Math. 112 Midterm I (2005–2006, Spring)

1) a) (10 pts.) Solve the equation $4\arctan x = 4\arcsin(\sin\frac{5\pi}{4}) + 2\pi$.

$$4\arctan x = 4\arcsin\left(\sin\frac{5\pi}{4}\right) + 2\pi$$

$$= 4\arcsin\left(-\frac{\sqrt{2}}{2}\right) + 2\pi$$

$$= 4\cdot\left(-\frac{\pi}{4}\right) + 2\pi = \pi$$

$$\Rightarrow \arctan x = \frac{\pi}{4} \Rightarrow \boxed{x = 1}$$

b) (10 pts.) Find all solutions to $\log_{\frac{1}{3}} 2x + \log_3(x^3 - x^2) = 0$.

$$\log_{\frac{1}{3}} 2x = \frac{\ln 2x}{\ln(3^{-1})} = -\frac{\ln 2x}{\ln 3}$$

$$\log_3(x^3 - x^2) = \frac{\ln(x^3-x^2)}{\ln 3}$$

Thus we get

$$-\frac{\ln 2x}{\ln 3} + \frac{\ln(x^3-x^2)}{\ln 3} = 0 \Rightarrow -\frac{1}{\ln 3}\left[\ln 2x - \ln(x^3-x^2)\right] = \ln 1$$

$$\Rightarrow \ln\left[\frac{2x}{x^3-x^2}\right] = \ln 1$$

$$\Rightarrow \frac{2x}{x^3-x^2} = 1$$

$$\Rightarrow x^3 - x^2 - 2x = 0$$

$$\Rightarrow x(x^2 - x - 2) = 0$$

$\Rightarrow$ $x=0$, $x=-1$ and $x=2$ are the solutions but for $x=0$ & $x=-1$ the given functions are <u>not</u> defined

$\Rightarrow$ $x=2$ is the only solution.

2)a)(10 pts.) Let $f(x)=\ln x+\arctan x$ be given and $x>0$. Find $\left.\frac{df^{-1}}{dx}\right|_{x=\frac{\pi}{4}}$.

$f'(x)=\frac{1}{x}+\frac{1}{1+x^2}$ and $f(a)=\frac{\pi}{4} \Rightarrow \boxed{a=1}$

$$\left.\frac{df^{-1}}{dx}\right|_{x=\frac{\pi}{4}}=\frac{1}{\left.\frac{df}{dx}\right|_{x=1}}=\frac{1}{1+\frac{1}{2}}=\frac{2}{3}$$

b) (10 pts.) Find $\frac{dy}{dx}$ if $y=3^{(x^3)}+x^{(3^x)}$.

Let $f(x)=3^{x^3}$ and $g(x)=x^{3^x}$.

$f(x)=e^{\ln(3^{x^3})}=e^{x^3\ln 3}\Rightarrow f'(x)=e^{x^3\ln 3}\cdot 3x^2\ln 3=3^{x^3}\cdot 3x^2\cdot\ln 3$

$g(x)=e^{\ln(x)^{3^x}}=e^{3^x\ln x}\Rightarrow g'(x)=e^{3^x\cdot\ln x}\left[3^x\ln 3\cdot\ln x+3^x\cdot\frac{1}{x}\right]$

$=x^{3^x}\left[3^x\left(\ln 3\cdot\ln x+\frac{1}{x}\right)\right]$

Thus $\frac{dy}{dx}=\frac{df}{dx}+\frac{dg}{dx}$

$=3^{x^3}3x^2\ln 3+x^{3^x}\left[3^x\left(\ln 3\cdot\ln x+\frac{1}{x}\right)\right]$

3) a)(10 pts.) Evaluate $\lim_{x\to\infty}\left(\cos\left(\frac{2}{x}\right)\right)^{x^2} \longrightarrow 1^{\infty}$ undetermined form

$$\lim_{x\to\infty}\left(\cos\left(\frac{2}{x}\right)\right)^{x^2} = \lim_{x\to\infty} e^{\ln\left(\cos\left(\frac{2}{x}\right)\right)^{x^2}} = e^{\lim_{x\to\infty} x^2 \cdot \ln\left(\cos\left(\frac{2}{x}\right)\right)}$$

$$= e^{L}$$

where $L = \lim_{x\to\infty} x^2 \cdot \ln\left(\cos\left(\frac{2}{x}\right)\right) \to \infty \cdot 0$ und. form.

$$= \lim_{x\to\infty} \frac{\ln\left(\cos\left(\frac{2}{x}\right)\right)}{\frac{1}{x^2}} \overset{\left[\frac{0}{0}\right]}{\underset{\text{L'HR}}{=}} \lim_{x\to\infty} \frac{\frac{1}{\cos\left(\frac{2}{x}\right)} \cdot \left(-\sin\left(\frac{2}{x}\right)\right) \cdot \left(-\frac{2}{\cancel{x^2}}\right)}{-\frac{2}{\cancel{x}^3}}$$

$$= \lim_{x\to\infty} \underbrace{\frac{-1}{\cos\left(\frac{2}{x}\right)}}_{\to -1} \cdot \underbrace{\frac{2 \cdot \sin\left(\frac{2}{x}\right)}{\frac{2}{x}}}_{\to 1} = -2 \Rightarrow e^{L} = e^{-2}.$$

b)(10 pts.) Evaluate $\lim_{x\to 2}\frac{\int_{2x}^{x^2}\frac{dt}{\ln t}}{\ln(3-x)} \to \frac{0}{0}$ use L'HR

$$= \lim_{x\to 2} \frac{\frac{1}{\ln(x^2)} \cdot 2x - \frac{1}{\ln(2x)} \cdot 2}{\frac{-1}{3-x}}$$

