Solutions of Midterm 1

1.Evaluate

(10 pts.) a) $\int \sqrt{4+e^x}\,dx$

Let $w=\sqrt{4+e^x}$,so $dw=\dfrac{e^x dx}{2\sqrt{4+e^x}}=\dfrac{w^2-4}{2w}dx$ .Then

$$\begin{aligned}\int \sqrt{4+e^x}\,dx &= \int \frac{2w^2}{w^2-4}dw\\ &= 2\int (1+\frac{4}{w^2-4})dw \quad \text{where } \frac{4}{w^2-4}=\frac{A}{w-2}+\frac{B}{w+2}, A=1, B=-1.\\ &= 2\int (1+\frac{1}{w-2}-\frac{1}{w+2})dw\\ &= 2\left[w+\ln\left|\frac{w-2}{w+2}\right|\right]+C\\ &= 2\left[\sqrt{4+e^x}+\ln\left(\frac{\sqrt{4+e^x}-2}{\sqrt{4+e^x}+2}\right)\right]+C.\end{aligned}$$

(10 pts.) b) $\int \dfrac{e^{\arctan x}}{(1+x^2)^{3/2}}dx$

$t=\arctan x \Leftrightarrow x=\tan t$ .Then $dx=\sec^2 t\,dt$ and $(1+x^2)^{3/2}=\sec^3 t$.

$$\begin{aligned}\int \frac{e^{\arctan x}}{(1+x^2)^{3/2}}dx &= \int \frac{e^t(\sec^2 t)dt}{\sec^3 t}\\ &= \int e^t \cos t\,dt\end{aligned}$$

Take $u=e^t$ and $dv=\cos t\,dt$ .Then $du=e^t dt$ and $v=\sin t$ .Thus

$\int e^t\cos t\,dt = e^t \sin t-\int e^t \sin t\,dt$

In the right integral,let $u=e^t$ and $dv=\sin t\,dt$,so $du=e^t dt$ and $v=-\cos t$ .Then

$\int e^t\cos t\,dt = e^t\sin t+e^t\cos t-\int e^t\cos t\,dt$ and

$\int e^t\cos t\,dt = \dfrac{e^t}{2}(\sin t+\cos t)+C$ .Thus

$\int \dfrac{e^{\arctan x}}{(1+x^2)^{3/2}}dx = \dfrac{e^{\arctan x}}{2}(\sin(\tan^{-1}x)+\cos(\tan^{-1}x))+C$.

2.Evaluate

(10 pts.) a) $\lim\limits_{x\to\pi^+}\dfrac{\ln(x-\pi)}{\ln(e^x-e^\pi)}:\left[\dfrac{\infty}{\infty}\right]$.We can use l'Hospital's Rule:

$\lim\limits_{x\to\pi^+}\dfrac{\ln(x-\pi)}{\ln(e^x-e^\pi)}=\lim\limits_{x\to\pi^+}\dfrac{\frac{d}{dx}(\ln(x-\pi))}{\frac{d}{dx}(\ln(e^x-e^\pi))}=\lim\limits_{x\to\pi^+}\dfrac{e^x-e^\pi}{e^x(x-\pi)}$ is also indeterminate(i.e.,$\left[\dfrac{0}{0}\right]$),but a second application of l'Hospital's Rule gives

$$\lim_{x\to\pi^+}\frac{\ln(x-\pi)}{\ln(e^x-e^\pi)}=\lim_{x\to\pi^+}\frac{e^x-e^\pi}{e^x(x-\pi)}=\lim_{x\to\pi^+}\frac{e^x}{e^x(x-\pi)+e^x}=1$$

(10 pts.) b) $\lim_{x\to 0}(\cos 2x)^{1/x^2} : 1^\infty$

$$L = \lim_{x\to 0}(\cos 2x)^{1/x^2} = \lim_{x\to 0} e^{\frac{\ln(\cos 2x)}{x^2}} = e^{\lim_{x\to 0}\frac{\ln(\cos 2x)}{x^2}} = e^{L_1} \text{ (continuity of exp)}$$

where

$$L_1 = \lim_{x\to 0}\frac{\ln(\cos 2x)}{x^2} : \left[\frac{0}{0}\right]$$

$$\overset{l'H}{=} \lim_{x\to 0}\left(\frac{\sin 2x}{2x}\right)\left(\frac{-2}{\cos 2x}\right) = -2$$

Thus, $L = e^{-2}$.

