1) Determine whether each of the following series is convergent or divergent. State clearly **the name** and **the conditions** of the test you are using. **Show all your work.**

a) $\sum_{n=1}^{\infty}\left(\frac{n!}{n^n}\right)^n$, $a_n = \left(\frac{n!}{n^n}\right)^n > 0 \ \forall n.$

$$0 < a_n = \left(\frac{n!}{n^n}\right)^n = \frac{1^n.2^n.\cdots.n^n}{n^n.n^n\cdots n^n} < \frac{1.n^n.n^n.\cdots.n^n}{n^n.n^n.n^n.\cdots.n^n} \leqslant \frac{1}{n^n} \leq \frac{1}{n^2} \leq \frac{1}{n^2} \text{ for } n \geqslant 2$$

and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ conv. $(p=2>1)$ $\xrightarrow[\text{Test}]{\text{Comparison}}$ $\sum_{n=1}^{\infty}\left(\frac{n!}{n^n}\right)$ converges.

OR

$$L = \lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\frac{n!}{n^n} = \lim_{n\to\infty}\frac{1.2.3.\cdots\cdots n}{n.n.n.\cdots n} = 0$$

where $0 < \frac{1}{n}\frac{2}{n}\frac{3}{n}\cdots\cdots\frac{n}{n} \leq \frac{1}{n}$. Then by Sandwich Thm. $\lim_{n\to\infty}\frac{n!}{n^n} = 0$

So $L = 0 < 1 \xrightarrow[\text{Test}]{\text{Root}} \sum_{n=1}^{\infty}\left(\frac{n!}{n^n}\right)^n$ is convergent.

b) $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^{1.1}}$, $a_n = \frac{1}{n(\ln n)^{1.1}} > 0 \ \forall n \geqslant 2.$

We use integral test. Let $f(x) = \frac{1}{x(\ln x)^{1.1}}$, $x \geqslant 2$. Then $f(x) > 0$

1) $f(x) > 0 \ \forall x \geqslant 2$,

2) $f(x)$ is continuous for $x \geqslant 2$,

3) $f(x)$ is decreasing for $x \geqslant 2$, (i.e., $f'(x) = \frac{-\left[(\ln x)^{1.1} + (1.1)(\ln x)^{0.1}\right]}{\left(x(\ln x)^{1.1}\right)^2} < 0$, $x \geqslant 2$) for $x \geqslant 2$. And

Thus, the conditions of the integral test are satisfied.

$$\int_2^{\infty}\frac{dx}{x(\ln x)^{1.1}} \underset{u=\ln x}{=} \int_{\ln 2}^{\infty}\frac{du}{u^{1.1}}$$ is convergent by the p-test for improper integrals $(p = 1.1 > 1)$.

So the series $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^{1.1}}$ is convergent by the integral test.