Fund. Thm. C

$$= \frac{\frac{4}{\ln 4} - \frac{2}{\ln 4}}{-1} = -\frac{1}{\ln 2}$$

4) a)(10 pts.) Evaluate $\int \frac{\sin 2x}{\sin x - 1 - \cos^2 x} dx. = \int \frac{2\sin x \cdot \cos x}{\sin x - 1 - (1-\sin^2 x)} dx = 2\int \frac{\sin x \cdot \cos x}{\sin^2 x + \sin x - 2} dx$

$u = \sin x$
$du = \cos x \, dx$

$$= 2\int \frac{u}{u^2+u-2} du = \frac{2}{3}\left[\int \frac{du}{u-1} - \int \frac{du}{u+2}\right]$$

$$\frac{u}{u^2+u-2} = \frac{A}{u-1} + \frac{B}{u+2} \Rightarrow A = \frac{1}{3}, \; B = -\frac{1}{3}$$

$$= \frac{2}{3}\left[\ln|u-1| - \ln|u+2|\right] + C$$

$$= \ln\left|\frac{\sin x - 1}{\sin x + 2}\right|^{2/3} + C$$

b) (10 pts.) Evaluate $\int (\arccos x)^2 dx.$

Integration by parts: $u = (\arccos x)^2 \Rightarrow du = 2\arccos x \cdot \left(\frac{-1}{\sqrt{1-x^2}}\right) dx$

$dv = dx \Rightarrow v = x$

$$I = \int (\arccos x)^2 dx = x(\arccos x)^2 - \underbrace{\int \arccos x \cdot \left(\frac{-2x}{\sqrt{1-x^2}}\right) dx}_{\text{II}}$$

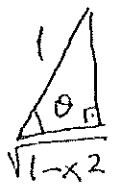
$$\text{II} = \int \arccos x \cdot \left(\frac{-2x}{\sqrt{1-x^2}}\right) dx = 2\sqrt{1-x^2} + \int 2\, dx = 2\sqrt{1-x^2} + 2x + C_1$$

$s = \arccos x \Rightarrow ds = \frac{-dx}{\sqrt{1-x^2}}$

$dt = \frac{-2x}{\sqrt{1-x^2}} dx \Rightarrow t = 2\sqrt{1-x^2}$

$$\Rightarrow I = x(\arccos x)^2 - 2\sqrt{1-x^2} - 2x + C$$

5)a)(10 pts.) Evaluate $\int \frac{(1-x^2)^{3/2}}{x^6}\,dx.$



$\sin\theta = x$ and $\theta = \arcsin x$ where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$\cos\theta\, d\theta = dx$

$$I = \int \frac{(1-\sin^2\theta)^{3/2}}{\sin^6\theta}\cos\theta\, d\theta = \int \frac{\cos^4\theta}{\sin^6\theta}\, d\theta = \int \cot^4\theta \cdot \csc^2\theta\, d\theta$$

$u = \cot\theta$

$du = -\csc^2\theta\, d\theta$

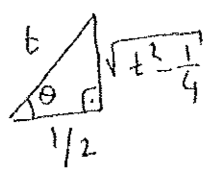
$$= -\int u^4\, du = -\frac{u^5}{5} + C = -\frac{\cot^5\theta}{5} + C = \frac{-1}{5}\left(\frac{\sqrt{1-x^2}}{x}\right)^5 + C$$

b)(10 pts.) Evaluate $\int_{1/2}^{1} \frac{dx}{\sqrt{x^2+x}} = \int_{1/2}^{1} \frac{dx}{\sqrt{(x+\frac{1}{2})^2 - \frac{1}{4}}} = \int_{1}^{3/2} \frac{dt}{\sqrt{t^2 - \frac{1}{4}}} = I$

$t = x + \frac{1}{2} \Rightarrow dt = dx$

$x = \frac{1}{2} \Rightarrow t = 1$ & $x = 1 \Rightarrow t = \frac{3}{2}$

$$I = \int_{1}^{3/2} \frac{dt}{\sqrt{t^2-\frac{1}{4}}} = \frac{1}{2}\int_{\pi/3}^{\operatorname{arcsec} 3} \frac{\sec\theta \cdot \tan\theta}{\sqrt{\frac{1}{4}(\sec^2\theta - 1)}}\, d\theta = \frac{1}{\cancel{2}}\int_{\pi/3}^{\operatorname{arcsec} 3} \frac{\sec\theta\, \cancel{\tan\theta}}{\frac{1}{\cancel{2}}\cancel{\tan\theta}}\, d\theta$$



$\sec\theta = 2t$ and $0 \le \theta < \frac{\pi}{2}$ (why?)

$t = \frac{\sec\theta}{2} \Rightarrow dt = \frac{\sec\theta\tan\theta}{2}\, d\theta$

$t = 1 \Rightarrow \theta = \frac{\pi}{3}$ & $t = \frac{3}{2} \Rightarrow \theta = \operatorname{arcsec} 3$

$$= \int_{\pi/3}^{\operatorname{arcsec} 3} \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| \Big|_{\pi/3}^{\operatorname{arcsec} 3}$$

$$= \ln\left(3 + \tan(\operatorname{arcsec} 3)\right) - \ln\left(\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right)$$

$$= \ln(3+\sqrt{8}) - \ln(2+\sqrt{3})$$

$$= \ln\left(\frac{3+\sqrt{8}}{2+\sqrt{3}}\right)$$