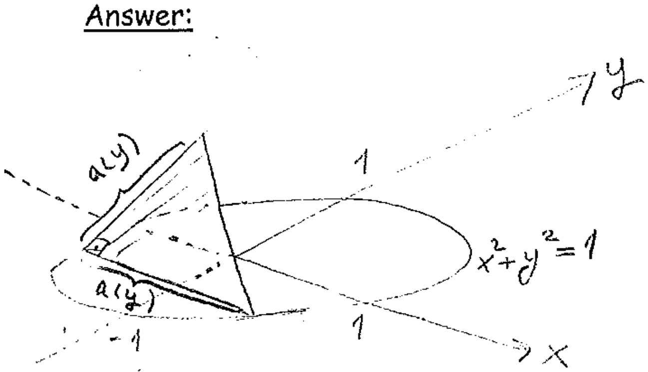
Surname:
Name: Key

Math.112(06) f (01)
Quiz 1

<u>Question:</u> The base of the solid is the disk $x^2 + y^2 \le 1$. The cross-sections by planes perpendicular to the $y-$axis between $y=-1$ and $y=1$ are isosceles right triangles with one leg in the disk. Find the volume of the solid.

<u>Answer:</u>



3

1 $\quad A(y) = \frac{1}{2}(a(y))^2 \longrightarrow$ cross-sectional area

3
$$= \frac{1}{2}\left[\sqrt{1-y^2} - \left(-\sqrt{1-y^2}\right)\right]^2$$
$$= 2(1-y^2)$$

3
$$V = \int_{-1}^{1} 2(1-y^2)\,dy = 4\int_{0}^{1}(1-y^2)\,dy = 4\left(y - \frac{y^3}{3}\right)\Big|_{y=0}^{1} = \frac{8}{3}.$$

integrand is even

yanlışsa
-1

Name : Spring 2009 - Math 112

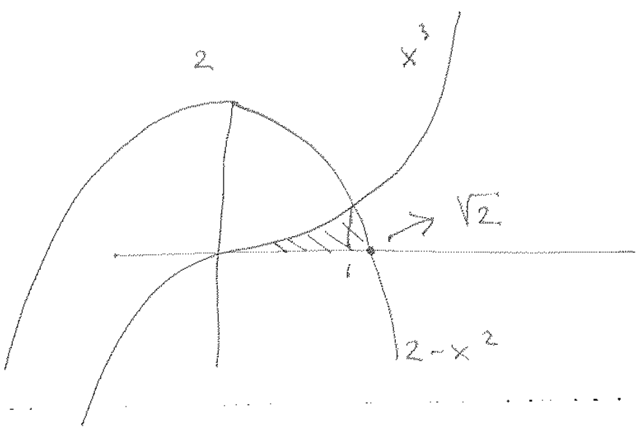
Quiz 1

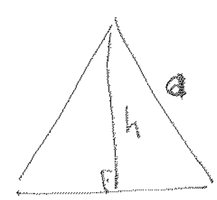
Find the volume of the solid whose cross-sections perpendicular to the x-axis are equilateral triangles whose heights run from the bottom of the region bounded by $y = 2 - x^2$, $y = x^3$, and x-axis to the top.

$$x^3 = 2 - x^2 \quad \text{iff} \quad x^3 + x^2 - 2 = 0$$

$$\text{iff} \quad (x-1)(x^2 + 2x + 2) = 0$$

$$\text{iff} \quad x = 1$$





$$\frac{a\sqrt{3}}{2} = h \iff a = \frac{2h}{\sqrt{3}}$$

$$\text{Area} = \frac{ah}{2} = \frac{h^2}{\sqrt{3}}$$

$$\text{Volume} = \int_0^1 \frac{(x^3)^2}{\sqrt{3}}\,dx + \int_1^{\sqrt{2}} \frac{(2-x^2)^2}{\sqrt{3}}\,dx$$

$$= \frac{1}{\sqrt{3}}\left[\frac{x^7}{7}\Big|_0^1 + \left(4x - \frac{2}{3}x^3 + \frac{x^5}{5}\right)\Big|_1^{\sqrt{2}} \right]$$

NAME: Key
STUDENT I.D.: 112

QUIZ 2
Section 06

Question: Let $f(x)$ be a positive continuous function on $[1,4]$. Let R_1 be the region bounded by the curves $y = f(x^2)$, $x = 1$, $x = 2$, and $y = 0$. Let R_2 be the region bounded by the curves $y = \sqrt{f(x)}$, $x = 1$, $x = 4$, and $y = 0$. Show that the volume of the solid S_1 obtained by rotating the region R_1 about the y-axis is equal to the volume of the solid S_2 obtained by rotating the region R_2 about the x-axis.

