Math. 112, MT2, (2009-10, Spring)

KEY

1. (2× 10 pts.) Evaluate

a) $\int \tan^{\frac{3}{2}} x \sec^4 x\,dx \xrightarrow[ds=\sec^2 x\,dx]{s=\tan x} \int s^{3/2}(s^2+1)\,ds$

& $\sec^2 x = \tan^2 x + 1$

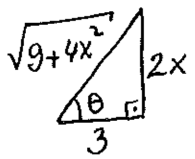
$$= \int (s^{7/2} + s^{3/2})\,ds$$

$$= \frac{2s^{9/2}}{9} + \frac{2s^{5/2}}{5} + C$$

$$= 2\left(\frac{\tan^{9/2} x}{9} + \frac{\tan^{5/2} x}{5}\right) + C.$$

b) $\int \frac{dx}{x\sqrt{4x^2+9}} \xrightarrow[2dx = 3\sec^2\theta\,d\theta]{2x = 3\tan\theta} \frac{1}{3}\int \frac{\sec\theta}{\tan\theta}\,d\theta$

$\sqrt{4x^2+9} = 3\sec\theta$



$$= \frac{1}{3}\int \csc\theta\,d\theta$$

$$= \frac{1}{3}\int \csc\theta \left(\frac{\csc\theta + \cot\theta}{\csc\theta + \cot\theta}\right) d\theta$$

$$= -\frac{1}{3}\ln|\csc\theta + \cot\theta| + C$$

$$= -\frac{1}{3}\ln\left|\frac{\sqrt{4x^2+9}}{2x} + \frac{3}{2x}\right| + C.$$

2.(4× 5 pts.)a) Let the sequence $\{a_n\}$ be defined by

$$a_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\cdots + \frac{1}{n^2}.$$

i) Show that $\{a_n\}$ is nondecreasing.

ii) Show that $\{a_n\}$ is bounded from above.(Hint: $n^2 > n^2 - n = n(n-1), n \geq 1$)

iii) Using the results of parts (a) and (b), what can you deduce about the convergence of $\{a_n\}$

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i) $a_{n+1} - a_n = \frac{1}{(n+1)^2} \geq 0 \implies \{a_n\}$ is nondecreasing.

ii) $0 \leq a_n = 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \leq 1 + (1 - \cancel{\frac{1}{2}}) + (\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}) + \cancel{\cdots} + (\cancel{\frac{1}{n-1}} - \frac{1}{n})$

$= 2 - \frac{1}{n} < 2$

where $\frac{1}{n^2} < \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$, $n \geq 2$

$\therefore \{a_n\}$ is bdd. from above (by 2).

iii) $\{a_n\}$ is nondecreasing and bdd. from above (by 2),

so $\lim_{n\to\infty} a_n = L \in \mathbb{R}$ (and $L \leq 2$).

b)Evaluate $\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{b\to\infty} \int_2^b \frac{dx}{x(\ln x)^2}$

$= \lim_{b\to\infty} \left(-\frac{1}{\ln x}\right)\Big|_{x=2}^{b}$

$= \lim_{b\to\infty} \left(\frac{1}{\ln 2} - \frac{1}{\ln b}\right)$

$= \frac{1}{\ln 2}$

$\boxed{\text{Remember: } \int \frac{dx}{x(\ln x)^2} = \int \frac{du}{u^2} \text{ where } u = \ln x \; = -\frac{1}{\ln x} + C}$

3. (2× 10 pts.) a) Find $\lim_{n\to\infty} \frac{5^n (n!)^3 n^5}{(3n)!}$

Consider the ~~nonnegative~~ positive series $\sum_{n=1}^{\infty} \frac{5^n (n!)^3 n^5}{(3n)!}$.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{5^{\cancel{n+1}}(\cancel{(n+1)!})^3 (n+1)^5}{\underset{(3n+1)(3n+2)(3n+3)}{\cancel{(3(n+1))!}}} \frac{\cancel{(3n)!}}{\cancel{5^n}\cancel{(n!)^3} n^5}$$

$$= \lim_{n\to\infty} \frac{5(n+1)^8}{(3n+1)(3n+2)(3n+3)n^5} = \frac{5}{27} < 1 \xRightarrow[\text{Test}]{\text{Ratio}} \sum_{n=1}^{\infty} \frac{5^n (n!)^3 n^5}{(3n)!} \text{ conv.}$$

$$\xRightarrow[\text{test}]{\text{n-th term}} \lim_{n\to\infty} \frac{5^n (n!)^3 n^5}{(3n)!} = 0$$

b) Find the value of c if $\sum_{n=2}^{\infty} (1+c)^{-n} = 2$.

$$\sum_{n=2}^{\infty} (1+c)^{-n} = \frac{(1+c)^{-2}}{1-(1+c)^{-1}} \text{ if } |1+c| > 1$$

$$\text{So, } \frac{(1+c)^{-2}}{1-(1+c)^{-1}} = 2 \quad \text{if} \quad c < -2 \text{ or } c > 0$$

$$\Downarrow$$

$$2c^2 + 2c - 1 = 0$$

$$\Rightarrow c_{1,2} = \frac{-1 \mp \sqrt{3}}{2}$$

$$\text{but } \frac{-1-\sqrt{3}}{2} \not< -2 \text{ or } \frac{-1-\sqrt{3}}{2} \not> 0$$

$$\therefore \quad c = \frac{-1+\sqrt{3}}{2} > 0.$$

4.(5× 4 pts.) In problem 4) (a)-(e),determine whether the statement is true or false.If it is true,explain why it is true.If it is false,explain why or give an example to show why it is false.

