BİLKENT UNIVERSITY
Mathematics Department

Math 112 Intermediate Calculus II
Spring Semester 2009 - 2010

FIRST EXAM

2:00 pm - 4:00 pm (120 minutes)	*March 6, 2010*

Name : ANSWERS

Id. No. : ....

Section : ....

IMPORTANT

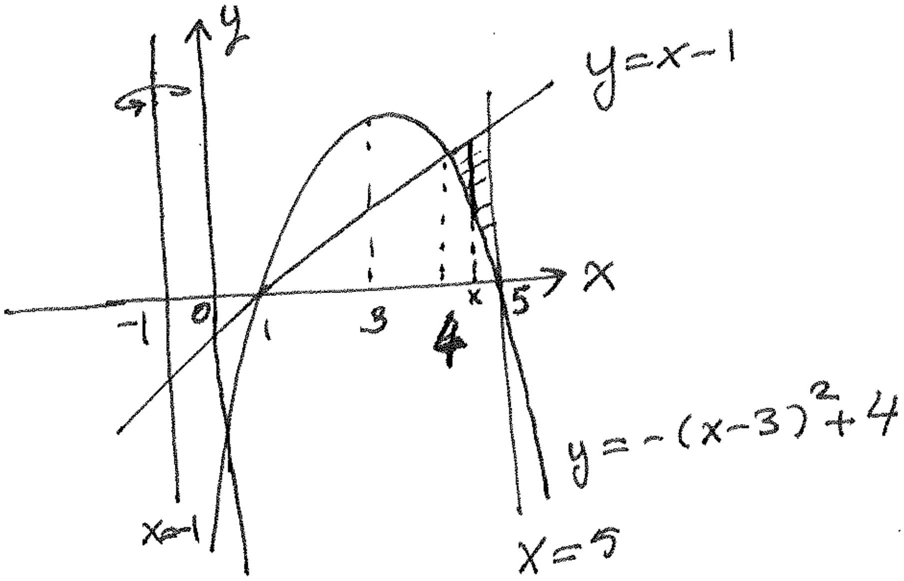
- This exam consists of 5 questions of the same weight.
- Each question is on a separate sheet. Please read the questions carefully and write your answers under the corresponding question. Be neat.
- Show all your work. Correct answers without sufficient explanation might not get the full credit.
- Calculators are not allowed.

Please do not write anything below this line.

Q1	Q2	Q3	Q4	Q5	Total
20 pt.	20 pt.	20 pt.	20 pt.	20 pt.	100 pt.

Question 1 (10+10=20 points)

a. Use the *Shell Method* to write an integral that gives the volume of the solid obtained by revolving the region bounded by the curves $y = -(x-3)^2 + 4$, $x = 5$, and $y = x - 1$ in the first quadrant about the line $x = -1$ (Do not evaluate the integral).



$$\begin{cases} y = -(x-3)^2 + 4 \\ y = x - 1 \end{cases}$$

$x = 1, 4$

$$V = 2\pi \int_4^5 \underbrace{(x+1)}_{\text{radius}} \left[(x-1) - \left(-(x-3)^2+4\right) \right] dx$$

b. Fill in the blanks:

The integral $I = \int_0^1 \sqrt{1 + 4x^2 e^{-2x^2}}\, dx$ can be thought of, geometrically, as the

