 4x + 3y + z = 13\,, \qquad 8x + 5y + 3z = 27\,,$$

(a) Using Gaussian elimination,
(b) Using Gauss–Jordan elimination.

Question 1.3: By any method, find all the solutions to

$$3x + 2y + z = 14\ , \qquad x + 6y + 5z = 20\ , \qquad 9x + 22y + 17z = 88\ .$$

Solutions

1.1: Part (a). The augmented matrix is $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 7 \\ 4 & 3 & 1 & 13 \\ 6 & -1 & -2 & 7 \end{array}\right]$.

Subtracting multiples of row 1 from rows 2 and 3 gives $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 7 \\ 0 & 1 & -1 & -1 \\ 0 & -4 & -5 & -14 \end{array}\right]$.

Adding 4 times row 2 to row 3 gives $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 7 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -9 & -18 \end{array}\right]$.

Multiplying rows 1 and 3 by factors, $\left[\begin{array}{ccc|c} 1 & 1/2 & 1/2 & 7/2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$.

The equations are $z = 2$ and $y = z - 1 = 1$ and $x = 7/2 - y/2 - z/2 = 7/2 - 1/2 - 1 = 2$. In conclusion, $(x, y, z) = (2, 1, 2)$.

Part (b). Subtracting multiples of row 3 from rows 1 and 2 gives $\left[\begin{array}{ccc|c} 1 & 1/2 & 0 & 5/2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$.

Subtracting half of row 2 from row 3 gives $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$.

Thus, again, $(x, y, z) = (2, 1, 2)$.

1.2: Part (a). We code the problem as $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 7 \\ 4 & 3 & 1 & 13 \\ 8 & 5 & 3 & 27 \end{array}\right]$.

Subtracting multiples of row 1 from rows 2 and 3 gives $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 7 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & -1 \end{array}\right]$.

Subtracting row 2 from row 3 gives $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 7 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

Dividing row 1 by 2 gives $\left[\begin{array}{ccc|c} 1 & 1/2 & 1/2 & 7/2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

Introducing a parameter r, we put $z = r$. Then $y = -1 + z = r - 1$ and

$$x = 7/2 - y/2 - z/2 = 7/2 - (r - 1)/2 - r/2 = 4 - r\ .$$

Hence $(x, y, z) = (4 - r, r - 1, r)$ where r is a parameter.

Part (b). Subtracting half of row 2 from row 1 gives $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

Hence, again, $(x, y, z) = (4 - r, r - 1, r)$.

1.3: The augmented matrix is $\left[\begin{array}{ccc|c} 3 & 2 & 1 & 14 \\ 1 & 6 & 5 & 20 \\ 9 & 22 & 17 & 88 \end{array}\right]$.

Interchanging rows 1 and 2 gives $\left[\begin{array}{ccc|c} 1 & 6 & 5 & 20 \\ 3 & 2 & 1 & 14 \\ 9 & 22 & 17 & 88 \end{array}\right]$.

Subtracting multiples of row 1 from the others, $\left[\begin{array}{ccc|c} 1 & 6 & 5 & 20 \\ 0 & -16 & -14 & -46 \\ 0 & -32 & -28 & -92 \end{array}\right]$.

Dividing row 2 by -16 gives $\left[\begin{array}{ccc|c} 1 & 6 & 5 & 20 \\ 0 & 1 & 7/8 & 23/8 \\ 0 & -32 & -28 & -92 \end{array}\right]$.

Adding 32 times tow 2 to row 3 gives $\left[\begin{array}{ccc|c} 1 & 6 & 5 & 20 \\ 0 & 1 & 7/8 & 23/8 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

Subtracting 6 times row 2 from row 1 gives $\left[\begin{array}{ccc|c} 1 & 0 & -1/4 & 11/4 \\ 0 & 1 & 7/8 & 23/8 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

The equations are $x - z/4 = 11/4$ and $y + 7z/8 = 23/8$. Introducing a parameter s, we can put $z = s$. The solutions are

$$(x, y, z) = (s/4 + 11/4, -7s/8 + 23/8, s)$$

where s is any real number.