Solution: The volume of S_1 is equal by the shell method to the integral

$$V_1 = \int_1^2 2\pi x f(x^2)dx,$$

and the volume of the solid S_2 is equal by the disc method to the integral

$$V_2 = \int_1^4 \pi\big(\sqrt{f(x)}\big)^2 dx = \int_1^4 \pi f(x)dx.$$

We are required to show that the integrals V_1 and V_2 are equal. Indeed, the substitution

$$u = x^2$$

transforms the integral V_1 to the integral V_2. To be more explicit,

$$V_1 = \int_1^2 2\pi x f(x^2)dx \underbrace{=}_{u=x^2,\ du=2xdx} \int_1^4 \pi f(u)du = \int_1^4 \pi f(x)dx = V_2.$$

Writing V_1 worths 4 points.
Writing V_2 worths 4 points.
Showing the equality of V_1 and V_2 worths 2 points.

Question: Find the length of the curve

$$\begin{cases} x = t^2 + \dfrac{1}{2t} \\ y = 4\sqrt{t} \quad , \quad 1/\sqrt{2} \le t \le 1 \end{cases}$$

Sol:

$$\frac{dx}{dt} = 2t - \frac{1}{2t^2} \quad 1 \qquad \frac{dy}{dt} = \frac{2}{\sqrt{t}} \quad 1$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4t^2 - 2\frac{1}{t} + \frac{1}{4t^4} + \frac{4}{t}$$

$$= 4t^2 + 2\frac{1}{t} + \frac{1}{4t^4}$$

$$= \left(2t + \frac{1}{2t^2}\right)^2 \quad 2$$

$$L = \int_{1/\sqrt{2}}^{1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \quad 4 \quad = \int_{1/\sqrt{2}}^{1} \left(2t + \frac{1}{2t^2}\right) dt$$

$$= \left(t^2 - \frac{1}{2t}\right)\Bigg|_{1/\sqrt{2}}^{1} \quad 1 \quad = \left(1 - \frac{1}{2}\right) - \left(\frac{1}{2} - \frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2} \quad 1$$

Quiz 3

474/21 Find the surface area of the surface generated by revolving the curve

$$C: \quad x = \frac{e^{y}+e^{-y}}{2}, \quad 0 \le y \le \ln 2$$

about the y-axis.

$$S = 2\pi \int_0^{\ln 2} x \, ds \qquad \text{where} \qquad ds = \sqrt{(dx)^2+(dy)^2}$$

$$= \sqrt{\left(\frac{dx}{dy}\,dy\right)^2+(dy)^2}$$

$$= \sqrt{\left(\frac{dx}{dy}\right)^2+1}\;dy$$

$$\frac{dx}{dy} = \frac{e^{y}-e^{-y}}{2}, \qquad \sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+\left(\frac{e^{y}-e^{-y}}{2}\right)^2}$$

$$= \sqrt{\frac{e^{2y}+2+e^{-2y}}{4}}$$

$$= \frac{e^{y}+e^{-y}}{2}$$

Then

$$S = 2\pi \int_0^{\ln 2} \frac{(e^{y}+e^{-y})^2}{4}\,dy$$

$$= \frac{\pi}{2} \int_0^{\ln 2} (e^{2y}+2+e^{-2y})\,dy$$

$$= \frac{\pi}{2}\left(\frac{e^{2y}}{2}+2y-\frac{e^{-2y}}{2}\right)\Bigg|_{y=0}^{\ln 2}$$

$$= \pi\left(\frac{15}{16}+\ln 2\right)$$

[redacted]

Quiz 4

Q) Show that $\tanh^{-1} x = \frac{1}{2} \ln \frac{1+x}{1-x}$, $|x|<1$

Answer: Let $y = \tanh^{-1} x$. Then $x = \tanh y$, or

$$x = \frac{e^{y} - e^{-y}}{e^{y} + e^{-y}} \quad \cdots (A) \qquad 1$$

Let's solve (A) for y.

$$(A) \Rightarrow \quad x e^{y} + x e^{-y} = e^{y} - e^{-y} \qquad 1$$

$$\underbrace{e^{-y}}_{\neq 0,\ \forall y} \underbrace{\left[x e^{2y} + x - e^{2y} + 1 \right]}_{1} = 0 \qquad 1$$

$$\Rightarrow \quad e^{2y}(x-1) = -(1+x)$$

or $\quad e^{2y} = \frac{1+x}{1-x} \qquad 1$

Then $\quad e^{y} = \mp \sqrt{\frac{1+x}{1-x}} \qquad 1$

Since $e^{y} > 0$ (1), $\quad e^{y} = \sqrt{\frac{1+x}{1-x}}$ (1), $|x|<1$.