(True) a) If $\sum_{n=1}^{\infty} b_n$ is convergent with positive terms, then $\sum_{n=1}^{\infty} (b_n)^n$ must also converge.

$\sum b_n$ conv. $\Rightarrow \lim_{n\to\infty} b_n = 0$ and $\lim_{n\to\infty} \sqrt[n]{(b_n)^n} = 0 < 1$

$\xRightarrow[\text{Test}]{\text{Root}} \sum (b_n)^n$ converges.

(False) b) $\sum_{n=1}^{\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}$ converges.

$\lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1 \neq 0 \xrightarrow[\text{test for div.}]{\text{n-th term}} \sum \frac{\sin\frac{1}{n}}{\frac{1}{n}}$ diverges.

(False) c) $\sum_{n=1}^{\infty} \frac{\arctan n}{n^3 + 59}$ diverges.

$0 \le \frac{\arctan n}{n^3+59} \le \frac{\pi}{2n^3}$ and $\sum \frac{\pi}{2n^3}$ conv., $p = 3 > 1$

$\xRightarrow[\text{Test}]{\text{Comparison}} \sum \frac{\arctan n}{n^3+59}$ converges

(False) d) $\sum_{n=1}^{\infty} \frac{n}{n^2 - \cos^2 n}$ converges.

$0 < n^2 - \cos^2 n \le n^2 \Rightarrow \frac{n}{n^2 - \cos^2 n} \ge \frac{1}{n}$ and $\sum \frac{1}{n}$ div., $p=1$,

$\xRightarrow[\text{Test}]{\text{Comparison}} \sum_{n=1}^{\infty} \frac{n}{n^2 - \cos^2 n}$ diverges.

e) $\sum_{n=1}^{\infty} \left(e^{\frac{1}{n}} - e^{\frac{1}{(n+1)}} \right) = e - 1$

$$S_n = \sum_{k=1}^{n} \left(e^{\frac{1}{k}} - e^{\frac{1}{k+1}}\right) = \left(e - \cancel{e^{\frac{1}{2}}}\right) + \left(\cancel{e^{\frac{1}{2}}} - \cancel{e^{\frac{1}{3}}}\right) + \cancel{\cdots} + \left(\cancel{e^{\frac{1}{n}}} - e^{\frac{1}{n+1}}\right)$$

$$= e - e^{\frac{1}{n+1}}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(e - e^{\frac{1}{n+1}}\right) = e - 1 = \sum_{n=1}^{\infty} \left(e^{\frac{1}{n}} - e^{\frac{1}{n+1}}\right)$$

5. (2× 10 pts.) Are they convergent? Give reasons for your answer.

a) $\sum_{n=1}^{\infty} \frac{4.7.10\cdots\cdots(3n+1)}{4^n(n+1)!} > 0$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\cancel{4.7.10\ldots(3n+1)}(3n+4)}{4^{\cancel{n}+1}(n+2)\cancel{!}} \frac{\cancel{4^n}\cancel{(n+1)!}}{\cancel{4.7.10\ldots 3n+1}} = \frac{3}{4} < 1$$

$$\xrightarrow[\text{Test}]{\text{Ratio}} \sum_{n=1}^{\infty} \frac{4.7.10\ldots(3n+1)}{4^n(n+1)!} \text{ converges.}$$

n^3+n+49 behaves like n^3 and

$3n^7+\ln n$ ~~behaves~~ behaves like n^7 for large n.

b) $\sum_{n=1}^{\infty} \frac{n^3+n+49}{\sqrt{3n^7+\ln n}}$

$$\Rightarrow \frac{n^3+n+49}{\sqrt{3n^7+\ln n}} \text{ behaves like } \frac{n^3}{n^{7/2}} = \frac{1}{n^{1/2}}$$

$$\lim_{n\to\infty} \frac{\frac{n^3+n+49}{\sqrt{3n^7+\ln n}}}{\frac{1}{n^{1/2}}} = \frac{1}{\sqrt{3}} \neq 0, \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} \text{ div., } p = \frac{1}{2} < 1$$

$$\xrightarrow{\text{LCT}} \sum_{n=1}^{\infty} \frac{n^3+n+49}{\sqrt{3n^7+\ln n}} \text{ diverges.}$$