(arc) length of the curve

$y = -e^{-x^2}$ with

$x \in [0,1]$, and

$I \geq d(P,Q) = \sqrt{(0-1)^2 + (-1 + e^{-1})^2}$

which is the length of the line segment joining the points

$P(0; -1)$

and

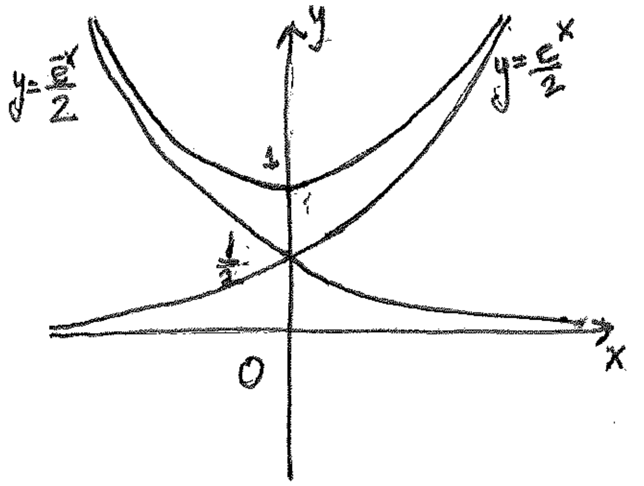
$Q(1; -e^{-1})$

that lie on the graph of

$y = -e^{-x^2}$

Question 2 (5+5+10=20 points)

a. Draw the graph of $y = \cosh x = \dfrac{e^x + e^{-x}}{2}$



b. Show that $\cosh^2 x - \sinh^2 x = 1$.

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

(Remember: $a^2 - b^2 = (a+b)(a-b)$)

$$= \frac{(e^x + \cancel{e^{-x}} + e^x - \cancel{e^{-x}})(\cancel{e^x} + e^{-x} - \cancel{e^x} + e^{-x})}{4}$$

$$= \frac{(2e^x)(2e^{-x})}{4}$$

$$= \frac{4}{4}$$

$$= 1$$

c. Find the area of the surface generated by revolving the curve $y = \cosh x$, $0 \le x \le \ln 2$ about the x-axis.

$$S = 2\pi \int_0^{\ln 2} \underbrace{\cosh x}_{r} \underbrace{\sqrt{1 + \sinh^2 x}\, dx}_{ds}$$

$$= 2\pi \int_0^{\ln 2} \underbrace{\cosh x \cosh x}_{\cosh^2 x}\, dx \qquad \text{from part b)}$$

$$= 2\pi \int_0^{\ln 2} \left(\frac{e^{x} + e^{-x}}{2} \right)^2 dx$$

$$= \frac{2\pi}{4} \int_0^{\ln 2} \frac{e^{2x} + 2 + e^{-2x}}{\cancel{4}}\, dx$$

$$= \frac{\pi}{2} \left(\frac{e^{2x}}{2} + 2x + \frac{e^{-2x}}{-2} \right) \Bigg|_{x=0}^{\ln 2}$$

$$= \frac{\pi}{2} \left[\left(\frac{e^{2\ln 2}}{2} + 2\ln 2 - \frac{e^{-2\ln 2}}{2} \right) - \left(\cancel{\frac{1}{2}} + 0 - \cancel{\frac{1}{2}} \right) \right]$$

$$= \frac{\pi}{2} \left(2 + 2\ln 2 - \frac{1}{8} \right)$$

$$= \pi \left(\frac{15}{16} + 2\ln 2 \right)$$

Question 3 (20 points) Find the length of the curve parametrized as follows:

$$C : x = \int_0^{t^2} \sqrt{1+u^3}\,du, \quad y = \frac{4\sqrt{2}}{7}t^{7/2}, \quad t \in [0, 1]$$

$$L = \int_{t=0}^{1} ds = \int_{t=0}^{1} \sqrt{(dx)^2 + (dy)^2}$$

$$= \int_{t=0}^{1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

where $\frac{dx}{dt} = 2t\sqrt{1+t^6}$ and $\frac{dy}{dt} = 2\sqrt{2}\,t^{5/2}$

$$= \int_0^1 \sqrt{\underbrace{4t^2(1+t^6) + 8t^5}_{4t^2(t^3+1)^2}}\, dt$$

$$= \int_0^1 2t(t^3+1)\,dt$$

$$= 2\int_0^1 (t^4 + t)\,dt$$

$$= 2\left(\frac{t^5}{5} + \frac{t^2}{2}\right)\Big|_{t=0}^{1}$$

$$= 2\left(\frac{1}{5} + \frac{1}{2}\right)$$

$$= \frac{7}{5}$$

Question 4 (10+10=20 points)

a. Let $f(x)$ be a differentiable function satisfying $f(2)=7$, $f(1)=0$, and $\int_1^2 \frac{f'(x)}{x}\mathrm{d}x = 8$. Compute $\int_0^{\ln 2} \frac{f(\mathrm{e}^x)}{\mathrm{e}^x}\mathrm{d}x$.