Hence, $\quad y = \ln \left(\frac{1+x}{1-x} \right)^{1/2}$ (1), or equivalently

$$\tanh^{-1} x = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right), \ |x|<1.$$

1

Quiz 5

Find a fcn. f, cont. $\forall x$ (and not everywhere zero), s.t.

$$8 - \frac{f^2(x)}{2} = \int_0^x \frac{f^2(t)\sin t}{2+\cos t}\,dt \quad \ldots (1)$$

$$-ff' = \frac{f^2 \sin x}{2+\cos x} \quad (2)$$

$$(2) \quad \frac{f'}{f} = -\frac{\sin x}{2+\cos x}, \quad f \neq 0$$

$$(2) \quad \ln|f| = \ln(2+\cos x) + c_1 \quad \text{where} \quad \int \frac{\overbrace{-\sin x\,dx}^{du}}{\underbrace{2+\cos x}_{u}} = \ln(2+\cos x) + c$$

$$(1) \quad |f(x)| = c_1 (2+\cos x)$$

$$(1) \Rightarrow \quad 8 - \frac{f^2(0)}{2} = 0 \Rightarrow f(0) = \mp 4 \quad (1)$$

$$f(0) = 3c_1 = \mp 4 \Rightarrow c = \mp \frac{4}{3} \quad (1)$$

$$f(x) = \mp \frac{4}{3}(2+\cos x). \quad (1)$$

Find the area of the surface generated by revolving the curve

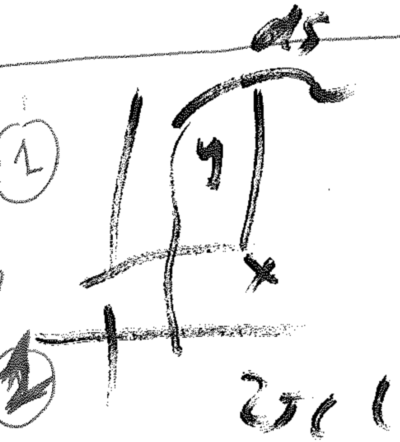
$$\left\{\begin{array}{l} x = \frac{t^2}{2} + t \\ y = 2t \\ \sqrt{5}-1 \le t \le \sqrt{12}-1 \end{array}\right\}$$

about the line $y = -2$

Solution

$$S = \int_{\sqrt{5}-1}^{\sqrt{12}-1} 2\pi (y+2)\, ds = \boxed{\int_{\sqrt{5}-1}^{\sqrt{12}-1} 2\pi (y+2) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

2



$\boxed{\frac{dx}{dt} = t + 1}$ 1 , $\boxed{\frac{dy}{dt} = 2}$ 1

$\boxed{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = t^2 + 2t + 5}$ 2

$$S = \boxed{\int_{\sqrt{5}-1}^{\sqrt{12}-1} 2\pi (2t+2) \sqrt{t^2+2t+5}\, dt}$$ 2

1

$$= \int_{4}^{9} 2\pi \sqrt{u}\, du = \boxed{\frac{4\pi}{3} u^{3/2} \Big|_9^4}$$

$\boxed{\begin{array}{l} u = t^2+2t+5 \\ du = (2t+2)\,dt \end{array}}$

1

1

Question: Let f be a continuous function on $[0,1]$. Show that

$$\int_0^1 \left(\int_0^x f(t)\,dt\right)dx = \int_0^1 (1-x) f(x)\,dx$$

Solution:

$$\int_0^1 \left(\int_0^x f(t)\,dt\right)dx = \left(x\int_0^x f(t)\,dt\right)\Big|_0^1 - \int_0^1 x f(x)\,dx \quad (1)$$

(1) $u = \int_0^x f(t)\,dt \Rightarrow du = f(x)\,dx$ (2)

$dv = dx \Rightarrow v = x$ (1)

(2)

$$= 1\int_0^1 f(t)\,dt - 0\int_0^0 f(t)\,dt - \int_0^1 x f(x)\,dx$$

$$= \int_0^1 f(t)\,dt - \int_0^1 x f(x)\,dx$$

(1)

$$= \int_0^1 f(x) - \int_0^1 x f(x)\,dx$$

$$= \int_0^1 (1-x) f(x)\,dx \quad (2)$$

Math 112; Section 01 (Ergün)

