Math. 112 (2008-9, Spring)

MT1 Solns.

(8+8+8 pts.) 1. a) Evaluate $\int e^{(e^x+x)}dx = \int e^{e^x}\cdot e^x\,dx \underset{dw=e^x dx}{\overset{w=e^x}{=}} \int e^w\,dw$

$$= e^w + C$$

$$= e^{e^x} + C$$

b) Evaluate $\int \sin^3 x\cos^{301} x\,dx = \int \sin^2 x\cos^{301} x\,\sin x\,dx$

$$= \int (1-\cos^2 x)\cos^{301} x\,\sin x\,dx$$

$$\underset{ds=-\sin x\,dx}{\overset{s=\cos x}{=}} -\int (1-s^2)s^{301}\,ds = -\left(\frac{s^{302}}{302}-\frac{s^{304}}{304}\right)+C$$

$$= -\left(\frac{\cos^{302}x}{302}-\frac{\cos^{304}x}{304}\right)+C$$

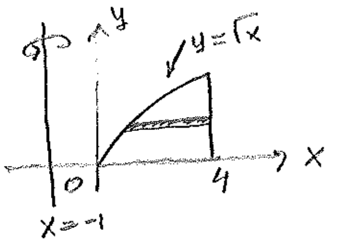
c) Write down the partial fraction decomposition for $\dfrac{x^6-3x^5+2x^4}{\left((x^2-2x+5)(x^2-1)\right)^2}$. **Don't evaluate the coefficients in the decomposition.**

$$\frac{x^6-3x^5+2x^4}{\left((x^2-2x+5)(x^2-1)\right)^2} = \frac{x^4\cancel{(x-1)}(x-2)}{(x^2-2x+5)^2(x-1)^{\cancel{2}}(x+1)^2} = \frac{x^4(x-2)}{(x^2-2x+5)^2(x-1)(x+1)^2}$$

$$= \frac{A_1x+B_1}{x^2-2x+5} + \frac{A_2x+B_2}{(x^2-2x+5)^2} + \frac{A_3}{x-1} + \frac{A_4}{x+1} + \frac{A_5}{(x+1)^2}.$$

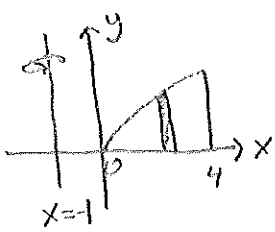
(6+6 pts.) 2. Consider the region R bounded by $y=\sqrt{x}$, $x=4$, and $y=0$. **Set up the integral** for the volume of the solid obtained by rotating R about the line $x=-1$ using

a) the disk method,



$$V = \pi\int_0^2 (4-(-1))^2\,dy - \pi\int_0^2 (y^2-(-1))^2\,dy$$

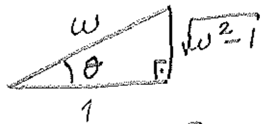
b) the shell method.



$$V = 2\pi\int_0^4 (x-(-1))(\sqrt{x})\,dx$$

(9+9 pts.) 3. Evaluate

a) $\int \frac{dx}{\sqrt{(x^2-2x)^3}}$ $= \int \frac{dx}{((x-1)^2-1)^{3/2}} \overset{x-1=w}{=} \int \frac{dw}{(w^2-1)^{3/2}} = \int \frac{\sec\theta \cancel{\tan\theta}\, d\theta}{\tan^{\cancel{3}2}\theta}$



$w = \sec\theta$

$$= \int \frac{1}{\cancel{\cos\theta}} \frac{\cos^{\cancel{2}}\theta}{\sin^2\theta}\, d\theta$$

$$\overset{u=\sin\theta}{=} \int \frac{du}{u^2}$$

$$= -\frac{1}{u} + C$$

$$= -\frac{1}{\sin\theta} + C$$

$$= -\frac{w}{\sqrt{w^2-1}} + C = \frac{1-x}{\sqrt{x^2-2x}} + C$$

b) $\int \frac{dx}{\cos x(1-\sin x)^2}$ $= \int \frac{\cos x\, dx}{\underset{\overset{\parallel}{(1-\sin^2 x)}}{\cos^2 x}(1-\sin x)^2}$

$$\overset{t=\sin x}{=} \int \frac{dt}{(1-t^2)(1-t)^2}$$

$$= \int \left(\frac{A_1}{1+t} + \frac{A_2}{1-t} + \frac{A_3}{(1-t)^2} + \frac{A_4}{(1-t)^3} \right) dt$$

where $A_1 = \frac{1}{8}$, $A_2 = \frac{1}{8}$, $A_3 = \frac{1}{4}$ and $A_4 = \frac{1}{2}$

$$= \frac{1}{8}\ln|1+t| - \frac{1}{8}\ln|1-t| + \frac{1}{4}\frac{1}{1-t} + \frac{1}{2}\frac{1}{2}\frac{1}{(1-t)^2} + A$$

$$= \frac{1}{8}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + \frac{1}{4}\frac{1}{1-\sin x} + \frac{1}{4}\frac{1}{(1-\sin x)^2} + A$$