$$\int_0^{\ln 2} \frac{f(e^x)}{e^x}\,dx \;\underset{\substack{ds = e^x dx\\ = s\,dx}}{\overset{s=e^x}{=\!=}}\; \int_1^2 \frac{f(s)}{s^2}\,ds$$

$x=0 \Rightarrow s=1$
$x=\ln 2 \Rightarrow s=2$

(integration by parts)

$$\underset{du = f'(s)ds,\ v=-\frac{1}{s}}{\overset{u=f(s),\ dv=\frac{ds}{s^2}}{=\!=}}\; \left.\frac{-f(s)}{s}\right|_{s=1}^{2} + \underbrace{\int_1^2 \frac{f'(s)}{s}\,ds}_{8}$$

$$= -\frac{\overset{7}{\overbrace{f(2)}}}{2} - \frac{-\overset{0}{\overbrace{f(1)}}}{1} + 8 = -\frac{7}{2}+8 = \frac{9}{2}$$

b. Evaluate $\int \arctan\sqrt{x}\,\mathrm{d}x \;\underset{dt=\frac{dx}{2\sqrt{x}}}{\overset{t=\sqrt{x}}{=\!=}}\; \int 2t \arctan t\, dt$

$$\underset{du=\frac{dt}{1+t^2},\ v=t^2}{\overset{u=\tan^{-1}t,\ dv=2t\,dt}{=\!=}}\; t^2\tan^{-1}t - \int \frac{t^2+1-1}{1+t^2}\,dt$$

$$= t^2\tan^{-1}t - \int\left(1-\frac{1}{1+t^2}\right)dt$$

$$= t^2\tan^{-1}t - t + \tan^{-1}t + C, \quad C\in\mathbb{R}.$$

$$\overset{t=\sqrt{x}}{=} (\tan^{-1}\sqrt{x})(x+1) - \sqrt{x} + C$$

Question 5 (6+14=20 points) Evaluate the followings.

a. $\displaystyle\int \frac{\cos 2\theta}{\cos\theta+\sin\theta}d\theta = \int \frac{\cos^2\theta-\sin^2\theta}{\cos\theta+\sin\theta}\,d\theta$

$$= \int (\cos\theta - \sin\theta)\,d\theta$$

$$= \sin\theta + \cos\theta + A, \quad A \in \mathbb{R}.$$

$e^{3t}\cdot e^{t}$

$\|$

b. $\displaystyle\int \frac{e^{4t}}{(e^t+2)(e^{2t}-1)}dt \underset{dw=e^t dt}{\overset{w=e^t}{=\!=}} \int \frac{w^3}{\underbrace{(w+2)(w^2-1)}_{w^3+2w^2-w+2}}\,dw$

$$\begin{array}{r|l} w^3 & w^3+2w^2-w+2 \\ -(w^3+2w^2-w+2) & 1 \\ \hline -2w^2+w-2 & \end{array}$$

$$\frac{2w^2-w+2}{(w+2)(w^2-1)} = \frac{A}{w+2}+\frac{B}{w-1}+\frac{C}{w+1}$$

$$\Rightarrow 2w^2-w+2 = A(w^2-1)+B(w+2)(w+1) + C(w+2)(w-1)$$

$w=1 \Rightarrow B=1/2$

$w=-1 \Rightarrow C=-5/2$

$w=-2 \Rightarrow A=4$

$$= \int\left(1-\frac{2w^2-w+2}{(w+2)(w^2-1)}\right)dw$$

$$= w - \int\left(\frac{\overset{=4}{A}}{w+2}+\frac{\overset{=1/2}{B}}{w-1}+\frac{\overset{-5/2}{C}}{w+1}\right)dw$$

$$= w - 4\ln|w+2| - \frac{1}{2}\ln|w-1| - \frac{5}{2}\ln|w+1| + C$$

$$\overset{w=e^t}{=} e^t - 4\ln(e^t+2) - \frac{1}{2}\ln|e^t-1| - \frac{5}{2}\ln(e^t+1) + C.$$