Math. 112, Midterm II, 2008–2009, Spring

(10+10 pts.)1. Determine whether

a) the improper integral $\int_0^\infty \frac{dx}{(x^5+x^2)^{\frac{1}{3}}}$ converges or diverges.

b) the sequence $\left\{\frac{3^n(n!)^2}{(2n)!}\right\}$ converges or diverges.

a) $\underbrace{\int_0^\infty \frac{dx}{(x^5+x^2)^{1/3}}}_{I} = \underbrace{\int_0^1 \frac{dx}{(x^5+x^2)^{1/3}}}_{I_1} + \underbrace{\int_1^\infty \frac{dx}{(x^5+x^2)^{1/3}}}_{I_2}$

$I_1 \leq \int_0^1 \frac{dx}{x^{2/3}}$ conv., $p = \frac{2}{3} < 1$ $\overset{DCT}{\Longrightarrow}$ I_1 converges, and

$I_2 \leq \int_1^\infty \frac{dx}{x^{5/3}}$ conv., $p = \frac{5}{3} > 1$ $\overset{DCT}{\Longrightarrow}$ I_2 converges.

Hence, I is convergent.

b) $\frac{a_{n+1}}{a_n} = \frac{3^{\cancel{n+1}}(\cancel{(n+1)!})^2}{\underbrace{\cancel{(2(n+1))!}}_{(2n+1)(2n+2)}} \cdot \frac{\cancel{(2n)!}}{\cancel{3^n(n!)^2}} = \frac{3(n+1)}{2(2n+1)} = 1 - \frac{n-1}{4n+2} \leq 1 \quad \forall n \geq 1$

$\Rightarrow a_{n+1} \leq a_n$, i.e., $\left\{\frac{3^n(n!)^2}{(2n)!}\right\}$ is nonincreasing.

$a_n \geq 0 \quad \forall n \geq 1$, i.e., the sequence is bdd. from below.

(monotone + bdd.) $\Rightarrow$ $\{a_n\}$ converges.

<u>second soln.</u> Consider the series $\sum_{n=1}^\infty \frac{3^n(n!)^2}{(2n)!}$.

If we apply the Ratio Test to this series, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{3}{4} < 1$.

The series converges. Hence, by the n-th term test for divergence $\lim_{n\to\infty} \frac{3^n(n!)^2}{(2n)!} = 0$, ie, the sequence converges.

(20 pts.)2. Find the sum of the series $\sum_{n=2}^{\infty} \frac{1}{(n-1)!(n+1)}$ by interpreting as a telescoping series .

$$\frac{1}{(n-1)!\,(n+1)} = \frac{n}{(n-1)!\,n\,(n+1)} = \frac{n}{(n+1)!} = \frac{n+1-1}{(n+1)!} = \frac{1}{n!} - \frac{1}{(n+1)!} .$$

$$\therefore \sum_{k=2}^{n} \frac{1}{(k-1)!\,(k+1)} = \sum_{k=2}^{n} \left(\frac{1}{k!} - \frac{1}{(k+1)!} \right)$$

$$= \left(\frac{1}{2!} - \cancel{\frac{1}{3!}} \right) + \left(\cancel{\frac{1}{3!}} - \cancel{\frac{1}{4!}} \right) + \cdots + \left(\cancel{\frac{1}{(n-1)!}} - \cancel{\frac{1}{n!}} \right) + \left(\cancel{\frac{1}{n!}} - \frac{1}{(n+1)!} \right)$$

$$= \frac{1}{2!} - \frac{1}{(n+1)!} , \text{ and } \lim_{n\to\infty} \frac{1}{(n+1)!} = 0 .$$

Hence , $\sum_{n=2}^{\infty} \frac{1}{(n-1)!\,(n+1)} = \frac{1}{2}$.

(10+10 pts.)3. a) Does the series $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$ converge?

b) Suppose that $\sum_{n=1}^{\infty} a_n$ converges and that $\lim_{n\to\infty} n^{1+\frac{1}{n}} |a_n| = \infty$. Does $\sum_{n=1}^{\infty} a_n$ converge absolutely? **Explain your answer clearly.**

a) $$\lim_{n\to\infty} \frac{\frac{1}{n^{1+\frac{1}{n}}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{1}{n \cdot n^{1/n}}}{\frac{1}{n}} = 1 \neq 0, \infty$$ and $\sum \frac{1}{n}$ div., harmonic ser.

where $n^{1/n} \to 1$ as $n \to \infty$.

