5.a) Assume f and g are continuous functions defined on $(0,\infty)$ such that

i) $f(x)$ grows faster than x^3 as $x \to \infty$, and

ii) $g(x)$ grows slower than e^x as $x \to \infty$.

Find $\lim\limits_{x\to\infty} \dfrac{(e^x - g(x))(x^3+1)}{e^{x+5}(f(x)+6)}$. **Explain your answer clearly.**

i) $\Rightarrow \lim\limits_{x\to\infty} \dfrac{x^3}{f(x)} = 0$ and ii) $\Rightarrow \lim\limits_{x\to\infty} \dfrac{g(x)}{e^x} = 0$

(i.e, $\lim\limits_{x\to\infty} f(x) = \infty$)

$$\lim_{x\to\infty} \frac{(e^x - g(x))(x^3+1)}{e^{x+5}(f(x)+6)} = \lim_{x\to\infty} \frac{\cancel{e^x}\left(1 - \frac{g(x)}{e^x}\right)}{\cancel{e^x}\cdot e^5} \, \frac{\cancel{f(x)}\left(\frac{x^3}{f(x)} + \frac{1}{f(x)}\right)}{\cancel{f(x)}\left(1 + \frac{6}{f(x)}\right)} = 0.$$

b) Which of the following functions $h(x) = \ln(x^2 + e^x)$, $k(x) = \ln(x^4 + e^{2x})$ grow faster as $x \to \infty$? Or do they grow at the same rate? **Explain your answer clearly.**

$$\lim_{x\to\infty} \frac{h(x)}{k(x)} = \lim_{x\to\infty} \frac{\ln(x^2+e^x)}{\ln(x^4+e^{2x})} \quad \left[\frac{\infty}{\infty}\right]$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{(2x+e^x)}{(x^2+e^x)} \, \frac{(x^4+e^{2x})}{4x^3+2e^{2x}}$$

$$= \lim_{x\to\infty} \frac{\cancel{e^x}\left(\frac{2x}{e^x}+1\right)}{\cancel{e^x}\left(\frac{x^2}{e^x}+1\right)} \, \frac{\cancel{e^{2x}}\left(\frac{x^4}{e^{2x}}+1\right)}{\cancel{e^{2x}}\left(\frac{4x^3}{e^{2x}}+2\right)} = \frac{1}{2}$$

since $\lim\limits_{x\to\infty} \dfrac{x^n}{e^x} = 0$ for any $n \in \mathbb{N}$

So the fcns. $h(x)$ and $k(x)$ grow at the same rate as $x \to \infty$