Quiz 4

Section 01 , Engin , Quiz5

Question $\int \ln(x^2+x+2)\,dx = ?$

$$\int \ln(x^2+x+2)\,dx = x\ln(x^2+x+2) - \underbrace{\int \frac{2x^2+x}{x^2+x+2}\,dx}_{I} \quad (1)$$

(1) $\begin{bmatrix} u=\ln(x^2+x+2), & du=\dfrac{2x+1}{x^2+x+2} \\ dv=dx, & v=x \end{bmatrix}$

(1) $$\frac{2x^2+x}{x^2+x+2} = 2 - \frac{x+4}{x^2+x+2} = \underbrace{2 - \frac{x+(1/2)}{x^2+x+2} - \frac{7/2}{x^2+x+2}}_{(2)}$$

As $(x^2+x+2)' = 2x+1 = 2(x+\frac{1}{2})$

Thus, $$I=\int \frac{2x^2+x}{x^2+x+2} = \int 2\,dx - \int \frac{x+(1/2)}{x^2+x+2}\,dx - \frac{7}{2}\int\frac{dx}{x^2+x+2} \quad (1)$$

$$= 2x - \frac{1}{2}\ln(x^2+x+2) - \frac{7}{2}\underbrace{\int \frac{dx}{x^2+x+2}}_{J}$$

(2) $$x^2+x+2 = (x+\tfrac{1}{2})^2 + \frac{7}{4} = \frac{7}{4}\left(\left(\sqrt{\tfrac{4}{7}}(x+1)\right)^2+1\right)$$

$$J=\int\frac{dx}{x^2+x+2} = \frac{4}{7}\int\frac{dx}{\left(\sqrt{\frac{7}{4}}(x+1)\right)^2+1} = \underbrace{\frac{4}{7}\sqrt{\frac{4}{7}}\tan^{-1}\left(\sqrt{\tfrac{7}{4}}(x+1)\right)+C}_{(1)}$$

Therefore,

$$\int \ln(x^2+x+2)\,dx = x\ln(x^2+x+2) \underbrace{-2x + \frac{1}{2}\ln(x^2+x+2) + \frac{7}{2}\frac{4}{7}\sqrt{\frac{4}{7}}\tan^{-1}\left(\sqrt{\tfrac{7}{4}}(x+1)\right)}_{(1)} + C'$$

Section 03

Question $\int \cos^{-1}(\sqrt{x})\,dx = ?$

Q5

Write down the partial fractional decomposition for

$$\frac{(2x^2+x-6)(x^4+2x^3)}{(2x^2-3x+10)(3x^2+5x-2)(3x-1)}$$

Factorization & Cancellation:

$$\left.\frac{(2x-3)(x+2)^{\cancel{2}}x^3}{(2x^2-3x+10)(3x-1)^2\cancel{(x+2)}}\right\} 4$$

2 pt for cancelling x+2

2 pt for correct factorization

Decomposition:

$$\underbrace{\frac{Ax+B}{2x^2-3x+10}}_{2} + \underbrace{\frac{C}{3x-1}}_{2} + \underbrace{\frac{D}{(3x-1)^2}}_{2}$$

Quiz 6

Evaluate $I = \int x^2 \tan^{-1} x \, dx$

$$u = \tan^{-1} x \quad , \quad dv = x^2 dx$$

$$du = \frac{dx}{1+x^2} \quad , \quad v = \frac{x^3}{3}$$

$$I = \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int \frac{x^3}{1+x^2}\,dx$$

$$x^3 \;\big|\; x^2+1$$

$$-x^3 \mp x \quad x$$

$$-x$$

$$= \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int\left(x - \frac{2x}{2(x^2+1)}\right)dx$$

$$= \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\left(\frac{x^2}{2} - \frac{1}{2}\ln(x^2+1)\right) + C$$

Question: $I = \int \dfrac{dx}{(3x^2-12x+8)^{5/2}} = ?$ Math 112 Sec 03

Sol: $3x^2-12x+8 = 3(x^2-4x)+8 = 3((x-2)^2-4)+8 = 3(x-2)^2-4$ (3)