$\overset{LCT}{\Longrightarrow} \sum \frac{1}{n^{1+\frac{1}{n}}}$ diverges.

b) Consider $\sum_{1}^{\infty} |a_n|$. Then

$$\lim_{n\to\infty} \frac{|a_n|}{\frac{1}{n^{1+\frac{1}{n}}}} = \lim_{n\to\infty} (n^{1+\frac{1}{n}})(|a_n|) = \infty \text{ (given)}$$

and

$\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$ diverges $\overset{LCT}{\Longrightarrow} \sum |a_n|$ diverges.

Hence, $\sum a_n$ can not be absolutely convergent and

$\sum a_n$ converges conditionally.

(10+10 pts.)4. Determine whether the following series converges or diverges.

Explain your answer clearly.

a) $\sum_{n=1}^{\infty}\left(1-\cos\left(\frac{1}{n^3}\right)\right)$ b) $\sum_{n=1}^{\infty}\frac{3}{e^{\cos\frac{1}{n}}+1}$

a) $0 \leq 1-\cos\left(\frac{1}{n^3}\right) = 2\sin^2\left(\frac{1}{2n^3}\right) \leq 2\left(\frac{1}{2n^3}\right)^2 = \frac{1}{2n^6}$ and

$\sum_{1}^{\infty}\frac{1}{2n^6}$ conv., $p=6>1$

$\overset{CT}{\Longrightarrow} \sum_{1}^{\infty}\left(1-\cos\left(\frac{1}{n^3}\right)\right)$ conv.

second soln.

$\lim_{n\to\infty}\frac{1-\cos\left(\frac{1}{n^3}\right)}{\frac{1}{n^3}} = 0$ and $\sum\frac{1}{n^3}$ conv., $p=3>1$

$\overset{LCT}{\Longrightarrow} \sum_{1}^{\infty}\left(1-\cos\left(\frac{1}{n^3}\right)\right)$ converges.

b) $\lim_{n\to\infty}\frac{3}{e^{\cos\frac{1}{n}}+1} = \frac{3}{e+1} \neq 0 \xrightarrow[\text{test for divergence}]{\text{n-th term}} \sum_{1}^{\infty}\frac{3}{e^{\cos\frac{1}{n}}+1}$ diverges.

(20 pts.)5. Find the radius and interval of convergence for the series

$$\sum_{n=1}^{\infty} a_n (x-3)^n \text{ where } a_n = \frac{1.3.5......(2n-1)}{2^n n!}.$$

Hint: $\frac{1}{2\sqrt{n}} \le a_n \le \frac{1}{\sqrt{2n+1}}$.

Consider $\sum_1^{\infty} a_n |(x-3)^n|$, and use the Ratio test. Then

$$L = \lim_{n\to\infty} \frac{(a_{n+1})|x-3|^{n+1}}{(a_n)|x-3|^n} = |x-3| \lim_{n\to\infty} \frac{\cancel{1.3.5\cdots(2n-1)}(2n+1)}{2^{\cancel{n+1}}(n+1)\cancel{!}} \frac{\cancel{2^n}\,\cancel{n!}}{\cancel{1.3.5\cdots(2n-1)}}$$

$$= |x-3| \left(\lim_{n\to\infty} \frac{2n+1}{2(n+1)} \right)^{\to 1} = |x-3|.$$

$L = |x-3| < 1 \Rightarrow \sum_1^{\infty} a_n (x-3)^n$ converges (absolutely), i.e.,

$\sum_1^{\infty} a_n (x-3)^n$ converges (abs.) if $x \in (2,4)$.

$L = |x-3| = 1 \Longrightarrow$ the Ratio Test fails, i.e., another test is needed.

$|x-3| = 1 \Rightarrow$ either $x-3=1$ or $x-3=-1$.

$\underline{x-3=1}$ or $x=4$: Consider $\sum a_n$.

$a_n \geq \frac{1}{2\sqrt{n}}$ and $\sum_1^{\infty} \frac{1}{2\sqrt{n}}$ div., $p = \frac{1}{2} < 1 \xRightarrow{CT} \sum a_n$ diverges.

$\underline{x-3=-1}$ or $x=2$: Consider $\sum_1^{\infty} (-1)^n a_n$.

$\frac{a_{n+1}}{a_n} = \frac{2n+1}{2(n+2)} < 1$, $a_n > 0$, and $a_n \to 0$ as $n \to \infty$ $\xRightarrow{AST} \sum_1^{\infty} (-1)^n a_n$ conv.

$\left(\frac{1}{2\sqrt{n}} \le a_n \le \frac{1}{\sqrt{2n+1}} \right)$, $\to 0$ (Sandwich Thm)

$\therefore$ $I = [2,4)$ and $R = 1$.