Comment: That was by Gauss–Jordan elimination. The Gaussian method would be to omit the last row operation, and to write down the equations $x+6y+5z = 20$ and $y+7z/8 = 23/8$. Again putting $z = s$, we would immediately obtain $y = -7s/8+28/8$. Then we would calculate x from

$$x = -6y - 5z + 20 = -6\left(\frac{-7s}{8} + \frac{28}{8}\right) - 5s + 20 = \frac{6.7 - 5.8}{8}s + \frac{-6.28 + 20.8}{8} = \frac{s}{4} + \frac{11}{4}\ .$$

2: Inverting matrices by Gauss–Jordan elimination

Question 2.1: By Gauss–Jordan elimination, find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 1 \\ 6 & -1 & -2 \end{bmatrix}.$$

Question 2.2: Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 16 \end{bmatrix}$. Evaluate A^{-1} by Gauss–Jordan elimination.

Solutions

2.1: We set up the problem as $\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & 1 & 0 & 1 & 0 \\ 6 & -1 & -2 & 0 & 0 & 1 \end{array}\right]$.

Subtracting multiples of row 1 from the other two rows, $\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & -4 & -5 & -3 & 0 & 1 \end{array}\right]$.

Adding 4 times row 2 to row 3 gives $\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -9 & -11 & 4 & 1 \end{array}\right]$.

Dividing row 3 by -9 gives $\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 11/9 & -4/9 & -1/9 \end{array}\right]$.

Adding multiples of row 3 to the other two rows, $\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & -2/9 & 4/9 & 1/9 \\ 0 & 1 & 0 & -7/9 & 5/9 & -1/9 \\ 0 & 0 & 1 & 11/9 & -4/9 & -1/9 \end{array}\right]$.

Subtracting row 2 from row 1 yields $\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 5/9 & -1/9 & 2/9 \\ 0 & 1 & 0 & -7/9 & 5/9 & -1/9 \\ 0 & 0 & 1 & 11/9 & -4/9 & -1/9 \end{array}\right]$.

Finally, dividing row 1 by 2 gives $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5/18 & -1/18 & 2/18 \\ 0 & 1 & 0 & -7/9 & 5/9 & -1/9 \\ 0 & 0 & 1 & 11/9 & -4/9 & -1/9 \end{array}\right]$.

So $A^{-1} = \frac{1}{18}\begin{bmatrix} 5 & -1 & 2 \\ -14 & 10 & -2 \\ 22 & -8 & -2 \end{bmatrix}$.

2.2: We set up the problem as $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 1 & 0 \\ 1 & 4 & 16 & 0 & 0 & 1 \end{array}\right]$.

Subtracting row 1 from rows 2 and 3 gives $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 3 & 15 & -1 & 0 & 1 \end{array}\right]$.

Subtracting 3 times row 2 from row 3 gives $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 6 & 2 & -3 & 1 \end{array}\right]$.

Dividing row 3 by 6 gives $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1/3 & -1/2 & 1/6 \end{array}\right]$.

Subtracting multiplies of row 3 from rows 1 and 2 gives $\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 2/3 & 1/2 & -1/6 \\ 0 & 1 & 0 & -2 & 5/2 & -1/2 \\ 0 & 0 & 1 & 1/3 & -1/2 & 1/6 \end{array}\right]$.

Subtracting row 2 from row 1 gives $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8/3 & -2 & 1/3 \\ 0 & 1 & 0 & -2 & 5/2 & -1/2 \\ 0 & 0 & 1 & 1/3 & -1/2 & 1/6 \end{array}\right]$.