Put $\sqrt{3}(x-2) = 2\sec\theta$ (2) Then $\sqrt{3}\,dx = 2\sec\theta\tan\theta\,d\theta$ (1)

and $3x^2-12x+8 = 3(x-2)^2-4 = 4(\sec^2\theta-1) = 4\tan^2\theta$ (1) So

$$I = \int \frac{(2/\sqrt{3})\sec\theta\tan\theta\,d\theta}{2^5\tan^5\theta} = \frac{1}{16\sqrt{3}}\int\frac{\sec\theta}{\tan^4\theta}\,d\theta \ (1) =$$

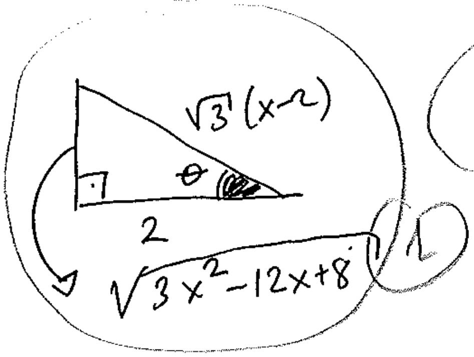
$\sec\theta = \frac{1}{\cos\theta}$, $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$$= \frac{1}{16\sqrt{3}}\int\frac{\cos^3\theta}{\sin^4\theta}\,d\theta = \frac{1}{16\sqrt{3}}\int\frac{(1-\sin^2\theta)\cos\theta}{\sin^4\theta}\,d\theta = \frac{1}{16\sqrt{3}}\int\frac{1-u^2}{u^4}\,du \ (1)$$

$u = \sin\theta$, $du = \cos\theta\,d\theta$

$$= \frac{1}{16\sqrt{3}}\int\left(\frac{1}{u^4}-\frac{1}{u^2}\right)du \ (1) = \frac{1}{16\sqrt{3}}\left(\frac{-1}{3}\frac{1}{u^3}+\frac{1}{u}\right)+C$$

$$= \frac{-1}{48\sqrt{3}}\frac{1}{\sin^3\theta} + \frac{1}{16\sqrt{3}}\frac{1}{\sin\theta} + C \ (1)$$



$$I = \frac{-1}{48\sqrt{3}}\left(\frac{\sqrt{3}(x-2)}{\sqrt{3x^2-12x+8}}\right)^3 + \frac{1}{16\sqrt{3}}\left(\frac{\sqrt{3}(x-2)}{\sqrt{3x^2-12x+8}}\right)+C \ (1)$$

$$= \frac{-1}{16}\frac{(x-2)^3}{(3x^2-12x+8)^{3/2}} + \frac{1}{16}\frac{(x-2)}{(3x^2-12x+8)^{1/2}}+C$$

Sec 01
Ergun

Question: $I=\int \frac{x\,dx}{(9x^2-36x+32)^{3/2}} = ?$

Sol As $\frac{d}{dx}(9x^2-36x+32) = 18x-36 = 18(x-2)$,

$$I = \underbrace{\int \frac{(x-2)\,dx}{(9x^2-36x+32)^{3/2}}}_{I_1} + \underbrace{\int \frac{2\,dx}{(9x^2-36x+32)^{3/2}}}_{I_2}$$ 1

$$I_1 = \frac{1}{18}\int \frac{du}{u^{3/2}} = \frac{1}{18}(-2)u^{-1/2} + C_1 = \frac{-1}{9\,(9x^2-36x+32)^{1/2}} + C_1$$ 1

$u = 9x^2-36x+32$

$du = 18(x-2)\,dx$ 1

2

$$9x^2-36x+32 = 9(x^2-4x)+32 = 9\left((x-2)^2 - 4\right)+32 = 9(x-2)^2 - 4$$

Put $3(x-2) = 2\sec\theta$ 2. Then, $3dx = 2\sec\theta\tan\theta\,d\theta$, and

$9x^2-36x+32 = 4\tan^2\theta$. So, I_2 becomes 2

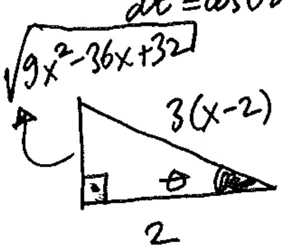
$$I_2 = \int \frac{2(2/3)\sec\theta\tan\theta\,d\theta}{2^3\tan^3\theta} = \frac{1}{6}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \frac{1}{6}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

$$= \frac{1}{6}\int \frac{dt}{t^2} = -\frac{1}{6}\,\frac{1}{t} + C_2 = \frac{-1}{6\sin\theta} + C_2$$

$t = \sin\theta$

$dt = \cos\theta\,d\theta$ 1

$$= \frac{-(x-2)}{2\sqrt{9x^2-36x+32}} + C_2$$



1

Consequently,

$$I = \frac{-1}{9\sqrt{9x^2-36x+32}} + \frac{-(x-2)}{2\sqrt{9x^2-36x+32}} + C$$

$$= \frac{16-9x}{18\sqrt{9x^2-36x+32}} + C$$

Quiz 7

Determine whether the integral $\int_1^\infty \frac{1}{\sqrt{e^x - x}}\,dx$ converges.