(15 pts.) 4. Let f be a positive continuously differentiable function on $[1,3]$. Suppose that the length of the curve $y = f(x)$ on $[1,3]$ is 2, and that the surface area of the surface generated by revolving $y = f(x)$, for $1 \le x \le 3$, about the x-axis is 7π. Calculate the surface area of the surface generated by revolving $y = f(x)$, for $1 \le x \le 3$, about the line $y = -1$.

$$L = \int_1^3 \sqrt{1 + (f'(x))^2}\, dx = 2$$

$$S_1 = 2\pi \int_1^3 f(x) \sqrt{1 + (f'(x))^2}\, dx = 7\pi$$

$$S = 2\pi \int_1^3 (f(x) - (-1)) \sqrt{1 + (f'(x))^2}\, dx$$

$$= \underbrace{2\pi \int_1^3 f(x) \sqrt{1 + (f'(x))^2}\, dx}_{S_1} + 2\pi \underbrace{\int_1^3 \sqrt{1 + (f'(x))^2}\, dx}_{L}$$

$$= 7\pi + 2\pi(2) = 11\pi.$$

(16 pts.) 5. A certain function f is defined and differentiable for all x and satisfies

i) $0 < f(x) < 1$, all x,

ii) $\int_1^x f(t)dt = x + \ln(f(x)), \forall x.$

(ii) + FTOC $\Rightarrow$ $f(x) = 1 + \dfrac{f'(x)}{f(x)}$

$\Rightarrow f^2(x) = f(x) + f'(x)$

or $f'(x) = f(x)(f(x) - 1)$

$\Rightarrow \dfrac{df}{f(f-1)} = dx$ where $\dfrac{1}{f(f-1)} = \dfrac{-1}{f} + \dfrac{1}{f-1}$

$\Rightarrow \int \left(\dfrac{-1}{f} + \dfrac{1}{f-1}\right) df = \int dx$

$\Rightarrow \ln\left|\dfrac{f-1}{f}\right| = x + c_1$

$\Rightarrow \ln\left(\dfrac{1-f}{f}\right) = x + c_1$ since $-1 < f(x) - 1 < 0$

$\Rightarrow \dfrac{1-f}{f} = C_1 e^x$

$\Rightarrow f(x) = \dfrac{1}{1 + C_1 e^x}$

ii) $\Rightarrow 0 = 1 + \ln(f(1))$

$\Rightarrow f(1) = e^{-1}$

$\Rightarrow f(1) = \dfrac{1}{1 + C_1 e} = \dfrac{1}{e}$

$\Rightarrow C_1 = \dfrac{e-1}{e}$

So, $f(x) = \dfrac{1}{1 + \frac{e-1}{e} e^x}$

or

$f(x) = \dfrac{1}{1 + (e-1)e^{x-1}}$

(15 pts.) 6. Let A denote the value of the integral $\int_0^{\pi} \frac{\cos x}{(x+2)^2}\,dx$. Compute $\int_0^{\pi/2} \frac{\sin x \cos x}{x+1}\,dx$ in terms of A.

$$\int_0^{\pi/2} \frac{\sin x \cos x \, dx}{x+1} = \frac{1}{2}\int_0^{\pi/2} \frac{\sin 2x}{x+1}\,dx$$

$$\underset{ds=2dx}{\overset{2x=s}{=}} \frac{1}{4}\int_0^{\pi} \frac{\sin s \, ds}{\frac{s}{2}+1}$$

$$= \frac{1}{2}\int_0^{\pi} \frac{\sin s \, ds}{s+2}$$

$$u = \frac{1}{s+2}, \quad dv = \sin s\, ds$$

$$= \frac{1}{2}\left[\left.\frac{-\cos s}{s+2}\right|_0^{\pi} - \underbrace{\int_0^{\pi} \frac{\cos s \, ds}{(s+2)^2}}_{A} \right]$$

$$= \frac{1}{2}\left(\frac{1}{\pi+2} + \frac{1}{2} - A\right)$$

OR

$$\int_0^{\pi} \frac{\cos x}{(x+2)^2}\,dx \overset{t=\frac{x}{2}}{=} \int_0^{\pi/2} \frac{\cos 2t}{(2t+2)^2}(2dt)$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{\cos 2t}{(t+1)^2}\,dt \qquad u = \cos 2t, \quad dv = \frac{dt}{(t+1)^2}$$

$$= \frac{1}{2}\left[-\left.\frac{\cos 2t}{t+1}\right|_{t=0}^{\pi/2} - 2\int_0^{\pi/2} \frac{\overbrace{\sin 2t}^{2\sin t\cos t}}{t+1}\,dt \right]$$

$$A = \frac{1}{2}\left(\frac{1}{\frac{\pi}{2}+1} + 1 - 4\int_0^{\pi/2} \frac{\sin t \cos t}{t+1}\,dt\right)$$

$$\Rightarrow 2A = \frac{2}{\pi+2} + 1 - 4\int_0^{\pi/2} \frac{\sin t \cos t}{t+1}\,dt$$

$$\Rightarrow \int_0^{\pi/2} \frac{\sin t \cos t}{t+1}\,dt = \frac{1}{4}\left(\frac{2}{\pi+2} + 1 - 2A\right)$$

$$= \frac{1}{2}\left(\frac{1}{\pi+2} + \frac{1}{2} - A\right)$$