Therefore, $A^{-1} = \dfrac{1}{6}\begin{bmatrix} 16 & -12 & 2 \\ -12 & 15 & -3 \\ 2 & -3 & 1 \end{bmatrix}$.

3: Evaluating determinants

Question 3.1: Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 1 \\ 6 & -1 & -2 \end{bmatrix}$.

(a) Evaluate $\det(A)$ by calculating suitable cofactors.

(b) Evaluate $\det(A)$ using row or column operations.

Solutions

3.1: Part (a). We have

$$\det(A) = \begin{vmatrix} 2 & 1 & 1 \\ 4 & 3 & 1 \\ 6 & -1 & -2 \end{vmatrix} = 2\begin{vmatrix} 3 & 1 \\ -1 & -2 \end{vmatrix} - \begin{vmatrix} 4 & 1 \\ 6 & -2 \end{vmatrix} + \begin{vmatrix} 4 & 3 \\ 6 & -1 \end{vmatrix}$$

$$= 2(-6+1) - (-8-6) + (-4-18) = -10 + 14 - 22 = -18\ .$$

Part (b). Adding multiples of row 1 from the other rows, $\det(A) = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & -4 & -5 \end{vmatrix}$.

Adding 4 times row 2 to row 3 gives $\det(A) = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -9 \end{vmatrix} = 2.1.(-9) = -18.$

4: Inverting matrices by the method of minors and cofactors

Question 4.1: Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 1 \\ 6 & -1 & -2 \end{bmatrix}$.

(a) By the method of minors and cofactors, evaluate A^{-1}.

(b) Using A^{-1}, give another method for solving the linear system in Question 1.1.

Question 4.2: Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$. By the method of minors and cofactors, evaluate A^{-1}.

Solutions

4.1: The matrix of minors is $\begin{bmatrix} -6+1 & -8-6 & -4-18 \\ -2+1 & -4-6 & -2-6 \\ 1-3 & 2-4 & 6-4 \end{bmatrix} = \begin{bmatrix} -5 & -14 & -22 \\ -1 & -10 & -8 \\ -2 & -2 & 2 \end{bmatrix}$.

The matrix of cofactors is $\begin{bmatrix} -5 & 14 & -22 \\ 1 & -10 & 8 \\ -2 & 2 & 2 \end{bmatrix}$.

The adjugate matrix is $\operatorname{adj}(A) = \begin{bmatrix} -5 & 1 & 2 \\ 14 & -2 & 2 \\ -22 & 8 & 2 \end{bmatrix}$. We have $\det(A) = 2(-5) + 14 - 22 = -18$ from the $(1, 1)$ entry of the equality $A \operatorname{adj}(A) = \det(A)I$. Therefore, $A = \frac{1}{18} \begin{bmatrix} 5 & -1 & 2 \\ -14 & 10 & -2 \\ 22 & -8 & -2 \end{bmatrix}$.

Part (b). We have $A^{-1} \begin{bmatrix} 7 \\ 13 \\ 7 \end{bmatrix} = \frac{1}{18} \begin{bmatrix} 35 - 13 + 14 \\ -98 + 130 - 14 \\ 154 - 104 - 14 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

4.2: The matrix of minors is $\begin{bmatrix} 2.9 - 4.3 & 1.9 - 4.1 & 1.3 - 2.1 \\ 1.9 - 1.3 & 1.9 - 1.1 & 1.3 - 1.1 \\ 1.4 - 1.2 & 1.4 - 1.1 & 1.2 - 1.1 \end{bmatrix} = \begin{bmatrix} 6 & 5 & 1 \\ 6 & 8 & 2 \\ 2 & 3 & 1 \end{bmatrix}$.

Taking the transpose an introducing alternating $\pm$ signs, $\operatorname{adj}(A) = \begin{bmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{bmatrix}$.

We have $\det(A) = 1.6 + 1(-5) + 1.1 = 2$. So $A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)} = \frac{1}{2} \begin{bmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{bmatrix}$.