Note that

$$\lim_{x\to\infty} \frac{\frac{1}{\sqrt{e^x-x}}}{e^{-x/2}} = 1 \qquad \text{and} \qquad \int_1^\infty e^{-x/2}\,dx = -2e^{-x/2}\Big|_1^\infty = \frac{2}{\sqrt{e}}.$$

Hence by limit comparison theorem the first integral also converges.

Is the improper integral $\int_0^\infty \frac{dx}{(x+x^4)^{1/3}}$ convergent?

Solution:

$$\int_0^\infty \frac{dx}{(x+x^4)^{1/3}} = \underbrace{\int_0^1 \frac{dx}{(x+x^4)^{1/3}}}_{\overset{\shortparallel}{I}} + \underbrace{\int_1^\infty \frac{dx}{(x+x^4)^{1/3}}}_{\overset{\shortparallel}{J}} \quad 1$$

$0 \leq \frac{1}{(x+x^4)^{1/3}} \leq \frac{1}{x^{1/3}}$ 2 for $x \in (0,1]$. As $\int_0^1 \frac{dx}{x^{1/3}}$ is convergent, 1 the direct comparision test implies 1 that I is convergent.

$0 \leq \frac{1}{(x+x^4)^{1/3}} \leq \frac{1}{x^{4/3}}$ 2 for $x \in [1,\infty)$. As $\int_1^\infty \frac{dx}{x^{4/3}}$ is convergent, 1 the direct comparision test implies that J is convergent. 1

Consequently, $\int_0^\infty \frac{dx}{(x+x^4)^{1/3}}$ is convergent. 1

Sec 1. Engin

Quiz

$$I = \int \frac{4-(2x+3)^{1/2}}{(2x+3)+(2x+3)^{2/3}}\,dx = ?$$

Sol. Let $\boxed{u^6 = (2x+3)}$ (2) Then $\boxed{6u^5\,du = 2dx}$ (1) and

$$I = \int \frac{4-u^3}{u^6+u^4}\,3u^5\,du = 3\int \frac{4-u^3}{u^2+1}\,u\,du$$

$$= \boxed{3\int \frac{-u^4+4u}{u^2+1}\,du} \quad (2)$$

$$\begin{array}{r|l} -u^4+4u & u^2+1 \\ \hline -u^4-u^2 & -u^2+1 \\ \hline u^2+4u & \\ u^2+1 & \\ \hline 4u-1 & \end{array}$$

So $\boxed{\dfrac{-u^4+4u}{u^2+1} = (-u^2+1) + \dfrac{4u-1}{u^2+1}}$ (2)

(1)

So, $I = \boxed{3\int \left(-u^2+1 + \frac{4u}{u^2+1} - \frac{1}{u^2+1}\right) du}$

$$= \boxed{3\left(\frac{-u^3}{3} + u + 2\ln(u^2+1) - \arctan u\right) + C} \quad (2)$$

$$= -u^3 + 3u + 6\ln(u^2+1) - 3\arctan u + C$$

$$= -(2x+3)^{1/2} + 3(2x+3)^{1/6} + 6\ln\left((2x+3)^{1/3}+1\right) - 3\arctan\left((2x+3)^{1/6}\right) + C$$

Sec 01. Engin

Quiz 8

Determine whether $\int_1^\infty \frac{e^x}{x}\,dx$ converges or not.

$e^x > 1 \quad , \ x \geq 1 \implies \frac{e^x}{x} > \frac{1}{x} \quad , \ x \geq 1$

and $\int_1^\infty \frac{dx}{x}$ diverges , $p=1$

So, $\int_1^\infty \frac{e^x}{x}\,dx$ diverges by the DCT.

OR

$\frac{e^x}{x} > \frac{x}{x} = 1$ and $\int_1^\infty dx$ diverges $\xRightarrow{DCT} \int_1^\infty \frac{e^x}{x}\,dx$ diverges.

OR, you can use LCT.

