Math.112 ,HW 8

Due on Apr. 13,2009 up to 12:50 p.m.

1. Find the sum of the series $\sum_{n=2}^{\infty}\frac{2n+1}{(n^2-1)(n+2)n}$.

2. Evaluate $\lim_{n\to\infty}\frac{\sum_{k=1}^{n}\frac{1}{k}}{\ln n}$.

(Hint: Remember the proof of the Integral Test)

Determine whether the following series is convergent

3. $\sum_{n=1}^{\infty} n^2 e^{-\sqrt{n}}$,

4. $\sum_{n=3}^{\infty}\frac{1}{\sqrt{n}\ln n+\sin^2 n}$.

(1) $$\frac{2n+1}{(n-1)(n+1)(n+2)n} = \frac{A}{(n-1)n} + \frac{B}{(n+1)(n+2)} \quad \text{7 pts} \quad \Longrightarrow \quad A=\frac{1}{2} \ \& \ B=-\frac{1}{2} \quad \text{2 pts}$$

$$S_n = \sum_{k=2}^{n}(b_k - b_{k+2}) = (b_2 - b_4) + (b_3 - b_5) + \dots + (b_n - b_{n+2})$$
$$= b_2 + b_3 - b_{n+1} - b_{n+2} \quad +$$

where $b_k = \frac{1/2}{(n-1)n}$

$$+ \lim_{n\to\infty} S_n = \lim_{n\to\infty}(b_2 + b_3 - b_{n+1} - b_{n+2}) = \lim_{n\to\infty}\left(\frac{1}{3} - \frac{1/2}{n(n+1)} - \frac{1/2}{(n+1)(n+2)}\right)$$
$$= \frac{1}{3}$$

9 pts. 7 pts.

OR

$$a_n = \frac{2n+1}{(n-1)n(n+1)(n+2)} = \frac{\overset{\frac{1}{2}}{A}}{n-1} + \frac{\overset{-\frac{1}{2}}{B}}{n} + \frac{\overset{-\frac{1}{2}}{C}}{n+1} + \frac{\overset{\frac{1}{2}}{D}}{n+2} = b_n - b_{n+1}$$

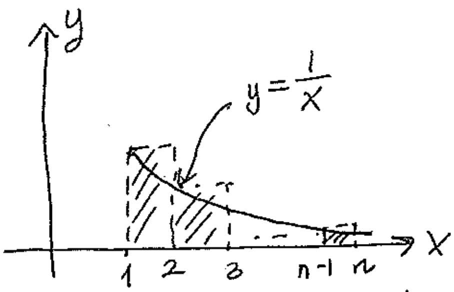
where $b_n = \frac{1/2}{n-1} - \frac{1/2}{n+1} = \frac{1}{n^2-1}$

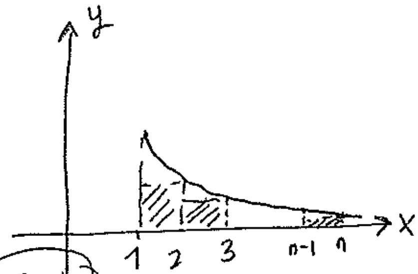
$$S_n = \sum_{k=2}^{n}(b_k - b_{k+1}) = (b_2 - b_3) + (b_3 - b_4) + \dots + (b_n - b_{n+1})$$
$$= b_2 - b_{n+1}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}(b_2 - b_{n+1}) = \lim_{n\to\infty}\left(\frac{1}{3} - \frac{1}{n^2+2n}\right) = \frac{1}{3}.$$

- partial fraction decomposition (5 pts.)
- coefficients (4 pts.)
- S_n (9 pts.)
- limit (7 pts.)

(2)





(A) $1+\frac{1}{2}+\cdots+\frac{1}{n} \le \ln n+1$ (5 pts.)

$$\frac{1}{2}+\cdots+\frac{1}{n} \le \underbrace{\int_1^n \frac{dx}{x}}_{\ln n} \le 1+\frac{1}{2}+\cdots+\frac{1}{n-1}$$

(B) $\frac{1}{n}+\ln n \le 1+\frac{1}{2}+\cdots+\frac{1}{n}$ (5 pts)

(A) & (B) $\Rightarrow \frac{1}{n}+\ln n \le \sum_{k=1}^{n}\frac{1}{k} \le \ln n+1$ (5 pts)

Thus, $\frac{1}{n\ln n}+1 \le \frac{\sum_{k=1}^{n}\frac{1}{k}}{\ln n} \le 1+\frac{1}{\ln n}$ (5 pts)

$\to 1$ (5 pts.)

Hence, $\lim_{n\to\infty}\frac{\sum_{k=1}^{n}\frac{1}{k}}{\ln n}=1$ by the Sandwich Theorem.

(3) Use the integral test since $f(x)=x^2e^{-\sqrt{x}}$ is cont. (2 pts) (total 8 pts.)
(2 pts) positive and decreasing for $x>16$. Indeed, (4 pts)

$f'(x)=2xe^{-\sqrt{x}}\left(1-\frac{\sqrt{x}}{4}\right)<0$ if $x>16$.

(2 pts.)

$$\int_{16}^{\infty} x^2e^{-\sqrt{x}}\,dx \overset{\sqrt{x}=y}{=} \int_4^{\infty} 2y^5e^{-y}\,dy = \lim_{b\to\infty}\int_4^b 2y^5e^{-y}\,dy$$

$$=\lim_{b\to\infty}\left[-2e^{-y}\left(y^5+5y^4+20y^3+60y^2+120y+120\right)\Big|_4^b\right]<\infty$$

$\Rightarrow \int_{16}^{\infty} x^2e^{-\sqrt{x}}\,dx$ converges

$\overset{\text{Int. Test}}{\Longrightarrow} \sum_{n=1}^{\infty} n^2e^{-\sqrt{n}}$ converges. (5 pts.)

③

OR, Find $p>0$ s.t. $\lim_{n\to\infty} \dfrac{\frac{n^2}{e^{\sqrt{n}}}}{\left(\frac{1}{\sqrt{n}}\right)^p} = 0$, and $\sum \frac{1}{n^{p/2}}$ conver. 8pts. 2pts.

Then, $\lim_{n\to\infty} \dfrac{n^{2+p/2}}{e^{\sqrt{n}}} \underset{\sqrt{n}=x}{=} \lim_{x\to\infty} \dfrac{x^{4+p}}{e^x} = 0$, $\forall p>2$. 7pts. and $\sum \frac{1}{n^{p/2}}$ conv. $(p>2)$ 2pts.

Compare $\sum n^2 e^{-\sqrt{n}}$ with $\sum \frac{1}{n^{p/2}}$, $p>2$.

L. Comp. Thm 2pts $\Longrightarrow \sum n^2 e^{-\sqrt{n}}$ converges. 4 pts.

④ $\ln x < x$ for $x \geq 1$. 3pts. So $\ln n = \ln(\sqrt{n})^2 = 2\ln\sqrt{n} \leq 2\sqrt{n}$. 5 pts. 2 pts.

Hence $\sqrt{n}\ln n + \sin^2 n \leq 2n+1 < 3n$, $n>3$. 4 pts.

$\therefore \dfrac{1}{\sqrt{n}\ln n + \sin^2 n} > \dfrac{1}{3n}$ 2pts and $\sum \frac{1}{3n}$ div., harmonic series 2pt

3pts. Comp. Thm. $\Longrightarrow \sum \dfrac{1}{\sqrt{n}\ln n + \sin^2 n}$ diverges. 4pts.