3. (10 pts.) a) Find $y(\frac{3\pi}{4})$ if $y = x - \sin^{-1}(\sin x)$

$$y(\frac{3\pi}{4}) = \frac{3\pi}{4} - \sin^{-1}(\sin\frac{3\pi}{4}) = \frac{3\pi}{4} - \sin^{-1}(\frac{1}{\sqrt{2}}) = \frac{3\pi}{4} - \frac{\pi}{4} = \frac{\pi}{2}.$$

(10 pts.) b) Show that $\sin^{-1}\left(\frac{x-1}{x+1}\right) = 2\tan^{-1}\sqrt{x} - \frac{\pi}{2}$

Take $f(x) = \sin^{-1}\left(\frac{x-1}{x+1}\right) - 2\tan^{-1}\sqrt{x} + \frac{\pi}{2}$.

$$f'(x) = \frac{1}{\sqrt{1-\left(\frac{x-1}{x+1}\right)^2}}\frac{d}{dx}\left(\frac{x-1}{x+1}\right) - 2\frac{1}{1+\left(\sqrt{x}\right)^2}\frac{d}{dx}\left(\sqrt{x}\right) \quad = 0 \Rightarrow f(x) = C \text{ and}$$

$f(0) = -\frac{\pi}{2} + \frac{\pi}{2} = 0$. Therefore, $f(x) = 0$.

4. (10 pts.) a) Let f be a twice differentiable function such that $f'(1) = f'(5) = 0$. Show that $\int_1^5 f''(x)f(x)dx \le 0$.

Let $u = f(x) \quad dv = f''(x)dx$

Then $du = f'(x)dx \quad v = f'(x)$

$$\int_1^5 f''(x)f(x)dx = f(x)f'(x)\Big|_1^5 - \int_1^5 (f'(x))^2 dx = -\int_1^5 (f'(x))^2 dx \le 0.$$

Remember: $f \ge 0$ for $a \le x \le b \Rightarrow \int_a^b f(x)dx \ge 0$.

(10 pts.) b) Suppose for a certain function f it is known that $f'(x) = e^{x^2}$, $f(2) = 7$.

Evaluate $\int_0^2 f(x)dx$.

Let $u = f(x) \quad dv = dx$

Then $du = f'(x)dx \quad v = x$

$$\int_0^2 f(x)dx = xf(x)\Big|_0^2 - \int_0^2 xf'(x)dx$$

$$= 2f(2) - \frac{1}{2}\int_0^2 2xe^{x^2}\,dx$$

$$= 2.7 - \frac{1}{2}(e^{x^2})\Big|_0^2$$

$$= \frac{29 - e^4}{2}.$$

5. (5 pts.) a) Find the domain of $h(x) = \arccos(x^2 - 2x)$.

$$D = \{x : -1 \le x^2 - 2x \le 1\} = \{x : 0 \le |x - 1| \le \sqrt{2}\} = \{x : 1 - \sqrt{2} \le x \le 1 + \sqrt{2}\}.$$

(15 pts.) b) Let g be a differentiable one-to-one function, f be a differentiable function.

Given that $g(3) = 5, g(1) = 2, g'(3) = 7, g'(1) = -3, f'(24) = \frac{1}{2}, f'(8) = -\frac{1}{3}$ and

$y = f(x^3.g^{-1}(x^2 + 1))$ .Evaluate $\left.\frac{dy}{dx}\right|_{x=2}$.

$$\frac{dy}{dx} = f'(x^3.g^{-1}(x^2 + 1))\frac{d}{dx}(x^3.g^{-1}(x^2 + 1))$$

$$= f'(x^3.g^{-1}(x^2 + 1))\left[3x^2 g^{-1}(x^2 + 1) + x^3\frac{d}{dx}(g^{-1}(x^2 + 1))\right]$$

where

$$\frac{d}{dx}\left(g^{-1}(x^2 + 1)\right) = \frac{2x}{g'(g^{-1}(x^2 + 1))}, \left. g^{-1}(x^2 + 1)\right|_{x=2} = g^{-1}(5) = 3, \text{ and}$$

$$\left.\frac{d}{dx}\left(g^{-1}(x^2 + 1)\right)\right|_{x=2} = \frac{4}{g'(3)} = \frac{4}{7}$$

$$\left.\frac{dy}{dx}\right|_{x=2} = f'(3.8).\left[3.4.3 + (8)\frac{4}{7}\right] = \frac{142}{7}$$