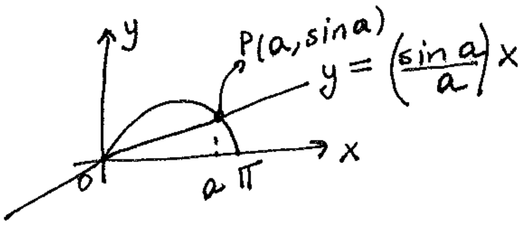
Quiz 8

Estimate the convergence behavior of

$$\int_0^{\infty} \frac{|\sin x|}{x^2}\,dx .$$

$$\int_0^{\infty} \frac{|\sin x|}{x^2}\,dx = \int_0^{a} \frac{|\sin x|}{x^2}\,dx + \int_a^{\infty} \frac{|\sin x|}{x^2}\,dx$$

where $a \in (0, \pi)$.



$$\sin x \geq \left(\frac{\sin a}{a}\right) x$$

$$\frac{|\sin x|}{x^2} = \frac{\sin x}{x^2} \geq \frac{\frac{\sin a}{a}}{x}, \quad x \in (0, a]$$

& $\displaystyle\int_0^{a} \frac{\frac{\sin a}{a}}{x}\,dx$ div., $p = 1$

$$\overset{DCT}{\Longrightarrow} \int_0^{a} \frac{\sin x}{x^2}\,dx \text{ diverges.}$$

$$\frac{|\sin x|}{x^2} \leq \frac{1}{x^2}, \quad x \in [a, \infty) \quad \text{and} \quad \int_a^{\infty} \frac{dx}{x^2} \text{ conv.}, \ p = 2 > 1$$

$$\overset{DCT}{\Longrightarrow} \int_a^{\infty} \frac{|\sin x|}{x^2}\,dx \text{ converges.}$$

Hence, $\displaystyle\int_0^{\infty} \frac{|\sin x|}{x^2}\,dx$ diverges.

QUIZ 8 Kopyası

Find exact value of

$$\sum_{n=0}^{\infty} \left(\arctan n - \arctan(n+2) \right)$$

Let $f(n) = \arctan n$

$$S_n = f(0) - \cancel{f(2)} + f(1) - \cancel{f(3)}$$
$$+ \cancel{f(2)} - \cancel{f(4)} + \cancel{f(3)} - \cancel{f(5)}$$
$$+ \text{- - - —}$$
$$+ \cancel{f(n-2)} - \cancel{f(n)} + \cancel{f(n-1)} - f(n+1)$$
$$+ \cancel{f(n)} - f(n+2)$$

$$\therefore S_n = f(0) + f(1) - f(n+1) - f(n+2)$$

$$\lim_{n \to \infty} S_n = 0 + \frac{\pi}{4} - \frac{\pi}{2} - \frac{\pi}{2}$$

Quiz 9

Calculate $\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$.

$$\sum_{n=1}^{\infty} \frac{n+1-1}{(n+1)!} = \sum_{n=1}^{\infty} \left(\frac{1}{n!} - \frac{1}{(n+1)!} \right) \quad \text{-- } 4 \text{ pts.}$$

$$S_n = \sum_{k=1}^{n} \left(\frac{1}{k!} - \frac{1}{(k+1)!} \right) \quad 2 \text{ pts.}$$

$$= \left(\frac{1}{1!} - \cancel{\frac{1}{2!}} \right) + \left(\cancel{\frac{1}{2!}} - \cancel{\frac{1}{3!}} \right) + \cdots + \left(\cancel{\frac{1}{n!}} - \frac{1}{(n+1)!} \right)$$

$$= 1 - \frac{1}{(n+1)!} \quad 2 \text{ pts.}$$

$$\underbrace{\lim_{n \to \infty} S_n = \lim_{n \to \infty} \left(1 - \cancel{\frac{1}{(n+1)!}} \right) = 1 = \sum_{n=1}^{\infty} \frac{n}{(n+1)!}}_{2 \text{ pts.}}$$

quiz2 9

Let f be a positive function with $\lim_{x \to \infty} f(x) = \frac{\pi}{2}$.

Show that $\sum \frac{f(n)}{n}$ is divergent.

Sol Let $a_n = \frac{f(n)}{n}$ and $b_n = \frac{1}{n}$. Then,

$a_n > 0$ and $b_n > 0$, and $\sum b_n$ is divergent.

2 2

Moreover,

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} f(n) = \frac{\pi}{2} \quad 4$$

Therefore, it follows by the limit comparision test that $\sum a_n$ is divergent. 2

Sol 2: As $\lim_{x \to \infty} f(x) = \frac{\pi}{2} > 1$, $f(n) \geq 1$ eventually 3

Thus, $\frac{f(n)}{n} \geq \frac{1}{n}$ 3

As $\sum \frac{1}{n}$ is divergent, by the limit comparision test $\sum \frac{f(n)}{n}$ is divergent.

