5) a) Determine whether each of the following series is convergent or divergent.State clearly the name and the conditions of the test you are using.

i) $\sum a_n^2$ where $a_n > 0$ and $\sum a_n$ converges.

Remember the Limit Comparison Test

$$L = \lim_{n\to\infty} \frac{a_n^2}{a_n} = \lim_{n\to\infty} a_n = 0 \quad \text{since } \sum a_n \text{ converges}$$

$\therefore \sum a_n^2$ converges from the LCT.

ii) $\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^n$

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \frac{1}{e} \neq 0 \xrightarrow[\text{test for divergence}]{\text{n-th term}} \sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^n \text{ diverges.}$$

b) Find the radius of convergence and interval of convergence of $\sum_{n=1}^{\infty}\frac{(-2)^n(x-3)^n}{\sqrt{2n+3}}$.(Do not forget the end points.)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 2|x-3| \lim_{n\to\infty}\sqrt{\frac{2n+3}{2n+5}} = 2|x-3| < 1 \Rightarrow |x-3| < \frac{1}{2}$$

$$\Rightarrow R = \frac{1}{2}$$

<u>end points:</u>

$|x-3| = \frac{-1}{2}$ (i.e., $x = \frac{5}{2}$) : $\sum_{n=1}^{\infty}\frac{1}{\sqrt{2n+3}}$ is divergent by the LCT with the divergent series $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ $(p = \frac{1}{2} < 1)$.

$|x-3| = \frac{1}{2}$ (i.e., $x = \frac{7}{2}$) : $\sum \frac{(-1)^n}{\sqrt{2n+3}}$ converges by the AST (i.e., $u_n = \frac{1}{\sqrt{2n+3}} \geq 0$, u_n is nonincreasing since $\frac{u_{n+1}}{u_n} = \sqrt{1 - \frac{2}{2n+5}} < 1$, and $u_n \to 0$ as $n \to \infty$).

Thus, $I = \left(\frac{5}{2}, \frac{7}{2}\right]$.