2 2

Quiz 9

Consider the sequence $\{a_n\} = \{\cos(\frac{b_n}{5})\}$

where $\sum_{n=1}^{\infty} b_n = \frac{\pi}{2}$. Find $\lim_{n\to\infty} a_n$.

-1

(5pts.) $\left(\sum_{n=1}^{\infty} b_n = \frac{\pi}{2} \right.$, i.e, $\sum_{n=1}^{\infty} b_n$ converges $\xRightarrow[\text{test for div.}]{n\text{-th term}} \lim_{n\to\infty} b_n = 0.$

(5pts) $\lim_{n\to\infty} \cos\left(\frac{b_n}{5}\right) = 1$

Quiz 9

Use Integral test to determine whether

$$\sum_{1}^{\infty} \frac{(\ln n)^{4/3}}{n^{3/2}} \quad \text{converges.}$$

$$\frac{(\ln x)^{4/3}}{x^{3/2}} > 0 \quad \& \text{ decreasing if } x > c = \lceil e^{8/9} \rceil = 3$$

$$\sum_{1}^{\infty} \frac{(\ln n)^{4/3}}{n^{3/2}} = \frac{(\ln 1)^{4/3}}{1} + \frac{(\ln 2)^{4/3}}{2^{3/2}} + \sum_{n \geq 3} \frac{(\ln n)^{4/3}}{n^{3/2}}$$

$$\sum_{n \geq 3}^{\infty} \frac{(\ln n)^{4/3}}{n^{3/2}} \sim \int_{3}^{\infty} \frac{(\ln x)^{4/3}}{x^{3/2}} dx = \underbrace{-\frac{2}{\sqrt{x}} (\ln x)^{4/3} \Big|_{3}^{\infty}}_{A} \underbrace{+ \frac{8}{3} \int_{3}^{\infty} \frac{(\ln x)^{1/3}}{x^{3/2}} dx}_{B}$$

L'Hospital's Rule

$$A \overset{LR}{=} \frac{2}{\sqrt{3}} (\ln 3)^{4/3} < \infty$$

$$B = \frac{8}{3} \Big(\underbrace{-\frac{2}{\sqrt{x}} (\ln x)^{1/3} \Big|_{3}^{\infty}}_{\text{similar to } A \;\; \underbrace{< \infty}_{\text{finite}}} + \frac{2}{3} \underbrace{\int_{3}^{\infty} \frac{1}{(\ln x)^{2/3} x^{3/2}} dx}_{C} \Big)$$

$$C = \int_{3}^{\infty} \frac{1}{\underbrace{(\ln x)^{2/3} x^{3/2}}_{> 0}} dx < \int_{3}^{\infty} \frac{1}{x^{3/2}} dx < \infty$$

$\therefore$ Series converges.

Quiz 10

Is the series $\sum (-1)^n \tan(\frac{1}{n})$ conditionally convergent?

Explain your answer

Sol: Let $a_n = (-1)^n \tan(\frac{1}{n})$. Then $|a_n| = \tan(\frac{1}{n})$ for all n

As $\lim_{n\to\infty} \frac{\tan(\frac{1}{n})}{\frac{1}{n}} = 1$, (1) the limit comparison test

implies that $\sum |a_n|$ diverges (1) because $\sum \frac{1}{n}$ diverges. (1)

In other words, $\sum a_n$ is <u>not</u> absolutely convergent.

(<u>OR</u> as $\tan x \geq x$ for all x,

$|a_n| = \tan(\frac{1}{n}) \geq \frac{1}{n} \geq 0$. As $\sum \frac{1}{n}$ diverges, by the

comparison test $\sum |a_n|$ diverges.)

Let $f(x) = \tan(\frac{1}{x})$. Then $f'(x) = \left(\sec^2 \frac{1}{x}\right) \frac{-1}{x^2} < 0$ for $x > 0$.

Therefore, $\tan(\frac{1}{n})$ is decreasing. (1)

Moreover, as $\tan(\frac{1}{n}) > 0$ (1) and as $\tan(\frac{1}{n}) \to 0$, (1)

it follows by the Alternating Series Test that (1)

$\sum a_n$ converges.

Since $\sum |a_n|$ diverges and since $\sum a_n$ converges,

$\sum a_n$ converges conditionally (1)